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Research Article

THE GENERALIZED HYERS-ULAM STABILITY OF SEXTIC FUNCTIONAL EQUATION IN VARIOUS MATRIX SPACES

L. Sajedi*, M. Imanazary Bonab and Y. Ganbarzadeh Bonab

Department of Math and Computer Engeering, Ajabshir Branch, Islamic Azad University, Eastern Azarbyjan, Ajabshir, Iran

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ABSTRACT

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the following generalized sextic functional equation

$$Df(x, y) := f(mx+y) + f(mx-y) + f(x+my) + f(x-my) - (m^4+m^2)[f(x+y+f(x-y)] - 2(m^6-m^4-m^2+1)[f(x)+f(y)]$$

in matrix fuzzy normed spaces. Furthermore, using the fixed point method, we also prove the Hyers-Ulam stability of the above functional equation in matrix random normed spaces.

Keywords: Hyers-Ulam stability; fixed point method; matrix fuzzy normed space; matrix random normed spaces; sextic functional equation.

Author Correspondence, e-mail: le.sajedi@gmail.com

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1. INTRODUCTION

The stability problem of functional equations originated from a question of Ulam [13] in1940, concerning the stability of group homeomorphisms. In the next year, Hyers [7] gave the first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' result was the generalized by Aoki [1] for additive mappings and by Rassias [11] for linear mappings. A generalization of the Rassias's theorem was obtained by Găvruta [6] who replaced $\varepsilon(\| x\|^p$

$+\| y\|^{p})$ the by a general control function $\varphi(x, y)$. Since then, the stability of several functional equations has been extensively investigated by several mathematicians. Furthermore some stability results of functional equations and inequalities were investigated in matrix normed spaces, matrix paranormed spaces and matrix fuzzy normed spaces.

In this paper, we consider the following sextic functional equation

$$f(mx+y)+f(mx-y)+f(x+my)+f(x-my)$$
$$= (m^4+m^2)[f(x+y)+f(x-y)]+2(m^6-m^4-m^2+1)[f(x)+f(y)] \quad (1.1)$$

The main purpose of this paper is to apply the fixed point method to investigate the Hyers-Ulam stability of functional equation (1.1) in matrix fuzzy spaces and matrix random normed spaces.

Definition 1.1 ([3]) Let X be a set. A function $d: X \times X \rightarrow [0, \infty]$ is called a generalized metric on X if d satisfies

(1) $d(x, y)= 0$ if only if $x = y$;

(2) $d(x, y)= d(y, x),\ \forall x, y \in X$;

(3) $d(x, z)\leq d(x, y)+ d(y, z),\ \forall x, y, z \in X$.

Theorem 1.2 ([3]) Let (X, d) be a complete generalized metric space and $J: X\rightarrow X$ be a strictly contractive mapping with Lipschitz constant $L< 1$.Then, for all $x\in X$, either $d(J^n x, J^{n+1}x)= \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that

(a) $d(J^n x, J^{n+1}x)< \infty$ for all $n \geq n_0$;

(b) The sequence $J^n x$ converges to a fixed point y^* of J;

(c) y^* is the unique fixed point of J in the set $Y= \{y\in X: d(J^{n0} x, y)< \infty\}$;

(d) $d(y, y^*)\leq d(y, Jy)$ for all $y\in Y$.

Katsaras [8] defined a fuzzy norm on a vector space to construct a fuzzy vector topological structure on the space. Some mathematicians have defined fuzzy norms on a vector space from various points of view in particular, Bag and Samanta [2], following Cheng and Mordeson [5], investigated some properties of fuzzy normed spaces. We use the definition of fuzzy normed spaces given in to investigate a fuzzy version of the generalized Hyers-Ulam stability for the functional equation (1.1) in the fuzzy normed vector space setting.

Definition 1.3 ([2]) Let X be a real vector space. A function $N: X\times\mathbb{R} \to [0, 1]$ is called a fuzzy norm on X if for all $x, y\in X$ and all s, t$\in\mathbb{R}$,

(N1) $N(x, t) = 0$ for $t\leq 0$;

(N2) $x= 0$ if and only if $N(x, t) = 1$ for all $t> 0$;

(N3) $N (cx, t) = N(x, \frac{t}{|c|})$ if $c\neq 0$;

(N4) $N(x+ y, s+ t)\geq min \{N(x, s), N(y, t)\}$;

(N5) $N(x, \cdot)$ is a non− decreasing function of $\mathbb{R}$ and $lim_{n\to\infty} N(x, t) = 1$;

(N6) for $x\neq 0$, $N(x, \cdot)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed vector space (briefly, FNS).

Definition 1.4 ([2]) Let (X, N) be a FNS. A sequence $\{x_n\}$ in X is said to converge or be convergent if there exists an $x\in X$ such that $lim_{n\to\infty} N (x_n - x, t) = 1$ for all t > 0. In this case, x is called the limit of the sequence $\{x_n\}$ in X and we denote it by $N - lim_{n\to\infty} x_n = x$.

Definition 1.5 ([2]) Let (X, N) be a FNS. A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon >$ 0 and each $t>0$ there exists an $n_0\in N$ such that for all $n \geq n_0$ and all $p > 0$, we have $N (x_{n+p} - x_n, t) > 1-\epsilon$.

It is well known that every convergent sequence in a FNS is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

We say that a mapping $f : X \to Y$ between FNS X and Y is continuous at a point $x\in X$ if for each sequence $\{x_n\}$ converging to $x_0\in X$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f: X\to Y$ is continuous at each $x\in X$, then $f: X \to Y$ is said to be continuous on X.

We will also use the following notations:

$e_j = (0, ..., 0, 1, 0, ..., 0)$;

E_{ij} is that (i, j)-component is 1 and the other components are zero;

$E_{ij} \otimes x$ is that (i, j)-component is x and the other components are zero; for $x\in M_n(X)$, y$\in M_k(X)$,

$$x \oplus y= \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$$

Let $(X, \|\cdot\|)$ be a normed space. Note that $(X, \{\|\cdot\|_n\})$ is a matrix normed space if and only if $(M_n(X), \|\cdot\|_n)$ is a normed space for each positive integer n and $\|AxB\|_k \leq \|A\|\|B\|\|x\|_n$ holds

holds for $A \in M_{k,n}$ $x = (x_{ij}) \in M_n(X)$ and $B \in M_{n,k}$ and that $(X, \{\|\cdot\|_n\})$ is a matrix Banach space if and only if X is a Banach space and $(X, \{\|\cdot\|_n\})$ is a matrix normed space.

Let E, F be vector spaces. For a given mapping $h: E \to F$ and a given positive integer n, define $h_n: M_n(E) \to M_n(F)$ by

$$h_n([x_{ij}]) = [h(x_{ij})]$$

for all $[x_{ij}] \in M_n(E)$.

Definition 1.6 ([9]) Let (X, N) be a fuzzy normed space.

(1) (X, N_n) is called a matrix fuzzy normed space if for each positive integer n, $(M_n(X), N_n)$ is a fuzzy normed space and $N_k(AxB, t) \geq N_n(x, t\|A\|\cdot\|B\|)$ for all $t > 0$, $A \in M_{k,n}(\mathbb{R})$, $B \in M_{n,k}(\mathbb{R})$ and $x = [x_{ij}] \in M_n(X)$ with $\|A\|.\|B\| \neq 0$.

(2) $(X, \{N_n\})$ is called a matrix fuzzy Banach space if (X, N) is a fuzzy Banach space and $(X, \{N_n\})$ is a matrix fuzzy normed space.

2. HYERS-ULAM STABILITY OF SEXTIC FUNCTIONAL EQUATION IN MATRIX FUZZY NORMED SPACES

Throughout this section, let $(X, \{N_n\})$ be a matrix fuzzy normed space, $(Y, \{N_n\})$ be a matrix fuzzy Banach space and let n be a fixed positive integer. Using thefixed point method, we prove the Hyers-Ulam stability of the sextic functional equation (1.1) in matrix fuzzy normed spaces. We need the following Lemma:

Lemma 2.1 ([10]) Let $(X, \{N_n\})$ be a matrix fuzzy normed space. Then

(1) $N_n(E_{kl} \otimes x, t) = N(x, t)$ for all $t > 0$ and $x \in X$;

(2) For all $[x_{ij}] \in M_n(X)$ and $t = \sum_{i,j=1}^{n} t_{ij}$,

$$N(x_{kl}, t) \geq N_n([x_{ij}], t) \geq \min\{N(x_{ij}, t_{ij}) : i, j = 1, 2, ..., n\},$$

$$N(x_{kl}, t) \geq N_n([x_{ij}], t) \geq \min\{N(x_{ij}, \frac{t}{n^2}) : i, j = 1, 2, ..., n\},$$

(3) $\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} x_{ij_n} = x_{ij}$ for $x_n = [x_{ij_n}]$, $x = [x_{ij}] \in M_k(X)$ for a mapping $f: X \to Y$, define $Df: X^2 \to Y$ and $Df_n: M_n(X^2) \to M_n(Y)$ by [12]

$Df(a, b) := f(ma + b) + f(ma - b) + f(a + mb) + f(a - mb) - (m^4 + m^2)[f(a + b) + f(a - b)]$

$$-2\,(m^6 - m^4 - m^2 + 1)[f(a) + f(b)]$$

$$Df_n([x_{ij}], [y_{ij}]) := f_n(m[x_{ij}] + [y_{ij}]) + f_n(m[x_{ij}] - [y_{ij}]) + f_n([x_{ij}] + m[y_{ij}]) + f_n([x_{ij}] - m[y_{ij}])$$
$$- (m^4 + m^2)[f_n([x_{ij}] + [y_{ij}]) + f_n([x_{ij}] - [y_{ij}])] - 2(m^6 - m^4 - m^2 + 1)[f_n([x_{ij}] + f_n([y_{ij}])$$

for all $a, b \in X$ and all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Theorem 2.2 Let $\varphi: X^2 \to [0, \infty)$ be a function such that there exists $\alpha < 1$ with

$$\varphi(a, b) \le \frac{\sim}{m^6}\varphi(ma, mb) \tag{2.1}$$

for all $a, b \in X$. Let $f: X \to Y$ be a mapping satisfying

$$N_n(Df_n([x_{ij}], [y_{ij}]), t) \ge \frac{t}{t + \sum_{i,j=1}^{n} ¨(x_{ij}, y_{ij})} \tag{2.2}$$

for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then $C(a) = N - \lim_{k\to\infty} m^{6k} f(\frac{a}{m^k})$ exists for each $a \in X$ and defines a sextic mapping $C: X \to Y$ such that

$$N_n(f_n([x_{ij}]) - C_n([x_{ij}]), t) \ge \sum_{i,j=1}^{n} \frac{2m^6(1-\sim)t}{2m^6(1-\sim)t + n^2 \sim \sum_{i,j=1}^{n} ¨(x_{ij}, 0)} \tag{2.3}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof Let $n = 1$ in (2.2). Then (2.2) is equivalent to

$$N(Df(a, b), t) \ge \frac{t}{t + ¨(a, b)} \tag{2.4}$$

for all $t > 0$ and $a, b \in X$. Letting $b = 0$ in (2.4), we get

$$N(f(ma) - m^6 f(a), \frac{t}{2}) \ge \frac{t}{t + ¨(a, 0)} \tag{2.5}$$

in (2.5), we get

$$N(f(a) - m^6 f(\frac{a}{m}), t) \ge \frac{t}{t + \frac{1}{2}¨(\frac{a}{m}, 0)} \ge \frac{t}{t + \frac{\sim}{2m^6}¨(a, 0)} \tag{2.6}$$

for all $t > 0$ and $a \in X$.

Consider the set $S: = \{g: X \to Y\}$ and introduce the generalized metric on S

$$d(g, h) = \inf\{¬ \in \mathbb{R}_+ : N(g(a) - h(a), ¬\, t) \ge \frac{t}{t + ¨(a, 0)}, \forall a \in X, \forall t > 0\}$$

where, as usual, $inf\ \varphi = +\infty$. It is easy to show that (S, d) is complete ([8], Lemma 2.1). Now we consider the linear mapping $J: S \to S$ such that $Jg(a) = m^6 g(\frac{a}{m})$, for all $a \in X$. Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$N(g(a) - h(a), \varepsilon t) \geq \frac{t}{t + \ddot{\ }(a,0)},$$

for all $a \in X$, for all $t > 0$. Then

$$N(Jg(a) - Jh(a), \sim \varepsilon t) = N(m^6 g(\frac{a}{m}) - m^6 h(\frac{a}{m}), \sim \varepsilon t)$$

$$= N(g(\frac{a}{m}) - h(\frac{a}{m}), \frac{\sim}{m^6}\varepsilon t) \geq \frac{\frac{\sim t}{m^6}}{\frac{\sim t}{m^6} + \ddot{\ }(\frac{\sim}{m},0)} \geq \frac{\frac{\sim t}{m^6}}{\frac{\sim t}{m^6} + \frac{\sim}{m^6}\ddot{\ }(a,0)} = \frac{t}{t + \ddot{\ }(a,0)}$$

for all $a \in X$ and $t > 0$. So, $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq \alpha\varepsilon$. This means that $d(Jg, Jh) \leq \alpha d(g, h)$, for all $g, h \in S$. It follows from (2.6) that $d(g, Jg) \leq \frac{\sim}{2m^6}$. By Theorem 1.2, there exists a mapping $C: X \to Y$ satisfying the following:

(1) C is a fixed point of J, i.e.

$$C(\frac{a}{m}) = \frac{1}{m^6} C(a)$$

for all $a \in X$. The mapping C is a unique fixed point of J in the set $M = \{g \in S: d(f, g) < \infty\}$.

(2) $d(J^k f, C) \to 0$ as $k \to \infty$. This implies the equality

$$N - \lim_{k \to \infty} m^{6k} f(\frac{a}{m^k}) = C(a),$$

for all $a \in X$.

(3) $d(f, C) \leq \frac{1}{1-\sim} d(f, Jf)$, which implies the inequality

$$D(f, C) \leq \frac{\sim}{2m^6(1-\sim)}. \tag{2.7}$$

By (2.4),

$$N(m^{6k}f(\frac{ma+b}{m^k}) + m^{6k}f(\frac{ma-b}{m^k}) + m^{6k}f(\frac{a+mb}{m^k}) + m^{6k}f(\frac{a-mb}{m^k}) - m^{6k}(m^4+m^2)[f(\frac{a+b}{m^k})$$

$$+ f(\frac{a-b}{m^k})] - 2m^{6k}(m^6 - m^4 - m^2 + 1)[f(\frac{a}{m^k}) + f(\frac{b}{m^k})], m^{6k}t) \geq \frac{t}{t + \ddot{\ }(\frac{a}{m^k}, \frac{b}{m^k})}$$

$$N(m^{6k}f(\frac{ma+b}{m^k})+ m^{6k}f(\frac{ma-b}{m^k})+ m^{6k}f(\frac{a+mb}{m^k})+ m^{6k}f(\frac{a-mb}{m^k})- m^{6k}(m^4+m^2)\,[f(\frac{a+b}{m^k})$$

$$+ f(\frac{a-b}{m^k})] -2m^{6k}(m^6 - m^4 - m^2 + 1)\,[f(\frac{a}{m^k})+f(\frac{b}{m^k})], m^{6k}t) \geq \frac{\frac{t}{m^k}}{\frac{t}{m^k}+\frac{\sim^k}{m^{6k}}\ddot{}\,(a,b)}$$

for all $a, b \in X$ and $t > 0$. Since $lim_{k\to\infty} \frac{\frac{t}{m^k}}{\frac{t}{m^k}+\frac{\sim^k}{m^{6k}}\ddot{}\,(a,b)} = 1$, for all $a, b \in X$ and $t > 0$.

$$N(C(ma+ b)+C(ma- b)+C(a+ mb)+ C(a- mb) - (m^4+ m^2)[C(a+ b)+C(a- b)]$$

$$-2\,(m^6 - m^4 - m^2 + 1)\,[C(a) + C(b)], t) = 1$$

for all $a, b \in X$ and $t > 0$.Thus

$$C(ma+ b)+C(ma- b)+C(a+ mb)+ C(a- mb) - (m^4+ m^2)[C(a+ b)+C(a- b)]$$

$$-2\,(m^6 - m^4 - m^2 + 1)\,[C(a) + C(b)] = 0.$$

So, the mapping $C: X \to Y$ is sextic. By Lemma (2.1) and (2.7),

$$N_n(f_n([x_{ij}]) - C_n([x_{ij}]), t) \geq min\,\{ N(f([x_{ij}]) - C([x_{ij}]), \frac{t}{n^2}): i, j=1,2,\ldots, n\}$$

$$\geq min\,\{\frac{2m^6(1-\sim)t}{2m^6(1-\sim)t+n^2\sim\ddot{}\,(x_{ij},0)}: i, j=1,2,\ldots, n\} \geq \frac{2m^6(1-\sim)t}{2m^6(1-\sim)t+n^2\sim\sum_{i,j=1}^{n}\ddot{}\,(x_{ij},0)}$$

for all $x = [x_{ij}] \in M_n(X)$. Thus $C: X\to Y$ is a unique sextic mapping satisfying (2.3), as desired.

Corollary 2.3 Let r, θ *be* positive real numbers with $r< 6$. Let $f: X\to Y$ be a mapping satisfying

$$N_n(Df_n([x_{ij}], [y_{ij}]), t) \geq \frac{t}{t+\sum_{i,j=1}^{n}\bar{}\,(\|x_{ij}\|^r+\|y_{ij}\|^r)} \qquad (2.8)$$

for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then $C(a) = N - lim_{k\to\infty} m^{6k} f(\frac{a}{m^k})$ exists for each $a \in X$ and defines a sextic mapping $C: X \to Y$ such that

$$N_n(f_n([x_{ij}])-C_n([x_{ij}]),\ t) \geq \sum_{i,j=1}^{n}\frac{2m^6(1-m^{r-6})t}{2m^6(1-m^{r-6})t+n^2m^{r-6}\sum_{i,j=1}^{n}\bar{}\,\|x_{ij}\|^r}$$

for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Proof The proof follows from Theorem 2.2 by taking $\varphi(a, b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$ and $\alpha = m^{r-6}$. We get the desired result.

Theorem 2.4 Let $\varphi: X^2 \to [0, \infty)$ be a function such that there exists $\alpha < 1$ with

$$\varphi(a, b) \le m^6 \alpha \varphi\left(\frac{a}{m}, \frac{b}{m}\right) \tag{2.9}$$

for all $a, b \in X$. Let $f: X \to Y$ be a mapping satisfying (2.2) for all $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then $C(a) = N - lim_{k \to \infty} \frac{1}{m^{6k}} f(m^k a)$ exists for each $a \in X$ and defines a sextic mapping $C: X \to Y$ such that

$$N_n(f_n([x_{ij}]) - C_n([x_{ij}]), t) \ge \sum_{i,j=1}^{n} \frac{2m^6(1-\alpha)t}{2m^6(1-\alpha)t + n^2 \sum_{i,j=1}^{n} \varphi(x_{ij}, 0)} \tag{2.10}$$

for all $x = [x_{ij}] \in M_n(X)$.

Proof Let (S, d) be the generalized metric space defined in the proof of Theorem 2.2. Now we consider the linear mapping $J: S \to S$ such that

$$Jg(a) = \frac{1}{m^6} g(ma),$$

for all $a \in X$. It follows from (2.5) that $d(f, Jf) \le \frac{1}{2m^6}$. So $D(f, C) \le \frac{1}{2m^6(1-\alpha)}$

The rest of the proof is similar to the proof of Theorem 2.2.

Corollary 2.5 Let r, θ be positive real numbers with $r > 6$. Let $f: X \to Y$ be a mapping satisfying *(2.8)* for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$. Then $C(a) = N-lim_{k \to \infty} \frac{1}{m^{6k}} f(m^k a)$ exists for each $a \in X$ and defines a sextic mapping $C: X \to Y$ such that

$$N_n(f_n([x_{ij}]) - C_n([x_{ij}]), t) \ge \sum_{i,j=1}^{n} \frac{2m^6(m^{r-6}-1)t}{2m^6(m^{r-6}-1)t + n^2 m^{r-6} \sum_{i,j=1}^{n} \theta \|x_{ij}\|^r} \tag{2.11}$$

for all $t > 0$ and $x = [x_{ij}], y = [y_{ij}] \in M_n(X)$.

Proof The proof follows from Theorem 2.4 by taking $\varphi(a, b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$ and $\alpha = m^{6-r}$. We get the desired result.

3. STABILITY OF FUNCTIONAL EQUATION (1.1) IN MATRIX RANDOM NORMED SPACES: A FIXED POINT APPROACH

In the sequel, we adopt the usual terminology, notions and conventions of the theory of random normed spaces as in [4]. Throughout this paper, let Δ^+ denote the set of all probability distribution functions $F: \mathbb{R} \cup \{\infty, +\infty\} \to [0, 1]$ such that F is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1$. It is clear that the set $D^+ = \{F \in \Delta^+ : l^- F(+\infty) = 1\}$, where $l^- f(x)$ denotes the left limit of the function f at the point x, that is, $l^- f(x) = lim_{t\to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \le G$ if and only if $F(t) \le G(t)$ for all $t \in \mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function ε_0 given by

$$\varepsilon_0(t) = \begin{cases} 0, & t \le 0, \\ 1, & t > 0. \end{cases}$$

Definition 3.1 A function $T: [0, 1]^2 \to [0, 1]$ is a continuous triangular norm (briefly, a continuous t-norm) if T satisfies the following conditions:

(a) T is commutative and associative;

(b) T is continuous;

(c) $T(a, 1) = a$ for all $a \in [0, 1]$;

(d) $T(a, b) \le T(c, d)$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Definition 3.2 A random normed space (briefly, RN-space) is a triple (X, μ, T), where X is a vector space, T is a continuous t-norm and $\mu: X \to D^+$ is a mapping such such that the following conditions hold:

(a) $\mu_x(t) = \varepsilon_0(t)$ for all $x \in X$ and $t > 0$ if and only if $x = 0$;

(b) $\mu_{\alpha x}(t) = \mu_x\left(\frac{t}{|\alpha|}\right)$ for all $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, $x \in X$ and $t \ge 0$;

(c) $\mu_{x+y}(t + s) \ge T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and $t, s \ge 0$.

Every normed space $(X, \|\cdot\|)$ defines a random normed space (X, μ, T) where

$$\mu_x(t) = \frac{t}{t + \|x\|}$$

for all $t > 0$, and T is the minimum t-norm defined by $T(a, b) = \min\{a, b\}$. This space is called the induced random normed space.

Definition 3.3 Let (X, μ, T) be an RN-space.

(1) A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ for every $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $lim_{n\to\infty}$ $(\varepsilon) > 1 - \lambda$ whenever $n \geq N$.

(2) A sequence $\{x_n\}$ in X is called a Cauchy sequence if, for every $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer N such that $lim_{n\to\infty}$ «$_{x_n - x_m}$ $(\varepsilon) > 1 - \lambda$ whenever $n \geq m \geq N$.

(3) A RN-space (X, μ, T) is said to be complete if and only if every Cauchy sequence in X is convergent to a point in X.

We note that if (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \to x$ then $lim_{n\to\infty}$ «$_{x_n}$ $(t) = \mu_x(t)$ almost everywhere (see [30]).

We introduce the concept of matrix random normed space.

Definition 3.4 Let (X, μ, T) be a random normed space. Then

(1) $(X, \{\mu^{(n)}\}, T)$ is called a matrix random normed space if for each positive integer n, $(M_n(X), \mu^{(n)}, T)$ is a random normed space and «$^{(k)}_{AxB}$ $(t) \geq$ «$^{(n)}_{x}$ $\left(\frac{t}{\| A \| \cdot \| B \|}\right)$ for all $t > 0$, $A \in M_{k,n}(\mathrm{R})$, $x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}(\mathrm{R})$ with $\|A\| \cdot \|B\| \neq 0$.

(2) $(X, \{\mu^{(n)}\}, T)$ is called a matrix random Banach space if (X, μ, T) is a complete random normed space and $(X, \{\mu^{(n)}\}, T)$ is a matrix random normed space.

For a matrix normed space $(X, \{\| \cdot \|_n\})$ let $\mu^{(n)}{}_x(t) := \frac{t}{t + \| x \|_n}$ for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$. Then

$$\text{«}^{(k)}_{AxB}\ (t) = \frac{t}{t + \| AxB \|_k} \geq \frac{t}{t + \|A\| \|x\|_n \|B\|}$$

$$\frac{\frac{t}{\|A\| \cdot \|B\|}}{\frac{t}{\|A\| \cdot \|B\|} + \|x\|_n} = \mu^{(n)} x \left(\frac{t}{\|A\| \cdot \|B\|}\right)$$

for all $t > 0$, $A \in M_{k.n}(\mathrm{R})$, $x = [x_{ij}] \in M_n(X)$ and $B \in M_{n,k}(\mathrm{R})$ with $\|A\| \cdot \|B\| = 0$. If $T(a, b) = min\{a, b\}$, then $(X, \{\mu^{(n)}\}, T)$ is a matrix random normed space.

Lemma 3.5 *Let* $(X, \{\mu^{(n)}\}, T)$ *be a matrix random normed space. Let* $\mu^{(1)} = \mu$. *For all* $[x_{ij}] \in M_n(X)$, *we have*

$$\mu_{x_{kl}}(t) \geq \mu^{(n)}_{[x_{ij}]}(t) \geq \min\{\mu_{x_{ij}}(\frac{t}{n^2}): i, j = 1, 2, ..., n\}.$$

Proof Since $E_{kl} \otimes x = e_k^* x e_l$ and $\|e_k^*\| = \|e_l\| = 1$, $\mu^{(n)}_{E_{kl}\otimes x}(t) \geq \mu_x(t)$ holds. Since $e_k(E_{kl} \otimes x)e_l^* = x$ and $\mu^{(n)}_{E_{kl}\otimes x}(t) \leq \mu_x(t)$ we get $\mu^{(n)}_{E_{kl}\otimes x}(t) = \mu_x(t)$. For all $[x_{ij}] \in M_n(X)$ and $t = \sum_{i,j=1}^{(n)} t_{ij}$, $\mu_{x_{kl}}(t) = \mu_{e_k[x_{ij}]e_l^*}(t) \geq \mu^{(n)}_{[x_{ij}]}(\frac{t}{\|e_k\| \cdot \|e_l\|}) = \mu^{(n)}_{[x_{ij}]}(t)$ holds. Thus

$$\mu^{(n)}_{[x_{ij}]}(t) = \mu^{(n)}_{\sum_{i,j=1}^{n} E_{ij}\otimes x_{ij}}(t) \geq \min\{\mu^{(n)}_{E_{ij}\otimes x_{ij}}(t_{ij}): i, j = 1, 2, ..., n\}$$

$$= \min\{\mu_{x_{ij}}(t_{ij}): i, j = 1, 2, ..., n\}$$

where $t = \sum_{i,j=1}^{(n)} t_{ij}$. So $\mu^{(n)}_{[x_{ij}]}(t) \geq \min\{\mu_{x_{ij}}(\frac{t}{n^2}): i, j = 1, 2, ..., n\}$.

Throughout this section, let X be a normed space and $(Y, \mu^{(n)}, T)$ a matrix random Banach space.

Theorem 3.6 *Let* $\varphi: X^2 \to [0, \infty)$ be a function such that there exists $\alpha < 1$ with

$$\varphi(a, b) \leq \frac{\alpha}{m^6}\varphi(ma, mb)$$

for all $a, b \in X$. Let $f: X \to Y$ be a mapping satisfying

$$\mu^{(n)}_{Df_n([x_{ij}],[y_{ij}])}(t) \geq \frac{t}{t + \sum_{i,j=1}^{n} \varphi(x_{ij}, y_{ij})} \tag{3.1}$$

for all $x = [x_{ij}]$, $y = [y_{ij}] \in M_n(X)$. *Then* $C(a) = \lim_{k\to\infty} m^{6k} f(\frac{a}{m^k})$ exists for each $a \in X$ and defines a sextic mapping $C: X \to Y$ such that

$$\mu_{f_n([x_{ij}])-C_n([x_{ij}])}(t) \geq \frac{2m^6(1-\alpha)t}{2m^6(1-\alpha)t + n^2\alpha\Sigma^n_{i,j=1}\varphi(x_{ij}, 0)} \tag{3.2}$$

for all $t > 0$ *and* $x = [x_{ij}] \in M_n(X)$.

Proof Let $n = 1$. Then (3.1) is equivalent to

$$\text{«}_{f(ma+b)+f(ma-b)+f(a+mb)+f(a-mb)-(m^4+m^2)[f(a+b)+f(a-b)]-2(m^6-m^4-m^2+1)[f(a)+f(b)]}(t) \geq \frac{t}{t+\text{¨}(a,b)} \quad (3.3)$$

for all $t > 0$ and $a, b \in X$.

Letting $b = 0$ in (3.3), we get

$$\text{«}_{f(ma)-m^6 f(a)}\left(\frac{t}{2}\right) \geq \frac{t}{t+\text{¨}(a,0)} \quad (3.4)$$

Let $a = \dfrac{a}{m}$ in (3.4), we get

$$\text{«}_{f(a)-m^6 f(\frac{a}{m})}(t) \geq \frac{t}{t+\frac{1}{2}\text{¨}(\frac{a}{m},0)} \geq \frac{t}{t+\frac{\sim}{2m^6}\text{¨}(a,0)} \quad (3.5)$$

for all $t > 0$ and $a \in X$.

Consider the set $S: = \{g: X \to Y\}$ and introduce the generalized metric on S

$$d(g,h) = \inf\{\text{¬} \in \mathbb{R}_+ : \mu_{g(a)-h(a)}(\text{¬}\, t) \geq \frac{t}{t+\text{¨}(a,0)}, \forall a \in X, \forall t > 0\}$$

where, as usual, $\inf \varphi = +\infty$. It is easy to show that (S, d) is complete (see the proof of [18], Lemma 2.1).

Now we consider the linear mapping $J: S \to S$ such that $Jg(a) = m^6 g(\frac{a}{m})$, for all $a \in X$. Let $g, h \in S$ be given such that $d(g, h) = \varepsilon$. Then

$$\text{«}_{g(a)-h(a)}(\text{£}t) \geq \frac{t}{t+\text{¨}(a,0)},$$

for all $a \in X$, for all $t > 0$. Then

$$\text{«}_{Jg(a)-Jh(a)}(\sim\text{£}t) = \text{«}_{m^6 g(\frac{a}{m})-m^6 h(\frac{a}{m})}(\sim\text{£}t) = \text{«}_{g(\frac{a}{m})-h(\frac{a}{m})}\left(\frac{\sim}{m^6}\text{£}t\right)$$

$$\geq \frac{\frac{\sim t}{m^6}}{\frac{\sim t}{m^6}+\text{¨}(\frac{a}{m},0)} \geq \frac{\frac{\sim t}{m^6}}{\frac{\sim t}{m^6}+\frac{\sim}{m^6}\text{¨}(a,0)} = \frac{t}{t+\text{¨}(a,0)}$$

for all $a \in X$ and $t > 0$. So, $d(g, h) = \varepsilon$ implies that $d(Jg, Jh) \leq \alpha\varepsilon$. This means that $d(Jg, Jh) \leq \alpha d(g, h)$, for all $g, h \in S$. It follows from (3.5) that $d(f, Jf) \leq \frac{\sim}{2m^6}$.

By Theorem 1.2, there exists a mapping $C: X \to Y$ satisfying the following:

(1) *C* is a fixed point of *J*, i.e.

$$C\left(\frac{a}{m}\right) = \frac{1}{m^{6}} C(a) \tag{3.6}$$

for all $a \in X$. The mapping *C* is a unique fixed point of *J* in the set $M = \{g \in S: d(f, g) < \infty\}$.

(2) $d(J^{k}f, C) \to 0$ as $k \to \infty$. This implies the equality

$$lim_{k\to\infty} m^{6k} f\left(\frac{a}{m^{k}}\right) = C(a),$$

for all $a \in X$.

(3) $d(f, C) \leq \frac{1}{1-\sim} d(f, Jf)$, which implies the inequality

$$D(f, C) \leq \frac{\sim}{2m^{6}(1-\sim)}. \tag{3.7}$$

By (3.3),

$$\text{«}_{m^{6k} f\left(\frac{ma\pm b}{m^{k}}\right)+m^{6k} f\left(\frac{a\pm mb}{m^{k}}\right)-m^{6k}(m^{4}+m^{2}) f\left(\frac{a\pm b}{m^{k}}\right)-2m^{6k}(m^{6}-m^{4}-m^{2}+1)\left[f\left(\frac{a}{m^{k}}\right)+f\left(\frac{b}{m^{k}}\right)\right]}(m^{6k}t)$$

$$\geq \frac{t}{t + \ddot{}\left(\frac{a}{m^{k}}, \frac{b}{m^{k}}\right)},$$

$$\text{«}_{m^{6k} f\left(\frac{ma\pm b}{m^{k}}\right)+m^{6k} f\left(\frac{a\pm mb}{m^{k}}\right)-m^{6k}(m^{4}+m^{2}) f\left(\frac{a\pm b}{m^{k}}\right)-2m^{6k}(m^{6}-m^{4}-m^{2}+1)\left[f\left(\frac{a}{m^{k}}\right)+f\left(\frac{b}{m^{k}}\right)\right]}(t)$$

$$\geq \frac{\frac{t}{m^{k}}}{\frac{t}{m^{k}} + \frac{\sim^{k}}{m^{6k}}\ddot{}(a,b)},$$

for all $a, b \in X$ and $t > 0$. Since $lim_{k\to\infty} \frac{\frac{t}{m^{k}}}{\frac{t}{m^{k}} + \frac{\sim^{k}}{m^{6k}}\ddot{}(a,b)} = 1$, for all $a, b \in X$ and $t > 0$.

$$\text{«}_{C(ma\pm b)+C(a\pm mb)-(m^{4}+m^{2})C(a\pm b)-2(m^{6}-m^{4}-m^{2}+1)[C(a)+C(b)]}(t) = 1$$

$$C(ma \pm b) + C(a \pm mb) - (m^{4} + m^{2})C(a\pm b) - 2(m^{6} - m^{4} - m^{2} + 1)[C(a) + C(b)] = 0$$

So, the mapping $C: X \to Y$ is sextic.

By Lemma (3.5) and (3.6),

$$\text{«}^{(n)}_{f_n([x_{ij}])-C_n([x_{ij}])}(t) \geq min\left\{ \text{«}_{f(x_{ij})-C(x_{ij})}(\frac{t}{n^2}) : i, j = 1,2,...,n \right\}$$

$$\geq min\left\{ \frac{2m^6(1-\sim)t}{2m^6(1-\sim)t + n^2\sim \ddot{\ } (x_{ij},0)} : i, j = 1,2,...,n \right\} \geq \frac{2m^6(1-\sim)t}{2m^6(1-\sim)t + n^2\sim \sum_{i,j=1}^{n} \ddot{\ } (x_{ij},0)}$$

for all $x = [x_{ij}] \in M_n(X)$. Thus $C: X \to Y$ is a unique sextic mapping satisfying (3.2), as desired.

Corollary 3.7 Let r, θ be positive real numbers with $r > 6$. Let $f: X \to Y$ be a mapping satisfying

$$\text{«}^{(n)}_{Df_n([x_{ij}],[y_{ij}])}(t) \geq \frac{t}{t + \sum_{i,j=1}^{n} -(\|x_{ij}\|^r + \|y_{ij}\|^r)} \tag{3.8}$$

for all $t > 0$ and $x = [x_{ij}]$, $y = [y_{ij}] \in M_n(X)$. Then $C(a) = lim_{k\to\infty} m^6kf(\frac{a}{m^k})$ exists for each $a \in X$ *and* defines a sextic mapping $C: X \to Y$ such that

$$\text{«}_{f_n([x_{ij}])-C_n([y_{ij}])}(t) \geq \frac{2m^6(m^{r-6}-1)t}{2m^6(m^{r-6}-1)t + n^2m^{r-6}\sum_{i,j=1}^{n} -\|x_{ij}\|^r}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof The proof follows from Theorem 3.6 by taking $\varphi(a, b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$. Then we can choose $= m^{6-r}$ and we get the desired result.

Theorem 3.8 Let $\varphi: X^2 \to [0, \infty)$ be a function such that there exists $\alpha < 1$ with

$$\ddot{\ } (a,b) \leq m^6\sim \ddot{\ } (\frac{a}{m}, \frac{b}{m}) \tag{3.9}$$

for all $a, b \in X$. Suppose that $f: X \to Y$ is an mapping satisfying (3.1) for all $x = [x_{ij}]$, $y = [y_{ij}] \in M_n(X)$. Then $C(a) = lim_{k\to\infty} \frac{1}{m^{6k}} f(m^k a)$ exists for each $a \in X$ and defines a sextic mapping $C: X \to Y$ such that

$$\text{«}_{f_n([x_{ij}])-C_n([y_{ij}])}(t) \geq \sum_{i,j=1}^{n} \frac{2m^6(1-\sim)t}{2m^6(1-\sim)t + n^2\sum_{i,j=1}^{n} \ddot{\ } (x_{ij},0)}$$

for all $t > 0$ and $x = [x_{ij}] \in M_n(X)$.

Proof Let (S, d) be the generalized metric space defined in the proof of Theorem 3.6. Now we consider the linear mapping $J: S \to S$ such that

$$Jg(a) = m^6 g\left(\frac{a}{m}\right),$$

for all $a \in X$.

It follows from (3.4) that $d(f, Jf) \le \frac{1}{2m^6}$. So

$$d(f, C) \le \frac{1}{2m^6(1-\sim)}$$

The rest of the proof is similar to the proof of Theorem 3.6.

Corollary 3.9 Let r, θ be positive real numbers with $r < 6$. Let $f: X \to Y$ be a mapping satisfying *(3.7)* for all $t > 0$ and $x = [x_{ij}]$, $y = [y_{ij}] \in M_n(X)$. Then $C(a) = lim_{k\to\infty} \frac{1}{m^{6k}} f(m^k a)$ exists for each $a \in X$ and defines a sextic mapping $C: X \to Y$ such that

$$« _{f_n([x_{ij}])-C_n([y_{ij}])}(t) \ge \frac{2m^6(1-m^{r-6})t}{2m^6(1-m^{r-6})t + n^2 m^{r-6} \sum_{i,j=1}^{n} - \|x_{ij}\|^r}$$

for all $t > 0$ *and* $x = [x_{ij}] \in M_n(X)$.

Proof The proof follows from Theorem 3.8 by taking $\varphi(a, b) = \theta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$. Then we can choose $= m^{r-6}$ and we get the desired result.

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