Journal of Fundamental and Applied Sciences

ISSN 1112-9867

Available online at http://www.jfas.info

Research Article

Special Issue

MODIFICATION OF CRAMER'S RULE

O. Babarinsa[1,*] and H. Kamarulhaili[2]

[1]Department of Mathematical Sciences, Federal University Lokoja, 11554 Kogi, Nigeria

[2]School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM Penang, Malaysia

Published online: 17 October 2017

ABSTRACT

While Cramer's rule allows complete substitution of constant terms to the coefficient matrix in the system of linear equations, the modified methods of Cramer's rule consider the constant terms as well as the coefficients of the matrix at the same time. The methods are derived from one of the properties of determinants. Furthermore, we prove the two methods to be equivalent and provide MATLAB codes for the modified methods. However, the methods are not practically suitable for higher system of linear equations because they inherit inefficiency and instability of Cramer's rule.

Keywords: Cramer's rule; determinant; system of linear equation.

Author Correspondence, e-mail: babs3in1@gmail.com

doi: http://dx.doi.org/10.4314/jfas.v9i5s.39

1. INTRODUCTION

If for n linear equations in n unknowns $x_1, x_2, x_3, \ldots, x_n$ is defined by

$$\begin{array}{ccccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & + & \cdots & + & a_{1n}x_n & = & c_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & + & \cdots & + & a_{2n}x_n & = & c_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & + & \cdots & + & a_{3n}x_n & = & c_3 \\ \vdots & + & \vdots & + & \vdots & + & \cdots & + & \vdots & = & \vdots \\ a_{n1}x_1 & + & a_{n2}x_2 & + & a_{n3}x_3 & + & \cdots & + & a_{nn}x_n & = & c_n \end{array} \quad (1)$$

Equation (1) can equivalently be written as matrix equation of the form,

$Ax_i = c$ (2)

where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \text{ and } c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}$$

the $n \times n$ matrix A (coefficient matrix) is nonsingular, c the constant term and the vector $x = (x_1, x_2, ..., x_n)^T$ is the column vector of the variables, $\forall A, c \in \Re$. Thus, the solutions of Equation (1) can be derived from an ancient method called Cramer's rule [1].

1.1. Theorem 1 (Cramer's Rule)

Let $Ax = c$ be a $n \times n$ system of linear equation and A a $n \times n$ matrix of x such that $\det(A) \neq 0$, then the unique solution $x_1, x_2, x_3, ..., x_n$ to the system in Equation (1) is given by

$$x_i = \frac{\det(A_{i|c})}{\det(A)} \quad (3)$$

where $A_{i|c}$ is the matrix obtained from A by substituting the column vector c to the ith column of A, for $i = 1,2,...,n$.

Historically, an Italian mathematician GerolamoCardanogave a rule for solving a system of two linear equations which called *regula de modo*-mother of rules. Though, his methods were practically based on 2×2 resultants. The rule later gave what we essentially known as Cramer's rule [2]. It was Colin MacLaurin [3], a Scottish mathematician that gave the first published results on resultants on solving two and three simultaneous equations in a book titled "*Treatise of Algebra*". In fact, in [4] showed that Cramer's rule was published two years earlier in Colin Maclaurin's posthumous. In [5]examined a manuscript that provides conclusive evidence that Maclaurin was teaching his students "Cramer's rule" over 20 years before Cramer published it. However, in [6] argued that the rule he chose to appropriate sign for each summand was wrong, though his assertion of "opposite" coefficient was right and this was corrected by Cramer by counting the number of transpositions, *dérangements*, in the permutation. In [7]pointed that for lack of good notation, Maclaurin missed the general rule for solving linear equations.

Regardless of its high complexity time, Cramer's rule is historically interesting and it is of theoretical importance for solving systems of linear equations [8]. It gives a clear representation of an individual component unconnected to all other components. Cramer's rule via Laplace expansion method of determinant has time complexity of $O(n.n!)$ and $O(n^3)$ when compared with other fast and concise methods such as K-Chio's method [9-10].

Cramer's rule has many disadvantages, it fails when the determinant of the coefficient matrix is zero, requires many calculations of determinants (if determinant values are calculated through minors) and is also numerically unstable [11]. Due to the disadvantages of Cramer's rule, in[12] expressed that Cramer's rule is unsatisfactory even for 2×2 linear systems because of round off error. However, in[13] gave counter example. Gauss elimination, Jacobi method and Gauss-Jordan elimination are efficient iterative and numerical methods that have succeeded Cramer's rule [14] including parallel Cramer's rule (PCR) for solving singular linear systems [15].

There are many previous work on Cramer's rule that made use of properties of determinants, especially cofactor in their proofs which includes Jacobi's proof [16] that led to Turdi's proof and rediscovered in [17]. Recently, Cramer's rule has been proved via adjoint matrix and the proof by identity matrix was adopted to solve a linear system of equation using elementary row operations make Cramer's rule invariant [18].

2. MODIFICATION OF CRAMER'S RULE

It may be a new proof of an old fact or it may be a new approach to several facts at the same time. If the new proof establishes same previously unsuspected connections between two ideas; it often leads to a generalization [19]. This paper provides two distinct approaches in solving system of linear equation. The new methods establish same previously unsuspected connections with Cramer's rule and derived from one of the properties of determinant. The formulas for the two methods make use of one to normalize it to standard Cramer's rule. The two methods are explained in this paper with proofs.

2.1.Method I

It is a well-established theorem that if the ith column in matrix A is a sum (difference) of the ith column of a matrix B and the ith column of a matrix C and all other rows in B and C are equal to the corresponding rows in A that is if two determinants differ by just one column [20-21] such that

$$A = \begin{bmatrix} b_{11} \pm c_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ b_{21} \pm c_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ b_{31} \pm c_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} \pm c_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ b_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ b_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \text{and}$$

$$C = \begin{bmatrix} c_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ c_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ c_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

For

$$A = B \pm C \quad (4)$$

then

$$\det(A) = \det(B) \pm \det(C)$$

2.1.1. Corollary 1

Let $Ax = c$ be a $n \times n$ system of linear equation and A is $n \times n$ matrix of x, if $\det(A) \neq 0$, then the $i$thentry x_i of the unique solution $x = x_1, x_2, x_3, \ldots, x_n$ is given by

$$x_i = \frac{\det(A_{i+c})}{\det(A)} - 1 \quad (5)$$

where A_{i+c} is the matrix obtained from A by adding the constant terms of vector c to the i thcolumn of A, for $i = 1, 2, \ldots, n$.

2.1.2.Proof

We adopt the assumptions of Cramer's rule as we let $\det(A)$ be determinant of the system for coefficient matrix such that $\det(A) \neq 0$ and equivalently extend Equation (4) to more general form by substituting c in the ith column of matrix A as

$$A_{i+c} = A \pm A_{i|c} \quad (6)$$

where

$$A_{i\pm c}=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1i}\pm c_{1i} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2i}\pm c_{2i} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3i}\pm c_{3i} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{ni}\pm c_{ni} \end{bmatrix},\quad A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1i} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2i} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3i} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{ni} \end{bmatrix} \text{ and}$$

$$A_{i|c}=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & c_{1i} \\ a_{21} & a_{22} & a_{23} & \cdots & c_{2i} \\ a_{31} & a_{32} & a_{33} & \cdots & c_{3i} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & c_{ni} \end{bmatrix}$$

we can deduce from Equation (6) that

$$\det(A_{i\pm c})=\det(A)\pm\det(A_{i|c}) \quad (7)$$

and by considering the positive sign of the above equation according to Corollary (1) we have

$$\det(A_{i+c})=\det(A)+\det(A_{i|c}) \quad (8)$$

Thus,

$$\det(A_{i|c})=\det(A_{i+c})-\det(A) \quad (9)$$

Hence, substitute Equation (9) in Equation (3)

$$\begin{aligned} x_i &= \frac{\det(A_{i|c})}{\det(A)} \\ &= \frac{\det(A_{i+c})-\det(A)}{\det(A)} \\ &= \frac{\det(A_{i+c})}{\det(A)}-1 \end{aligned}$$

The MATLAB code on single physical processor for method I is provided in Fig. 1.

```
function x=Method1(A,b)
      A=input('matrix A =');
      b=input('vector b =');
      n=size(A,1);
      m=size(A,2);
    if n =m
          Error ('The matrix is not square.');
          x=[];
   else
        detA=det(A);
       if det(A) =0
          x=zeros(n,1);
          for j=1:n
            if j =1 & j =n
                 Ab=[A(:,1:j-1)  b+A(:,j)  A(:,j+1:n)];
            elseif j==1
                 Ab=[b+A(:,1)  A(:,2:n)];
            elseif j==n
                 Ab=[A(:,1:n-1)  b+A(:,n)];
            end
             x(j)=(det(Ab)/detA) - 1;
         end
      else
         Error ('The matrix A has a zero determinant.');
            x=[];
      end
  end
```

Fig.1. MATLAB code for Method I

2.2.Method II

All assumptions of method 1 still hold except that the constant terms are subtracted from the coefficients of the variables in each column. Let $\det(A)$ be determinant of the system for coefficient matrix, provided that $\det(A)\neq 0$ and let $\det(A_{i-c})$ denotes the n th-order determinant from $\det(A)$ by subtracting the constant terms (nonhomogeneous terms $(c_1, c_2, ..., c_n)$ from the i th column of A, for $i = 1,2,..., n$.

2.2.1. Corollary 2

Let $Ax = c$ be $n\times n$ system of linear equation and A is $n\times n$ matrix of x, if $\det(A)\neq 0$, then the i thentry x_i of the unique solution $x = x_1, x_2, x_3, ..., x_n$ is given by

$$x_i = 1-\frac{\det(A_{i-c})}{\det(A)} \quad (10)$$

where A_{i-c} is the matrix obtained from A by subtracting the constant terms of vector c from the i th column of A, for $i = 1,2,..., n$.

2.2.2. Proof

By considering the minus sign of Equation (7) based on Corollary (2), we have

$\det(A_{i-c}) = \det(A) - \det(A_{i|c})$ (11)

Thus,

$\det(A_{i|c}) = \det(A) - \det(A_{i-c})$ (12)

Substituting Equation (12) in Equation (3), we have

$$x_i = \frac{\det(A_{i|c})}{\det(A)}$$
$$= \frac{\det(A) - \det(A_{i-c})}{\det(A)}$$
$$= 1 - \frac{\det(A_{i-c})}{\det(A)}$$

The MATLAB code for method II on single physical processor is provided in Fig. 2.

```
function x=Method2(A,b)
      A=input('matrix A =');
      b=input('vector b =');
      n=size(A,1);
      m=size(A,2);
    if n =m
          Error ('The matrix is not square.');
          x=[];
   else
        detA=det(A);
        if det(A) =0
            x=zeros(n,1);
           for j=1:n
              if j =1 & j =n
                   Ab=[A(:,1:j-1)  A(:,j)-b  A(:,j+1:n)];
             elseif j==1
                   Ab=[A(:,1)-b  A(:,2:n)];
             elseif j==n
                   Ab=[A(:,1:n-1)  A(:,n)-b];
             end
              x(j)= 1 -(det(Ab)/detA);
         end
      else
         Error ('The matrix A has a zero determinant.');
            x=[];
      end
  end
```

Fig.2.MATLAB code for Method II

2.2.3. Proposition 1

Given a $n \times n$ system of linear equation, $Ax = c$, where A is $n \times n$ matrix of x such that $\det(A) \neq 0$ for the distinct solution of x_i and c the column vector. If $x_i = \frac{\det(A_{i+c})}{\det(A)} - 1$ when the column vector c is added to the column of matrix A and $x_i = 1 - \frac{\det(A_{i-c})}{\det(A)}$ when the column vector c is subtracted from the column of matrix A, then

$$\frac{\det(A_{i+c})}{\det(A)}-1=1-\frac{\det(A_{i-c})}{\det(A)}$$

2.2.4. Proof

We consider Equation (5) of Corollary (1) to proof this proposition by substituting Equation (8) in it to have

$$x_i=\frac{\det(A)+\det(A_{i|c})}{\det(A)}-1$$

$$\therefore\ x_i=\frac{\det(A_{i|c})}{\det(A)} \quad (13)$$

Now, substitute Equation (12) in Equation (13) to get

$$x_i=\frac{\det(A)-\det(A_{i-c})}{\det(A)}$$

$$=1-\frac{\det(A_{i-c})}{\det(A)}$$

Similarly, Equation (10) in Corollary (2) can be used to proof Equation (5).

2.3.Numerical Example

Without loss of generality, we provide a numerical example in the given system of linear equations:

$$\begin{aligned}
2x_1+5x_2-9x_3+3x_4&=151\\
5x_1+6x_2-4x_3+2x_4&=103\\
3x_1-4x_2+2x_3+7x_4&=16\\
11x_1+7x_2+4x_3-8x_4&=-32
\end{aligned}$$

2.3.1. Method I

The method adds the constant terms to each of the column in coefficient matrix. Thus,

$$x_1=\frac{\begin{vmatrix}2+151 & 5 & -9 & 3\\ 5+103 & 6 & -4 & 2\\ 3+16 & -4 & 2 & 7\\ 11+(-32) & 7 & 4 & -8\end{vmatrix}}{\begin{vmatrix}2 & 5 & -9 & 3\\ 5 & 6 & -4 & 2\\ 3 & -4 & 2 & 7\\ 11 & 7 & 4 & -8\end{vmatrix}}-1=\frac{-9492}{-2373}-1=3$$

$$x_2 = \frac{\begin{vmatrix} 2 & 5+151 & -9 & 3 \\ 5 & 6+103 & -4 & 2 \\ 3 & -4+16 & 2 & 7 \\ 11 & 7+(-32) & 4 & -8 \end{vmatrix}}{\begin{vmatrix} 2 & 5 & -9 & 3 \\ 5 & 6 & -4 & 2 \\ 3 & -4 & 2 & 7 \\ 11 & 7 & 4 & -8 \end{vmatrix}} - 1 = \frac{-14238}{-2373} - 1 = 5$$

$$x_3 = \frac{\begin{vmatrix} 2 & 5 & -9+151 & 3 \\ 5 & 6 & -4+103 & 2 \\ 3 & -4 & 2+16 & 7 \\ 11 & 7 & 4+(-32) & -8 \end{vmatrix}}{\begin{vmatrix} 2 & 5 & -9 & 3 \\ 5 & 6 & -4 & 2 \\ 3 & -4 & 2 & 7 \\ 11 & 7 & 4 & -8 \end{vmatrix}} - 1 = \frac{23730}{-2373} - 1 = -11$$

$$x_4 = \frac{\begin{vmatrix} 2 & 5 & -9 & 3+151 \\ 5 & 6 & -4 & 2+103 \\ 3 & -4 & 2 & 7+16 \\ 11 & 7 & 4 & -8+(-32) \end{vmatrix}}{\begin{vmatrix} 2 & 5 & -9 & 3 \\ 5 & 6 & -4 & 2 \\ 3 & -4 & 2 & 7 \\ 11 & 7 & 4 & -8 \end{vmatrix}} - 1 = \frac{-18984}{-2373} - 1 = 7$$

2.3.2. Method II

This method subtracts the constant terms from the column being substituted to. Hence, the solutions are:

$$x_1 = 1 - \frac{\begin{vmatrix} 2-151 & 5 & -9 & 3 \\ 5-103 & 6 & -4 & 2 \\ 3-16 & -4 & 2 & 7 \\ 11-(-32) & 7 & 4 & -8 \end{vmatrix}}{\begin{vmatrix} 2 & 5 & -9 & 3 \\ 5 & 6 & -4 & 2 \\ 3 & -4 & 2 & 7 \\ 11 & 7 & 4 & -8 \end{vmatrix}} = 1 - \frac{4746}{-2373} = 3$$

$$x_2 = 1 - \frac{\begin{vmatrix} 2 & 5-151 & -9 & 3 \\ 5 & 6-103 & -4 & 2 \\ 3 & -4-16 & 2 & 7 \\ 11 & 7-(-32) & 4 & -8 \end{vmatrix}}{\begin{vmatrix} 2 & 5 & -9 & 3 \\ 5 & 6 & -4 & 2 \\ 3 & -4 & 2 & 7 \\ 11 & 7 & 4 & -8 \end{vmatrix}} = 1 - \frac{9492}{-2373} = 5$$

$$x_3 = 1 - \frac{\begin{vmatrix} 2 & 5 & -9-151 & 3 \\ 5 & 6 & -4-103 & 2 \\ 3 & -4 & 2-16 & 7 \\ 11 & 7 & 4-(-32) & -8 \end{vmatrix}}{\begin{vmatrix} 2 & 5 & -9 & 3 \\ 5 & 6 & -4 & 2 \\ 3 & -4 & 2 & 7 \\ 11 & 7 & 4 & -8 \end{vmatrix}} = 1 - \frac{-28476}{-2373} = -11$$

$$x_4 = 1 - \frac{\begin{vmatrix} 2 & 5 & -9 & 3-151 \\ 5 & 6 & -4 & 2-103 \\ 3 & -4 & 2 & 7-16 \\ 11 & 7 & 4 & -8-(-32) \end{vmatrix}}{\begin{vmatrix} 2 & 5 & -9 & 3 \\ 5 & 6 & -4 & 2 \\ 3 & -4 & 2 & 7 \\ 11 & 7 & 4 & -8 \end{vmatrix}} = 1 - \frac{14238}{-2373} = 7$$

3. CONCLUSION

The two methods show the flexibility of computing Cramer's rule and ensure that there is no loss of generality in the coefficient matrix. The methods are also show how property of determinant led to the modification of Cramer's rule. The presence of one in the formulae is to normalize the modified methods to classical Cramer's rule. These methods are more of theoretical and are impracticable nor efficient in numerical world because Cramer’s rule is also not efficient for larger system of linear equations. However, they do better in handling relative residual error for small ill-conditioned system than Cramer’s rule. Further modification on the methods may increase their efficiency and stability.

4. ACKNOWLEDGEMENTS

The authors of this paper are very grateful for the suggestions of the anonymous reviewers.

5.REFERENCES

[1] Debnath L. A brief historical introduction to matrices and their applications. International Journal of Mathematical Education in Science and Technology, 2013,45(3):360-377

[2] Cardano G., Witmer T. R., Ore O. The rules of algebra (Ars Magna). New York: Dover Publications Inc., 2007

[3] MacLaurin C. A treatise of algebra. London: A. Millar and J. Nourse, 1748

[4] Boyer C. Colin Maclaurin and Cramer's rule. Scripta Mathematica, 1966, 27(4):377-379

[5] Hedman B. An Earlier Date for "Cramer's Rule". Historia Mathematica, 1999, 26(4):365-368

[6] Kosinski A. Cramer's rule is due to Cramer. Mathematics Magazine, 2001, 74(4):310-312

[7] Günther S. Geschichte der Mathematik. Leipzig: G.J. Göschen, 1908

[8] Brunetti M. Old and new proofs of Cramer's rule. Applied Mathematical Sciences, 2014, 8(133):6689-6697

[9] Habgood K, Arel I. A condensation-based application of Cramer's rule for solving large-scale linear systems.Journal of Discrete Algorithms, 2012, 10:98-109

[10] Shafarevich I., Remizov A. Linear algebra and geometry. Berlin: Springer Verlag, 2012

[11] Debnath L. A brief historical introduction to determinant with applications. International Journal of Mathematical Education in Science and Technology, 2013, 44(3):388-407

[12] Moler C. Cramer's rule on 2-by-2 systems. ACM SIGNUM Newsletter, 1974, 9(4):13-14

[13] Dunham CB. Cramer's rule reconsidered or equilibration desirable. ACM SIGNUM Newsletter, 1980, 15(4):9-9

[14] Watkins D. Fundamentals of matrix computations. New Jersey: John Wiley and Sons, 2004

[15] Gu C, Wang G, Xu Z. PCR algorithm for the parallel computation of the solution of a class of singular linear systems. Applied Mathematics and Computation, 2006, 176(1):237-244

[16] Muir T. The theory of determinants in the historical order of development. London:

Macmillan and Company Limited, 1911

[17] Li H, Huang T, Gu T, Liu X. From Sylvester's determinant identity to Cramer's rule. 2014, https://arxiv.org/pdf/1407.1412.pdf

[18] Ufuoma O. A new and simple method of solving large linear systems based on Cramer's rule but employing Dodgson's condensation. In World Congress on Engineering and Computer Science, pp. 23-25

[19] Halmos P. The heart of mathematics. The American Mathematical Monthly, 1980, 87(7):519-524

[20] Turnbull H. The theory of determinants, matrices, and invariants. New York: Dover Publications, 1960

[21] Aitken A. Determinants and matrices. Edinburgh: Interscience Publishers, 1956

How to cite this article:
Babarinsa O, Kamarulhaili H. Modification of cramer's rule. J. Fundam. Appl. Sci., 2017, *9(5S), 556-567.*