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THE DIRECT DISCRETE METHOD FOR THE ANALYSIS MAGNETOHYDRODYNAMIC EQUATIONS

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ABSTRACT

The Direct Discrete Method has been applied to derive the algebraic form of the Magnetohydrodynamic equations. It is shown that the method can be considered as a replacement for the various differential based discritization methods. This approach is based on the global variables that are used in two cell complexes- one dual of the other- to subdivide the 3D space. In this paper, using the discrete form of the differential operators, we discretized the magnetohydrodynamic equations directly into a set of algebraic equations. Using this method, an example of a 3D magnetohydrodynamic equilibrium problem as a case study has also been represented.

Keywords: Direct Discrete Method, magnetohydrodynamic equations, cell complex, discretization

1. INTRODUCTION

The laws of MHD phenomena are usually written in the form of differential equations as the starting point in order to find the numerical solutions.

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Then the differential equations need to be discretized into algebraic equations by some techniques based on the differential formulation. Different discretization techniques have been applied to MHD [1], such as: Finite Difference Method (FDM), Finite Element Method (FEM), Finite Volume Method (FVM), etc. [2].Even in FVM, the standard practice is to use the integrals of the field functions, which is an essential component of the differential formulation.

The geometrical content of physical laws, e.g., balance and circuital laws, are omitted in the limiting process of differential formulation to obtain point functions and that is the motivation to analyze the physical theories with DDM.

In this way it is possible to formulate the MHD laws in a system of algebraic equations that can be directly applied to formulate the MHD equations.

The DDM has been successfully applied to the numerical solutions of the different fields of computational physics, such as electromagnetism [3], [4], [5], fluid acoustics [6], elastodynamics, and heat conduction [7].

The approach that avoids the differential formulation to produce discretized equations and express them directly in an algebraic form originated from the basic physical law is known as Direct Discrete Method (DDM) or the Cell Method. In the following sections we discretized the set of magnetohydrodynamic (MHD) equations using DDM.

This paper is organized as follows: A description of the method including the concept of global quantity and the elements of the cell complex are introduced in section 2. The prime and dual cell complexes of space and time and the associated inner and outer orientations are also presented in this section. The incidence numbers constituting the discrete form of gradient, curl, and divergence operators are also defined in this section. Section 3 addresses the discrete forms of MHD equations . each of them are based on the physics originated from a the relative physical law. In section 4 the equations of balance including mass and momentum balance are derived. surface forces are investigated in section 5. An example of MHD equilibrium problem is also represented in section 6. Section 7 gives some concluding remarks.

2. Global quantities in the cell complexes

The global or integral variables are used in DDM. They are defined as the variables that are not local points. In contrast to a local field that is defined as a space density or time rate, they have been integrated over space or time elements. . In other words, we integrate a local variable over a spatial element or a time period to obtain a global variable as a measurable variable. Based on the role that a physical variable plays in DDM, it may be classified into one of the three types of variables; configuration variables that describe the configuration of the system, source variables that describe the sources of the system, and energy variables that are obtained by the product of a configuration by a source variable. These variables are related to spatial and temporal element. The points (P), lines (L), surfaces (S), and volumes (V), whose numbers are N_p , N_l , N_s ,and N_v, are denoted by 0-cells, 1-cells, 2-cells, and 3-cells of the spatial cell complex, respectively. A set of p-cells is conceived as a spatial cell complex, where p is the dimension of the cell. Given a cell complex (primal cell complex), one can construct another cell complex (dual cell complex), whose vertices are a set of points inside the primal 3-cells, e.g., their barycenters and the edges of the dual cell complex are constructed by connecting the adjacent barycenters with straight lines. Doing so, a Voronoi dual cell complex is made. If n denotes the dimension of the space, there is an (n-p)-cell of the dual cell complex for each p-cell of the primal one. For example there is a dual volume for a primal point in 3D.

Temporal elements are instants and intervals. The time interval between two primal instants (I) is a primal interval (T) and the middle instant of a primal interval is a dual instant$(\tilde{I})$.The interval between two dual instants is the dual interval $(\tilde{T})$. This implies that DDM considers a proper instant or interval for a certain variable. If a physical time dependent variable is not the rate of another variable, it will be global variable in time [8], [9].

We must assign inner or outer orientations to spatial and temporal elements. If the orientation of a space element lies on it, we ascribe an inner orientation to it. On the other hand, the outer orientation depends on the dimension of the space in which the element is embedded. They are concepts that depend on the nature of the quantity and can truly describe the geometrical orientation in the physical theory. The oddness condition says that a global physical variable, associated with a space or time element, changes its sign, if the element reverses its orientation. Inner orientation of a positive point is considered as a sink. Inner orientation of a line is considered along the line. Inner orientation of a surface and volume inherit their orientations from the line. A tilde is used to specify the outer orientation of the space and time elements. A point is considered as the inner orientation of the volume containing the point. The outer orientation of a line is considered as the inner orientation of a surface crossing the line. The outer orientation of a surface is considered as the inner orientation of the line crossing the surface.

The dependence of the global variables on the oriented geometrical elements is presented with a square bracket; a tilde is used for the outer orientation to distinguish from the inner one.

We should be very careful how to assign a global variable to a certain cell complex and its orientation. For example, magnetic flux is associated with 2-cells of primal cell complex and so on.

As a general rule the configuration variables are associated with cells with inner orientation, while the source variables are associated with cells with outer orientation. The outer orientation is assigned to the elements of Voronoi diagram as the dual cell complex $\tilde{\mathrm{K}}$ whose elements are $\tilde{\mathrm{P}}$,$\tilde{\mathrm{L}}$, $\tilde{\mathrm{S}}$, and $\tilde{\mathrm{V}}$. Since for any p-cell in $K=\{p_h, l_\alpha, s_\beta, v_k\}$, there is an (n-p)-cell in $\tilde{K}=\{\tilde{v}_h, \tilde{s}_\alpha, \tilde{l}_\beta, \tilde{p}_k\}$, where n is the dimension of the space, the outer orientation of $\tilde{\mathrm{V}}$ is specified as inward normal to its faces (related to the inner orientation of P). The outer orientations of dual cell complex are also defined similarly.

The relation between $\mathrm{p_h}$ and $\mathrm{l_\alpha}$ in K is denoted by an incidence number $\mathrm{g_{\alpha h}}$. If $\mathrm{p_h}$ is a face of $\mathrm{l_\alpha}$, depending on the their orientations to be compatible or incompatible, the value of $\mathrm{g_{\alpha h}}$ is $+1$or -1, respectively. If $\mathrm{p_h}$ is not a face of $\mathrm{l_\alpha}$, the value of $\mathrm{g_{\alpha h}}$ is 0. The $N_l \times N_p$incidence gradient matrix $[G]$ composed of $\mathrm{g_{\alpha h}}$ values links $\mathrm{l_\alpha}$with $\mathrm{p_h}$. The incidence number $\tilde{\mathrm{d}}_{\mathrm{h\alpha}}$links $\tilde{v}_h$ and $\tilde{\mathrm{s}}_\alpha$ and the arrays of the $N_s \times N_v$ incidence divergence matrix $[\tilde{D}]$, which corresponds to $\mathrm{g_{\alpha h}}$. The minus sign refers to the conventional sign for the inner orientation of P.

Similarly, the $N_s \times N_l$incidence curl matrix $[C]$ links $\mathrm{l_\alpha}$ with $\mathrm{s_\beta}$. The $\mathrm{N_s \times N_l}$ matrix $[\tilde{\mathrm{C}}^{\mathrm{T}}]$ links the dual elements $\tilde{\mathrm{s}}_\alpha$ and $\tilde{\mathrm{l}}_\beta$. The relations between these incidence matrices are as follows:

$$-[\mathrm{G}] = [\tilde{\mathrm{D}}^{\mathrm{T}}], \qquad (1.a)$$

$$[\mathrm{C}] = [\tilde{\mathrm{C}}^{\mathrm{T}}], \qquad (1.b)$$

$$[\mathrm{D}] = [\tilde{\mathrm{G}}^{\mathrm{T}}]. \qquad (1.c)$$

3. MHD equations in DDM

The set of equations governing MHD consists of the two coupled parts of electromagnetism and fluid dynamics that are derived in DDM in the following.

The electromagnetic part of MHD are the reduced Maxwell equations in which the MHD assumptions are considered. They are expressed in DDM as follows:

3.1. Magnetic Gauss's Law

The magnitude of resistance in high temperature plasmas is negligible and that's why the "infinite conductivity" approximation is applied in the dynamical phenomena and then we have an ideal MHD. The flux through any closed material loop is conserved and consequently the magnetic field lines behave like elastic bands frozen into the single fluid plasma. The Magnetic flux flowing through the boundaries of a prime 3-cell vanishes in any prime instant. This law can be written in terms of incidence numbers $d_{k\beta}$ and incidence matrix $[D]$ as follows:

$$\Phi_m[\partial V, t_n] = 0 \tag{2.a}$$

$$\sum_\beta d_{k\beta}\{\phi_m\} = 0, \tag{2.b}$$

$$[D]\{\phi_m\} = 0. \tag{2.c}$$

3.2. Faraday's Law

Faraday's law in its most general form embodies many of the MHD phenomena. The impact of the electromotive force related to a prime 2-cell in an interval is equal to the changes of magnetic flux flowing through it. This law can be written in terms of incidence numbers $c_{\beta\alpha}$and incidence matrix $[C]$ as follows:

$$\mathscr{E}[\partial S, \tau_n] = \phi[S, t_n^-] - \phi[S, t_n^+] \tag{3.a}$$

$$\sum_\alpha c_{\beta\alpha}\mathcal{U}_{e\alpha}[\tau_{n+1}, l_\alpha] + \{\phi[t_{n+1}, s_\beta] - \phi[t_n, s_\beta]\} = 0, \tag{3.b}$$

$$[\mathrm{C}]\{\mathcal{U}_\mathrm{e}\} = \Delta_\mathrm{t}\{\phi_\mathrm{m}\} \tag{3.c}$$

3.3. Ampere's Law

The impact of the magnetomotive force related to the boundaries of a dual 2-cell in an interval is equal to electric charge flowing through it during the interval. This law can be written in terms of incidence numbers $\tilde{c}_{\alpha\beta}$ and incidence matrix $[\tilde{C}]$ as follows:

$$\mathscr{F}[\partial\tilde{\mathrm{S}}, \tilde{\tau}_\mathrm{n}] = \mathrm{Q}^{\mathrm{flow}}[\tilde{\mathrm{S}}, \tilde{\tau}_\mathrm{n}] \tag{4.a}$$

$$\sum_\beta \tilde{c}_{\alpha\beta}\mathscr{F}_{\mathrm{m}\beta}(t_n) = \{I\}, \tag{4.b}$$

$$[\tilde{C}]\{F\} = \{I\} \qquad (4.c)$$

3.4. Electric Charge Conservation Law

The electric charge flowing through the boundaries of a dual 3-cell in an interval vanishes due to the quasi-steady approximation of MHD. This law can be written in terms of incidence numbers $\tilde{d}_{h\alpha}$and incidence matrix $[\tilde{D}]$ as follows:

$$Q^{flow}[\partial\tilde{v}_h, \tilde{t}_n] = 0 \qquad (5.a)$$

$$\sum_\beta \tilde{d}_{h\alpha}\{I_\alpha\} = 0, \qquad (5.b)$$

$$[\tilde{D}]\{I_\alpha\} = 0 \qquad (5.c)$$

The set equations (2) through (5) express the reduced Maxwell equations suited for MHD in DDM.

The constitutive equations link source with configuration variables. For the electromagnetism part of MHD equations, they are written as follows:

$$\frac{\phi_B[t_n, s_\beta]}{s_\beta} \approx \mu \frac{\mathscr{F}_m[\tilde{\tau}_n, \tilde{l}_\beta]}{\tilde{\tau}_n \tilde{l}_\beta} \qquad (6)$$

$$\frac{Q^{flow}[\tilde{\tau}_n, \tilde{s}_\alpha]}{\tilde{\tau}_n \tilde{s}_\alpha} \approx \frac{\sigma}{2}\left(\frac{\mathscr{E}[\tau_n, l_\alpha]}{\tau_n l_\alpha} + \frac{\mathscr{E}[\tau_{n+1}, l_\alpha]}{\tau_{n+1} l_\alpha}\right) \qquad (7)$$

Where μ and σ are the permeability and conductivity constants of the cell v_c, respectively.

4. Balance Equations

The balance equations are related to entering or leaving an entity from a volume and that's why they are always written in the dual cell complex $\tilde{K}$. There are two balance equations governing the MHD equations; mass and momentum balance equations. In the following subsections they are

4.1. Mass Balance Equation

The continuity of mass within a dual 3-cell for an incompressible fluid requires that the number of particles inside a given volume is changed only if there is a net flux of the particles flowing through the surface enclosing that volume. Therefore, mass continuity equation can be written as follows:

$$M^{cont}[\tilde{\boldsymbol{t}}_{n+1}, \tilde{\boldsymbol{v}}_h] - M^{cont}[\tilde{\boldsymbol{t}}_n, \tilde{\boldsymbol{v}}_h] + M^{flow}[\tilde{\boldsymbol{\tau}}_n, \partial\tilde{\boldsymbol{v}}_h] = 0 \qquad (8)$$

Where M^{flow} is the net mass flow during a dual interval [7].

MHD supposes that the time variation of the mass content of the single fluid plasma in a certain volume does not change in time. Therefore, the first two terms of (8) vanishes and then (8) in terms of incidence numbers $\tilde{d}_{h\alpha}$ and incidence matrix $[\tilde{D}]$ becomes as follows:

$$M^{flow}[\tilde{\boldsymbol{\tau}}_n, \partial\tilde{\boldsymbol{v}}_h] = 0 \qquad (9.a)$$

$$\sum_{\beta} \tilde{d}_{h\alpha}\{m_{\alpha}\} = 0\,, \qquad (9.b)$$

$$[\tilde{D}]\{m_{\alpha}\} = 0 \qquad (9.c)$$

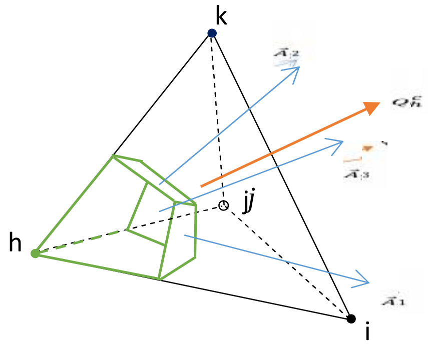
The mass current in an instant Q(I), is defined as the rate of flowing mass. The corresponding mass current density $\vec{q}(I)$ is then defined as the surface density of the flowing mass:

$$Q(I) = \frac{M^{flow}[\tilde{S}, \tilde{\tau}]}{\tilde{\tau}} = \boldsymbol{A}.\boldsymbol{q}(I) \qquad (10)$$

Where, $\vec{A}$ is the area vector through which M^{flow} is to flow. They include three faces of the boundaries of the dual polyhedron contained in the primal tetrahedron (Fig. 4). Since the centroids of the edges, faces, and volumes are used in $\tilde{K}$, it is easy to see that for the vertex *h* we have:

$$Q_h^c(t_n) = \boldsymbol{q}^c(t_n).(\boldsymbol{A}_1 + \boldsymbol{A}_2 + \boldsymbol{A}_3) = \frac{1}{3}\boldsymbol{q}^c(t_n).\boldsymbol{A}_h^c \qquad (11)$$

Where, A_h^c is the area vector of the face opposite to the vertex *h* .



Fig, 1. The mass current flowing from three faces of the dual polyhedron contained in the primal tetrahedron corresponding to the vertex h. The surface force originated from the vertex h is also shown.

Then,

$$M^{flow}[\partial\tilde{v}_h, \tilde{\tau}_n] = \tilde{\tau}\sum_{c\in\mathcal{J}(h)} Q_h^c(t_n) = 0 \qquad (12)$$

Where, $\mathcal{J}$(h) is the set of primal 3-cells with the vertex h in common.

In the next subsection, a global variable for the velocity field will be introduced to express the momentum balance equation in DDM.

4.2. Momentum Balance equation

In order to derive the momentum balance equation we need a global variable for the velocity field. The velocity itself is not a global variable in the spatial description, but it is a line density. Therefore in order to analyze the momentum balance equation the velocity has to be replaced by a global variable. The global variable of velocity circulation Γ, is defined as the integration of the velocity field along a line. Since the velocity field in MHD is rotational, the vorticities are always created and Γ is always nonzero along any closed path.

$$\Gamma[\boldsymbol{I},\boldsymbol{L}] = \int_{L} \vec{u}(t,\boldsymbol{P}).d\vec{l} \qquad (13)$$

The kinetic potential is defined as a global variable at every point and instant $\varphi_h(t_n)$, as the velocity circulation difference along a line. For every edge of the primal tetrahedron whose boundary points are h and k we have:

$$\boldsymbol{u}\,.\,\boldsymbol{L} = \varphi_h - \varphi_k \tag{14}$$

In other words, the velocity components can be evaluated in terms of the kinetic potential at the four vertices of the primal tetrahedron.

In order to find the kinetic potential inside every primal cell, it is sufficient to have the corresponding values at the primal 0-cells. This may be done by an interpolation function, e.g.

$$\varphi(t,x,y,z) = (a+bt) + (b_x + a_x t)x + (b_y + a_y t)y + (b_z + a_z t)z \tag{15}$$

If (14) is written for every vertex of the primal tetrahedron and subtract one equation from the preceding one, we have

$$\begin{bmatrix} L_{ix} & L_{iy} & L_{iz} \\ L_{jx} & L_{jy} & L_{jz} \\ L_{kx} & L_{ky} & L_{kz} \end{bmatrix}_c \begin{Bmatrix} u_x(t_n) \\ u_y(t_n) \\ u_z(t_n) \end{Bmatrix}_c = \begin{Bmatrix} \varphi_i(t_n) - \varphi_h(t_n) \\ \varphi_i(t_n) - \varphi_h(t_n) \\ \varphi_i(t_n) - \varphi_h(t_n) \end{Bmatrix}_c \tag{16}$$

Where, the three edges starting from the node h, are denoted by L_i, L_j, L_k.

Since the area vector associated to the face opposite to vertex i is given by

$$\vec{A}_i = \frac{1}{2}\vec{L}_k \times \vec{L}_j = \frac{1}{2}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ L_{kx} & L_{ky} & L_{kz} \\ L_{jx} & L_{jy} & L_{jz} \end{vmatrix}, \tag{17}$$

it turns out that the relation between the velocity and the kinetic potential values at the four vertices of a primal tetrahedron with volume v_c is as follows:

$$\boldsymbol{u}_c(t_n) = -\frac{1}{3v_c}\left[\boldsymbol{A}_h\varphi_h + \boldsymbol{A}_i\varphi_i + \boldsymbol{A}_j\varphi_j + \boldsymbol{A}_k\varphi_k\right] \tag{18}$$

The continuity equation for the kinetic potential under the divergence free assumption of the velocity field in MHD in terms of the incidence numbers $d_{k\beta}$ and incidence matrix $[D]$ can be written as follows:

$$\Phi[\partial l_\alpha, t_n] = 0 \tag{19.a}$$

$$\textstyle\sum_\beta d_{k\beta}\{\varphi\} = 0\ , \tag{19.b}$$

$$[D]\{\varphi\} = 0. \tag{19.c}$$

The relation between mass current density defined in (9) and velocity vector in the cell c, can be written as:

$$\boldsymbol{q}_c(t_n) = \rho_c \mathbf{u}_c(t_n) \tag{20}$$

Since it links a source variable with a configuration variable, it is a constitutive equation. Substituting (18) for $\vec{u}_c$ in (20) leads to

$$Q_h^c(t_n) = \frac{1}{3}\rho_c[-\frac{1}{3v_c}\left(\boldsymbol{A}_h\varphi_h + \boldsymbol{A}_i\varphi_i + \boldsymbol{A}_j\varphi_j + \boldsymbol{A}_k\varphi_k\right)] \,.\boldsymbol{A}_h^c \qquad (21)$$

This implies that all of the four vertices of the primal tetrahedron contribute to form its mass current.

Now, we are in a position to write the momentum balance equation. Momentum is a source variable and referred to the dual cell complex. Its associated balance equation in DDM is written as follows:

$$\boldsymbol{P}^{cont}[\tilde{t}_{n+1},\tilde{v}_h] - \boldsymbol{P}^{cont}[\tilde{t}_n,\tilde{v}_h] + \boldsymbol{P}^{flow}[\tilde{\tau}_n,\partial\tilde{v}_h] = \boldsymbol{I}^S[\tilde{\tau}_n,\partial\tilde{v}_h] \qquad (22)$$

Where, the momentum content $\boldsymbol{P}^{cont}$ and the momentum flow $\boldsymbol{P}^{flow}$ are satisfied in the following constitutive equations, respectively:

$$\boldsymbol{P}^{cont}[\tilde{\boldsymbol{t}}_n,\tilde{\boldsymbol{v}}_h] = \tilde{v}_h\rho_h(\tilde{t}_n)\vec{u}_h(\tilde{t}_n) \qquad (23)$$

$$\boldsymbol{P}^{flow}[\tilde{\boldsymbol{\tau}}_n,\boldsymbol{\partial}\tilde{v}_h] = \tilde{\tau}_n\left[q_c(t_n).\vec{A}\right]\vec{u}_c(t_n) = \tilde{\tau}_n\frac{1}{3}\sum_{c\,\in\mathcal{J}(h)}\rho_c\left(\vec{u}_c(t_n).\vec{A}_h^c\right)\vec{u}_c(t_n) \qquad (24)$$

The surface impulse $\vec{I}^S$ exerted on the boundaries of dual 3-cell can be derived from the acting normal forces F_h^c as follows:

$$F_h^c = -p_c(A_1 + A_2 + A_3) = -p_c\frac{A_h^c}{3} \qquad (25.a)$$

Therefore,

$$\boldsymbol{I}^S[\tilde{\tau}_n,\partial\tilde{v}_h] = -\tilde{\tau}_n\frac{1}{3}\sum_{c\in\mathcal{J}(h)}p_c(t_n)A_h^c \qquad (25.b)$$

Where is the pressure in cell .

Substituting (23), (24), and (25) to (22) gives:

$$\vec{P}^{cont}(\tilde{t}_{n+1}) - \vec{P}^{cont}(\tilde{t}_n) + \tilde{\tau}_n\frac{1}{3}\sum_{c\,\in\mathcal{J}(h)}\rho_c\left(\vec{u}_c(t_n).\vec{A}_h^c\right)\vec{u}_c(t_n) = -\tilde{\tau}_n\frac{1}{3}\sum_{c\,\in\mathcal{J}(h)}p_c(t_n)\vec{A}_h^c \qquad (26)$$

The RHS of (26) refers to the minus of discrete gradient of the pressure acting on dual faces contained in . Substituting the momentum contents from (23) to the (26) leads to the discrete form of momentum balance equation:

$$\rho_c\frac{\vec{u}_h(\tilde{t}_{n+1})-\vec{u}_h(\tilde{t}_n)}{\tilde{\tau}_n} + \frac{\rho_c}{3\tilde{v}_h}\sum_{c\,\in\mathcal{J}(h)}\left(\vec{u}_c(t_n).\vec{A}_h^c\right)\vec{u}_c(t_n) = -[G]\{p_h\} \qquad (27)$$

That is the equation of motion in MHD. If this equation is not satisfied in MHD, the plasma is accelerated to huge velocities and causing severe damage to the plasma equipment. In the next section we derive the appropriate terms for the effective pressure including the magnetic pressure in a 3-cell.

5. Surface forces

Although the notion of limit that is required in the differential formulation is not used in DDM, surface forces still prevail over the volume forces because the cell sizes are also small in the cell method and then the body forces can be neglected compared to the surface forces.

The surface forces acting on the dual faces with vertex h in common are related to the dual faces and dual intervals. All of the dual 2-cells located inside the primal 3-cell should be taken into account (Fig. 4).

In order to write the equilibrium condition on dual cells, we consider the surface forces (h) acting on the boundaries (dual faces) of a dual 3-cell located in a tetrahedron. Then, the total surface force from all of cells with h in common is:

$$\boldsymbol{T}(h) = \sum_{c\epsilon\mathcal{J}(h)} \boldsymbol{T}^c(h). \quad (28)$$

The total surface force originated from the four vertices of a primal tetrahedron is introduced as the following global vector:

$$= [\quad (h) \quad (h) \quad (h) \; \ldots\ldots \quad (\;) \quad (\;) \quad (\;)]$$

(29)

The three components of the internal surface force ^{c} acting on any area vector result in a uniform stress matrix . Uniformity of □ inside any primal 3-cell leads to a surface force vector that is a linear function of the area vector $\vec{□}_{□}$:

$$\begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = \begin{bmatrix} \tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \tau_{zz} \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} \quad (30)$$

This implies that the surface force vector can be identified in terms of the components of the area vector.

It follows that the stress tensor τ is a symmetric matrix. Then,

$$\boldsymbol{\sigma} = \frac{1}{2}(\quad + \quad) . \quad (31)$$

As a result, the effective stress tensor inside any 3-cell can be written in terms of six elements:

$$\boldsymbol{\sigma}_c = \begin{bmatrix} \boldsymbol{\sigma}_{xx} & \boldsymbol{\sigma}_{yy} & \boldsymbol{\sigma}_{zz} & \boldsymbol{\sigma}_{xy} & \boldsymbol{\sigma}_{yz} & \boldsymbol{\sigma}_{zx} \end{bmatrix}_c^T \quad (32$$

The surface force originated from the vertex h is equal to the sum of the forces acting on the dual area vectors □$_1$, □$_2$, and $_3$ (Fig. 5). It is easy to show that the sum of these area vectors is equal to $^h/_3$.

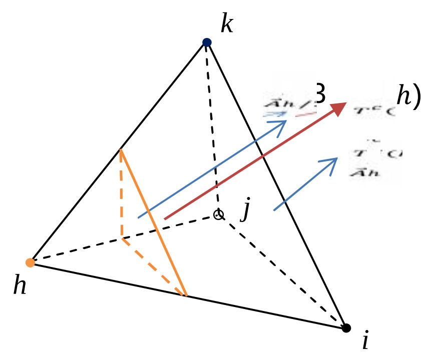


Fig. 2. The triangle opposed to the vertex h whose area is equal to $^h/_3$ is equivalent to the part of the boundaries of the dual polyhedron contained in the primal tetrahedron.

It follows that the global vector opposite to the vertex h in cell is related to $\boldsymbol{\sigma}_c$ as follows:

$$\begin{bmatrix} T_x(h) \\ T_y(h) \\ T_z(h) \end{bmatrix}_c = \frac{1}{3} \begin{bmatrix} A_{hx} & 0 & 0 & A_{hy} & 0 & A_{hz} \\ 0 & A_{hy} & 0 & A_{hx} & A_{hz} & 0 \\ 0 & 0 & A_{hz} & 0 & A_{hy} & A_{hx} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix}_c \quad (33)$$

Similarly, the total surface force originated from the four vertices of c is:

$$\begin{bmatrix} T_x(h) \\ T_y(h) \\ T_z(h) \\ T_x(i) \\ T_y(i) \\ T_z(i) \\ T_x(j) \\ T_y(j) \\ T_z(j) \\ T_x(k) \\ T_y(h) \\ T_z(h) \end{bmatrix}_c = \frac{1}{3} \begin{bmatrix} A_{hx} & 0 & 0 & A_{hy} & 0 & A_{hz} \\ 0 & A_{hy} & 0 & A_{hx} & A_{hz} & 0 \\ 0 & 0 & A_{hz} & 0 & A_{hy} & A_{hx} \\ A_{ix} & 0 & 0 & A_{iy} & 0 & A_{iz} \\ 0 & A_{iy} & 0 & A_{ix} & A_{iz} & 0 \\ 0 & 0 & A_{iz} & 0 & A_{iy} & A_{ix} \\ A_{jx} & 0 & 0 & A_{jy} & 0 & A_{jz} \\ 0 & A_{jy} & 0 & A_{jx} & A_{jz} & 0 \\ 0 & 0 & A_{jz} & 0 & A_{jy} & A_{jx} \\ A_{kx} & 0 & 0 & A_{ky} & 0 & A_{kz} \\ 0 & A_{hy} & 0 & A_{hx} & A_{hz} & 0 \\ 0 & 0 & A_{kz} & 0 & A_{ky} & A_{kx} \end{bmatrix}_c \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{zx} \end{bmatrix}_c \quad (34)$$

or:

$$\boldsymbol{T}_c = \mathbf{A}_c \boldsymbol{\sigma}_c \quad (35)$$

Where $\mathbf{A}_c$ in RHS of (35) is a 12×6 matrix.

The fluid motion in MHD is essentially driven by two forces; plasma pressure gradient $[\widetilde{D}]\boldsymbol{P}$, originated from the ion pressures in different directions, and the magnetic or Lorentz force $[\widetilde{D}]\boldsymbol{\sigma}_B$. If the plasma pressure is a scalar function, it is enough to operate the discrete gradient matrix on it to derive the relative force. In most cases, where these two forces are approximately of equal importance, in the generalized Ohm’s law they cancel out each other and equilibrium situations appear. This is called the ‘pressure-balance’ condition.

We can consider $\boldsymbol{\sigma}_B$ as two different parts, according to the relative sources constituting it; $\boldsymbol{\sigma}_{1B}$ coming from the fact that the magnetic field has a magnetic pressure and σ_{2B} from bending and parallel compression of the magnetic field. These two parts are the so called Maxwell stresses.

The first part can be easily added to the plasma pressure and does not involve in the vorticities of (13).

6. Magnetic stress tensors

In the previous section we turned out that the effective stress tensor in any 3-cell c is resulted as the sum of the plasma and magnetic pressures in that cell, i.e,

$$\boldsymbol{\sigma}_c = \boldsymbol{P} + \boldsymbol{\sigma}_{1B} + \boldsymbol{\sigma}_{2B} \quad (36)$$

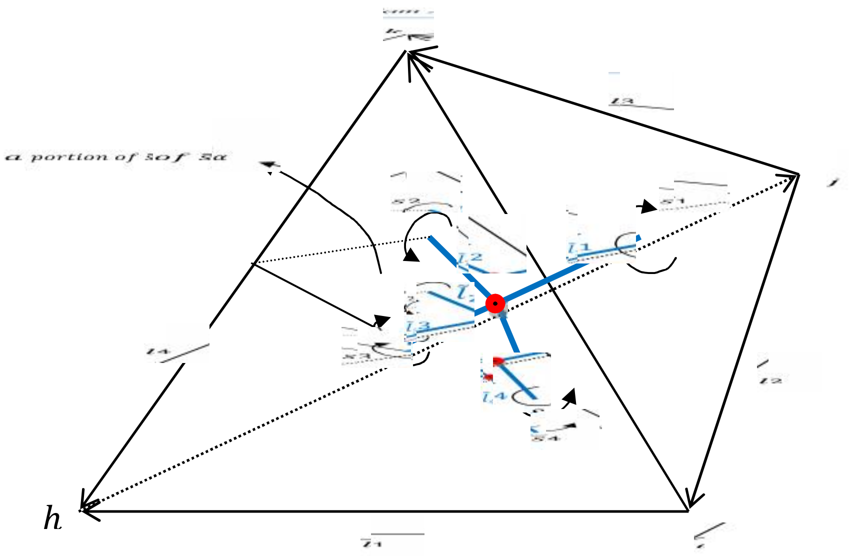


Fig. 3. A primal 3-cell and its primal 1-cells and 2-cells as well as dual 1-cells and a portion of a 2-cell with a local numbering are displayed. The barycenter of the tetrahedron as the dual node is displayed with a red dot.

In order to quantify the magnetic stress tensors, they should be written in their discrete forms involving their dependence on the spatial elements.

6.1. Magnetic pressure tensor

The magnetic field may be supposed to have a magnetic pressure that is denoted by $\boldsymbol{\sigma}_{1B}$. It is equivalent to the magnetic energy density. If the magnetic field has a spatial variation, then it may also be considered to have a bending and parallel compression, producing perpendicular and parallel forces. The stress tensor related to the latter force is denoted by $\boldsymbol{\sigma}_{2B}$. Therefore,

$$\boldsymbol{\sigma}_{1B} = \frac{1}{2\mu}\boldsymbol{B}_c{}^T\boldsymbol{B}_c, \qquad (37)$$

and

$$\boldsymbol{\sigma}_{2B} = \frac{1}{\mu}\boldsymbol{B}_c{}^T[\boldsymbol{G}]\boldsymbol{B}_c \qquad (38)$$

Where, $\boldsymbol{B}_c$ is the amplitude of the local magnetic induction that must be expressed in terms of the known parameters in a primal tetrahedron.

Arranging the area vectors of the tetrahedron shown in Fig. 3 gives rise to the following 3 × 3 matrices:

$$\boldsymbol{A}_h = \begin{pmatrix} \boldsymbol{s}_2 \\ \boldsymbol{s}_3 \\ \boldsymbol{s}_4 \end{pmatrix}, \boldsymbol{A}_i = \begin{pmatrix} \boldsymbol{s}_1 \\ \boldsymbol{s}_3 \\ \boldsymbol{s}_4 \end{pmatrix}, \boldsymbol{A}_j = \begin{pmatrix} \boldsymbol{s}_1 \\ \boldsymbol{s}_2 \\ \vec{s}_4 \end{pmatrix}, \boldsymbol{A}_k = \begin{pmatrix} \vec{s}_1 \\ \vec{s}_2 \\ \vec{s}_3 \end{pmatrix} \qquad (39)$$

The magnetic fluxes corresponding to the above area vectors are then written as follows:

$$\boldsymbol{\Phi}_h = \begin{pmatrix} \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}, \boldsymbol{\Phi}_i = \begin{pmatrix} \phi_1 \\ \phi_3 \\ \phi_4 \end{pmatrix}, \boldsymbol{\Phi}_j = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_4 \end{pmatrix}, \boldsymbol{\Phi}_k = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \qquad (40)$$

According to the area vectors and magnetic fluxes of (39) and (40), $\boldsymbol{B}_c$ in any cell c can be expressed by definition as either of the following forms:

$$\boldsymbol{B}_c = \boldsymbol{A}_h{}^{-1}\boldsymbol{\Phi}_h,\ \boldsymbol{B}_c = \boldsymbol{A}_i{}^{-1}\boldsymbol{\Phi}_i, \boldsymbol{B}_c = \boldsymbol{A}_j{}^{-1}\boldsymbol{\Phi}_j, \boldsymbol{B}_c = \boldsymbol{A}_k{}^{-1}\boldsymbol{\Phi}_k \qquad (40)$$

Inserting one of the expressions for the local magnetic induction of (40) into (37) and (39) determines both magnetic stress tensors.

The magnetic constitutive equation for a typical primal tetrahedron can be written as:

$$\boldsymbol{H}_c = \boldsymbol{\mu}_c{}^{-1}\boldsymbol{B}_c \qquad (41)$$

Where, $\boldsymbol{H}_c$ and $\boldsymbol{\mu}_c$ are the local magnetic intensity vector and permeability matrix in cell c, respectively.

The term $\boldsymbol{\sigma}_{2B}$ may be considered to operate in two orthogonal directions; normal and parallel to the magnetic induction lines. The normal component provides a bending force and the parallel one creates a parallel compression force along the magnetic induction lines. However, in cases where the magnetic induction lines are supposed to be straight and parallel, both of them vanish. In such cases where $\boldsymbol{\sigma}_{2B}$ is absent, the total force inside any primal cell can be simply written as the sum of the plasma gradient pressure and magnetic pressure.

7. An equilibrium problem

We consider a system with straight and parallel magnetic induction lines that is in equilibrium with the plasma pressure. Since $\boldsymbol{B}_c$ is parallel and constant, the so that $\boldsymbol{\sigma}_{2B}$ vanishes and it follows that the total force must be equal to zero, i.e:

$$[\boldsymbol{G}]\{\boldsymbol{P} + \boldsymbol{\sigma}_{1B}\} = \boldsymbol{0} \qquad (42)$$

And consequently,

$$\boldsymbol{P} + \boldsymbol{\sigma}_{1B} = \boldsymbol{constant} \qquad (43)$$

i.e. the sum of the total pressure coming from the plasma and the magnetic induction must be equal to a constant. This case takes place in a cylindrical plasma where the magnetic induction lines are directed along the axis of the cylinder. In this case the plasma pressure gradient and Lorentz force vector are both radial. They can cancel each other out to obtain an equilibrium condition.

8. CONCLUSION

A direct discrete formulation using DDM was applied for the MHD equations without passing through differential formulation. In this method, the complete set of MHD equations, including the equations of electromagnetism and fluid dynamics were derived. In doing so, we used the prime and dual cell complexes as replacements for the coordinate system, and the global variables instead of the field variables in differential formulation. In this way, the global variables were also classified into configuration, source, or energy variables. Naturally we associated the configuration variables to the prime cell complex and the source variables to its dual. The effective pressure in the equation of motion was written in the new approach as the discrete gradient of the plasma and magnetic pressure tensor that is originated from the Lorentz force.

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