NEWTON-SOR ITERATIVE METHOD FOR SOLVING THE TWO-DIMENSIONAL POROUS MEDIUM EQUATION

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ABSTRACT

In this paper, we consider the application of the Newton-SOR iterative method in obtaining the approximate solution of the two-dimensional porous medium equation (2D PME). The nonlinear finite difference approximation equation to the 2D PME is derived by using the implicit finite difference scheme. The developed nonlinear system is linearized by using the Newton method. At each temporal step, the corresponding linear systems are solved by using SOR iteration. We investigate the efficiency of the Newton-SOR iterative method by solving three examples of 2D PME and the performance is compared with the Newton-GS iterative method. Numerical results show that the Newton-SOR iterative method is better than the Newton-GS iterative method in terms of a number of iterations, computer time and maximum absolute errors.

Keywords: porous medium equation; finite difference scheme; Newton; Successive Over Relaxation, Gauss-Seidel.

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1. INTRODUCTION

Mathematical models that are developed in order to describe many natural and engineering phenomena are generally written in the forms of partial differential equation (PDE). The phenomena are usually modeled and described together with the appropriate initial and boundary conditions. The solution of the mathematical models can be obtained with respect to the given initial and boundary conditions.

With exceptional to the case where the exact solution can be obtained, most of the nonlinear PDE problems are difficult to be solved. The solution of the nonlinear heat and mass transfer problem, for instance plays a significant role in the study of various thermal and diffusion processes. It makes the study of the mechanisms in the particular complex physical phenomena easier to be conducted. In addition to that, a particular exact solution of PDE problem can be used to verify the accuracy and efficiency of various numerical methods. In the other words, PDE problems with exact solutions can be used in the experiment of developing new numerical methods which in turn enable researchers to study more complex PDE problems [1].

This paper considers the application of the Newton-Successive Over Relaxation (Newton-SOR) iterative method for solving the two-dimensional porous medium equation (2D PME) under the bounded spatial region $\Omega = [0,1] \times [0,1]$ and time interval $t = (0,1)$ which can be defined as [2]:

$$\frac{\partial u}{\partial t} = \alpha \left[\frac{\partial}{\partial x} \left(u^m \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(u^m \frac{\partial u}{\partial y} \right) \right], u(x,y,t) = g(x,y,t), \ (x,y) \in \Omega$$
$$u(x,y,0) = u_0(x,y).$$

$$(1)$$

where m and α are rational numbers, satisfying the exact solution $g(x,y,t)$ and subject to the initial condition $u_0(x,y)$.

From the recent literature survey on the application and the solution of Equation (1), in [3] studied the role of transient solidification processing parameters for the designing of the microstructure in the continuous casting process. They employed the inverse heat conduction method to solve the heat flow across the metal casting/mould interface phenomena, which is characterized by a macroscopic average metal/mould interfacial heat transfer coefficient based

on the measure of the temperature inside the casting. They used the explicit finite difference scheme onto the set of boundary conditions and the heat conduction equation that resembles Equation (1). The heat flow problem is solved by minimizing the standard least square norm between the estimated temperatures computed from the solution of the direct problem in which using the estimated values of the heat transfer coefficient components and the experimental temperatures.

In [4] studied the solution of the instability phenomenon based on the double phase flow of two immiscible fluids through the inclined homogeneous porous medium. They implemented the Crank-Nicolson finite difference scheme onto the derived nonlinear PDE that similar with the one-dimensional case of Equation (1) and solved it numerically.

Other than the use of finite difference scheme,Equation (1) which can also be known as the heat and mass transfer equation with power-law temperature-dependent thermal conductivity was studied by [5]. They applied an analytical approach called the Adomian's decomposition method for the solution. Besides that, the one-dimensional case of the PME problem has been studied numerically by [6-7] with the use finite difference scheme and iterative methods.

This paper aims to solve Equation (1) with the use finite difference scheme together with the application of the Newton-SOR iterative method. This study investigates the efficiency of the applied numerical technique in terms of the number of iterations, the computer time and the maximum absolute errors over the Newton-Gauss-Seidel (Newton-GS) iterative method [8].

2. FINITE DIFFERENCE APPROXIMATION EQUATION

In this paper, we discretize $u(x,y,t)$ on an equidistant spatial-temporal grid $(x_i, y_j, t_n) = (ih, jh, nk)$ with $1 \leq i, j \leq M$, $0 < n < T$ where $h = \Delta x = \Delta y$ and $k = \Delta t$ are spatial and temporal steps respectively. The implicit finite difference scheme is used due to its good stability and yields lesser computational cost compared to Crank-Nicolson scheme. When Equation (1) is discretized by using the implicit finite difference scheme, the resultant finite difference approximation equation can be written as

$$f\left(u_{i,j,n+1}\right)=u_{i,j,n+1}-A_1\left(u_{i,j,n+1}^m u_{i+1,j,n+1}-2u_{i,j,n+1}^{m+1}+u_{i,j,n+1}^m u_{i-1,j,n+1}\right)$$
$$-A_2\left(mu_{i,j,n+1}^{m-1}u_{i+1,j,n+1}^2-2mu_{i,j,n+1}^{m-1}u_{i+1,j,n+1}u_{i-1,j,n+1}+mu_{i,j,n+1}^{m-1}u_{i-1,j,n+1}^2\right)$$
$$-A_3\left(u_{i,j,n+1}^m u_{i,j+1,n+1}-2u_{i,j,n+1}^{m+1}+u_{i,j,n+1}^m u_{i,j-1,n+1}\right) \qquad (2)$$
$$-A_4\left(mu_{i,j,n+1}^{m-1}u_{i,j+1,n+1}^2-2mu_{i,j,n+1}^{m-1}u_{i,j+1,n+1}u_{i,j-1,n+1}+mu_{i,j,n+1}^{m-1}u_{i,j-1,n+1}^2\right)-u_{i,j,n}$$

where $A_1=A_3=\dfrac{\alpha k}{(h)^2}$, $A_2=A_4=\dfrac{\alpha k}{4(h)^2}$ with $k=\dfrac{1}{T}$ and $h=\dfrac{1}{M}$.

The implementation of Equation (2) to every interior mesh point in region Ω will result a sparse and large system of nonlinear equations as it relates every neighborhood points. Such nonlinear system can be generally written as

$$F\left(\underset{\sim}{U}_{n+1}\right)=0 \qquad (3)$$

where $\quad F\left(\underset{\sim}{U}_{n+1}\right)=\left(f_{1,1},f_{2,1},\ldots,f_{1,2},f_{2,2},\ldots,f_{M,M}\right)^T \quad$ and

$\underset{\sim}{U}_{n+1}=\left(u_{1,1,n+1},u_{2,1,n+1},\ldots,u_{1,2,n+1},u_{2,2,n+1},\ldots,u_{M,M,n+1}\right).$

Solving Equation(3) generally will face the high complexity problem. Therefore, it needs an efficient iterative method to be solved.

3. NEWTON-SOR ITERATIVE METHOD

In the year 1967, in [8] have studied the monotone iteration processes for the nonlinear elliptic boundary value problems and introduced the Newton-GS iterative method. This method uses the Gauss-Seidel iteration on the linear systems of Newton's method or the linearized system of equations. They have suggested that by using the optimum relaxation parameter of $\omega>1$ on Newton-GS iterative method, a faster convergence of the linearized system of equations can be achieved. More in [9] has extended the study of the convergence properties of the Newton-GS iterative method and produced the global convergence theorems for the Newton-GS iterative method and its related methods.

Actually, when a relaxation parameter $1<\omega<2$ is applied onto the Newton-GS iterative method such method can be extended and called as the Newton-SOR iterative method. The concept of the SOR iteration was introduced by [10].

Now, Newton-SOR iterative method will be applied to solve Equation (3). To begin, Newton method is used to transform Equation (3) into the corresponding linear system as follows:

$$J_F\left(\underset{\sim\, n+1}{U^{(\ell)}}\right)\underset{\sim\, n+1}{\Delta H^{(\ell)}} = -\underset{\sim}{F}\left(\underset{\sim\, n+1}{U^{(\ell)}}\right), \; \ell = 1,2,\dots \tag{4}$$

where $J_F\left(\underset{\sim\, n+1}{U^{(\ell)}}\right)$ is a pentadiagonal $M^2 \times M^2$ Jacobian matrix, $\underset{\sim\, n+1}{\Delta H^{(\ell)}}$ and $-\underset{\sim}{F}\left(\underset{\sim\, n+1}{U^{(\ell)}}\right)$ are column

matrices with iterative index (ℓ) and time level $n+1$.

Let the matrix $J_F\left(\underset{\sim\, n+1}{U^{(\ell)}}\right)$ be decomposed into

$$J_F\left(\underset{\sim\, n+1}{U^{(\ell)}}\right) = D - L - V \tag{5}$$

By using Equation (5), the SOR iteration can be implemented onto Equation (4) and the

resultant Newton-SOR iterative method can be written as

$$\underset{\sim\, n+1}{\Delta H^{(\ell+1)}} = (1-\omega)\underset{\sim\, n+1}{\Delta H^{(\ell)}} + \omega(D-L)^{-1}\left(V\underset{\sim\, n+1}{\Delta H^{(\ell)}} - \underset{\sim}{F}\left(\underset{\sim\, n+1}{U^{(\ell)}}\right)\right) \tag{6}$$

where ω represents the relaxation parameter and $\underset{\sim\, n+1}{\Delta H^{(\ell+1)}} = \underset{\sim\, n+1}{U^{(\ell+1)}} - \underset{\sim\, n+1}{U^{(\ell)}}$.

In practice, the optimum value of ω is determined by running the Algorithm 1 several times

until the least number of iterations is obtained. Moreover, when $\omega=1$, the Newton-SOR

iterative method reverts back to the Newton-GS iterative method.

Algorithm 1: Newton-SOR Iterative Method

i.   Compute A_1, A_2, A_3, and A_4,

ii.   Compute $-\underset{\sim}{F}\left(\underset{\sim\, n+1}{U^{(\ell)}}\right)$ and $J_F\left(\underset{\sim\, n+1}{U^{(\ell)}}\right)$,

iii.   Iterate Eq. (6),

iv.   If $\left|\underset{\sim\, n+1}{\Delta H^{(\ell+1)}} - \underset{\sim\, n+1}{\Delta H^{(\ell)}}\right| < \varepsilon$, compute the solution vectors $\underset{\sim\, n+1}{U^{(\ell+1)}} = \underset{\sim\, n+1}{\Delta H^{(\ell+1)}} + \underset{\sim\, n+1}{U^{(\ell)}}$,

v.   Check the convergence of the solution vectors $\left|\underset{\sim\, n+1}{U^{(\ell+1)}} - \underset{\sim\, n+1}{U^{(\ell)}}\right| < \varepsilon$,

vi.   Go to the next temporal step.

4. NUMERICAL EXPERIMENTS

In this section, numerical results are presented in order to observe the performance and the

efficiency of the proposed Newton-SOR iterative method against the control Newton-GS

iterative method. Three parameters which are the number of iterations (ℓ), the computer time

in seconds (s) and the maximum absolute errors (error) will be measured for a comparative

analysis between the Newton-SOR and the Newton-GSiterative methods. The following three 2D PME are used as the test problems.

Example 1 [2]:

We choose $\alpha = 0.2$ and $m = 1$ in Equation (1) becomes

$$\frac{\partial u}{\partial t} = 0.2 \left[\frac{\partial}{\partial x}\left(u \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(u \frac{\partial u}{\partial y}\right)\right]. \tag{7}$$

and the exact solution is given by $u(x, y, t) = x + y + (0.4)t$.

Example 2 [2]:

We take $\alpha = 0.2$ and $m = 2$, and Equation (1) becomes

$$\frac{\partial u}{\partial t} = 0.2 \left[\frac{\partial}{\partial x}\left(u^2 \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(u^2 \frac{\partial u}{\partial y}\right)\right]. \tag{8}$$

and the exact solution is given by $u(x, y, t) = (5(x + y + t))^{1/2}$.

Example 3 [11]:

We use $\alpha = 1.0$ and $m = 5$ and the problem becomes

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u^5 \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(u^5 \frac{\partial u}{\partial y}\right). \tag{9}$$

and the exact solution is $u(x, y, t) = \left(\dfrac{4(x + y + 2t)}{5}\right)^{1/4}$.

Throughout the simulations, the convergence parameter is set to be $\varepsilon = 10^{-10}$ and the size of matrices to be considered are 16, 32, 64, 128 and 256. The numerical results for the three examples are tabulated in Tables 1, 2 and 3.

Table 1. Comparison of number of iterations (ℓ), computer time in seconds (s) and maximum

absolute error (error) for Example 1

M	Method	ω	(ℓ)	(s)	Error
16	GS	1.00	136	0.35	8.87E-11
	SOR	1.19	100	0.28	3.93E-12
32	GS	1.00	438	2.35	2.97E-10
	SOR	1.43	207	1.43	1.59E-11
64	GS	1.00	1526	19.77	1.88E-09
	SOR	1.64	412	8.47	5.83E-11
128	GS	1.00	5459	258.55	8.96E-09
	SOR	1.80	813	55.72	8.16E-11
256	GS	1.00	19388	4275.66	3.85E-08
	SOR	1.89	1608	454.53	1.44E-10

Table 2. Comparison of number of iterations (ℓ), computer time in seconds (s) and maximum

absolute error (error) for Example 2

M	Method	ω	(ℓ)	(s)	Error
	GS	1.00	130	0.47	7.57E-11
	SOR	1.16	100	0.30	2.58E-11
	GS	1.00	400	1.96	2.31E-09
	SOR	1.37	208	1.48	1.26E-10
	GS	1.00	1380	19.26	1.31E-08
	SOR	1.60	420	8.68	2.97E-10
	GS	1.00	4901	248.82	4.95E-08
	SOR	1.77	836	56.86	5.66E-10
	GS	1.00	17458	4243.79	1.75E-07
	SOR	1.87	1658	478.09	1.50E-09

Table 3. Comparison of number of iterations (ℓ), computer time in seconds (s) and maximum absolute error (error) for Example 3

M	Method	ω	(ℓ)	(s)	Error
16	GS	1.00	739	0.82	1.10E-09
	SOR	1.55	291	0.50	2.84E-11
32	GS	1.00	2630	8.51	6.69E-09
	SOR	1.74	573	2.92	3.97E-11
64	GS	1.00	9478	113.63	3.56E-08
	SOR	1.86	1118	18.86	5.69E-11
128	GS	1.00	34098	1653.85	1.72E-07
	SOR	1.93	2178	131.63	6.65E-11
256	GS	1.00	121649	29234.80	7.78E-07
	SOR	1.96	4276	1163.89	6.26E-11

Based on Table 1, it can be observed that with the increasing size of matrices M, the Newton-SOR iterative method exhibits a lesser number of iterations and faster computer time compared to the Newton-GS iterative method when it is at the optimal relaxation parameter. Then, we have computed the percentage of improvement in terms of a number of iterations $(\Delta\ell)\%$ and computer time $(\Delta s)\%$ by the Newton-SOR iterative method against the Newton-GS iterative method using the expressions:

$$(\Delta\ell)\% = \left|\frac{\ell_{GS} - \ell_{SOR}}{\ell_{GS}}\right| \times 100\% \tag{10}$$

and

$$(\Delta s)\% = \left|\frac{s_{GS} - s_{SOR}}{s_{GS}}\right| \times 100\%, \tag{11}$$

and the result is tabulated in Table 4.

Table 4. Percentage of improvement in terms of number of iterations $(\Delta\ell)\%$ and computer time $(\Delta s)\%$ by the Newton-SOR method against the Newton-GS method

M	Number of iterations $(\Delta\ell)\%$
16	23.08% - 60.62%
32	48.00% - 78.21%
64	69.57% - 88.20%
128	82.94% - 93.61%
256	90.50% - 96.48%
M	**Computer time** $(\Delta s)\%$
16	20.00% - 39.02%
32	24.49% - 65.69%
64	54.93% - 83.40%
128	77.15% - 92.04%
256	88.73% - 96.02%

5. CONCLUSION

This paper investigates the application and the efficiency of the Newton-SOR iterative method in obtaining the approximate solution of the two-dimensional porous medium equation (2D PME). The numerical results obtained based on the tested examples 1, 2 and 3 show that the application of the Newton-SOR iterative method, which uses the optimal relaxation parameter, able to reduce the number of iterations and computer time for solving the 2D PME(see in Tables 1, 2, 3 and 4). Besides that, the accuracy of the numerical solutions by the Newton-SOR iterative method is better than the Newton-GS iterative method. It can be seen in Tables 1, 2 and 3 that the maximum absolute errors produced by the Newton-SOR iterative method is smaller than by the Newton-GS iterative method. This work will be extended to study the uses of two different relaxation parameters or the Modified SOR [12].

6. ACKNOWLEDGEMENTS

7. REFERENCES

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