Growth Rates of Autoregressive Processes

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Abstract

For any d, m *and* $\mathrm{k} \in \mathbb{N}$, *let* $\mathrm{GL(k)}$ *denote the set of invertible real* $\mathrm{k \times k}$ *matrices,* $\mathrm{M(m,k)}$ *the set of real* $\mathrm{m \times k}$ *matrices and* $\mathbf{G}(\mathrm{d,m})$ *the set of matrices of the form* $\begin{bmatrix} \tilde{A} & \tilde{B} \\ 0_{m\times d} & \tilde{C} \end{bmatrix}$, *where* $\tilde{A} \in \mathrm{GL(d)}$ *and* $\tilde{C} \in \mathrm{GL(m)}$. *Let* $(A,B,C) := (A_i, B_{i+1}, C_{i+1})$ *be an i.i.d. sequence in* $\mathrm{GL(d)} \times \mathrm{M(d,m)} \times \mathrm{GL(m)}$. *Set* $A_n \cdots A_m := I$ *for* $m > n$, *where* I *denotes the identity matrix and consider the random sequence* $(R_n(A,B,C))$ *defined as follows:*

$$R_0(A,B,C) := B_1, \; R_n(A,B,C) := \sum_{k=0}^{n} A_n \cdots A_{n-k+1} B_{n-k+1} C_{n-k} \cdots C_1, \; n \geq 1. \tag{1}$$

The process $R_n(A,B,C)$ *is an autoregressive prcess and obeys* $R_n(A,B,C) = A_n R_{n-1}(A,B,C) + B_{n+1}C_n \cdots C_1$. *Let* $V \in \mathrm{M(m,k)}$, $k \geq 1$. *In this paper, the almost sure asymptotic behaviour of* $\left(\frac{1}{n} \ln \| R_n(A,B,C)V \|\right)$ *is studied. Conditions are given under which* $\lim_{n\to\infty} \frac{1}{n} \ln \| R_n(A,B,C)V \|$ *exists* $\mathbb{P}$*-a.s. It is shown that under these conditions, the value of this limit depends on the location of* V *in a filtration of* $\mathrm{M(m,k)}$. *Let* λ_{μ_M} *be the upper Lyapunov exponent associated with* μ_M, μ_M *denoting the common distribution of the the elements of the sequence* $M \subset \mathbf{G}(\mathrm{d,m})$, $M := (M_i)$ *where*

$$M_i := \begin{bmatrix} A_i & B_{i+1} \\ 0_{m\times d} & C_{i+1} \end{bmatrix}. \tag{2}$$

It is also shown that if μ_M *is irreducible (in a sense to be defined later), then* $\lim_{n\to\infty} \frac{1}{n} \ln \| R_n(A,B,C)V \| = \lambda_{\mu_M}$ *almost surely, for all non-zero* V *for which the process* $(R_n(A,B,C)V)$ *is defined.*

Key words: *Growth rate, Lyapunov exponent.*

Resumé

Soit d, m *et* $\mathrm{k} \in \mathbb{N}$, $\mathrm{GL(k)}$ *l'ensemble des matrices reelles et invertibles d'ordre k,* $\mathrm{M(m,k)}$ *l'ensemble des matrices reelles de dimension* $\mathrm{m \times k}$ *et* $\mathbf{G}(\mathrm{d,m})$ *l'ensemble des matrices reelles de forme* $\begin{bmatrix} \tilde{A} & \tilde{B} \\ 0_{m\times d} & \tilde{C} \end{bmatrix}$, *où* $\tilde{A} \in \mathrm{GL(d)}$ *et* $\tilde{C} \in \mathrm{GL(m)}$. *Soit* $(A,B,C) := (A_i, B_{i+1}, C_{i+1})$ *une suite aleatoire indpendent et indentiquement distribuée des valeurs dans* $\mathrm{GL(d)} \times \mathrm{M(d,m)} \times \mathrm{GL(m)}$. *Pour* $m, n \in \mathbb{N}$, *posez* $A_n \cdots A_m := I$ *si* $m > n$, *où* I *denote la matrix identité et considerez la suite aleatoire definie par (1). Le processus* $(R_n(A,B,C))$ *est autoregressif et il Satisfait* $R_n(A,B,C) = A_n R_{n-1}(A,B,C) + B_{n+1}C_n \cdots C_1$. *Soit* $V \in \mathrm{M(m,k)}$, $k \geq 1$. *Dans cet article, le comportement asymptotique du processus reelle* $\left(\frac{1}{n} \ln \| R_n(A,B,C)V \|\right)$ *est etudiée. Des conditions sont données sous lesquelles* $\lim_{n\to\infty} \frac{1}{n} \ln \| R_n(A,B,C)V \|$ *existe* $\mathbb{P}$*-p.s. Il est demontré que lorsque ces conditions sont verifiées, alors la valeur de la limit dépend de la position de* V *dans une filtration de* $\mathrm{M(m,k)}$. *Soit* λ_{μ_M} *l' exposant de Lyapunov le plus grand associé avec* μ_M *où chaque élément de la suite* M *(donnée par (2)) a distribution* μ_M. *Il est demontré que lorsque* μ_M *est irreductible (dans un sens à definir), alors* $\lim_{n\to\infty} \frac{1}{n} \ln \| R_n(A,B,C)V \| = \lambda_{\mu_M}$ $\mathbb{P}$*-p.s pour tout* V *non-zero, tel que le processus* $(R_n(A,B,C)V)$ *est defini.*

Mots Clés: *Taux de Croissance, Exposant de Lyapunov.*

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1 Introduction

In the last decades, there has been much interest in various generalisations of autoregressive processes. Some examples include moving averages, random coefficient models with different types of noise etc. Stationarity properties (see e.g. [12], [7], [1]), almost sure convergence, convergence in distribution and asymptotic distributions in certain cases (see e.g. [6], [8], [9], [10]) have also been a subject of interest. This while, little attention has been paid to questions related to the growth rates of such processes. These questions are taken up in this paper. Let $(X_t)_{t\geq 0}$ be a process with values in a normed space with norm $\| . \|$. The growth rate of X_t is understood to be $\lim\limits_{t\to\infty} \frac{1}{t} \ln \| X_t \|$ if it exists.

The object of interest here is the growth rate of the autoregressive process $R_n(A, B, C)$ in (1); (henceforth we simply write R_n for $R_n(A, B, C)$).

The study of the growth rate of the process is motivated by questions arising in various fields of science. Consider the following(strongly simplified) one dimensional example: Assume that for $n \geq 1$ there are R_{n-1} individuals in an environment at time $n-1$ and that in the time intervall $[n-1, n)$ the population in the environment increases by a ranom factor of A_n and a random number B_{n+1} individuals migrate into and out of the environment. The total number R_n of indivduals in the environment at time n is therefore given by $R_n = A_n R_{n-1} + B_{n+1}$. The following question may be asked: At what speed does the number of individuals in the environment ultimately increase? One way of solving this problem(which is done here in a generalised setting) is to study the growth rate of the process (R_n). Assume that it is shown that under certain conditions, $\lim\limits_{n\to\infty} \frac{1}{n} \ln \| R_n \| = \lambda$, where $\lambda \in I\!R$. Then under these conditions, if $\lambda < 0$ it would be expected that the population dies out ultimately, if $\lambda > 0$ the population would be expected to explode ultimately and if $\lambda = 0$, this suggests that the population would oscillate ultimately around some number.

In what follows, the asymptotic behaviour of $(\frac{1}{n} \ln \| R_n V \|)$ is described, when $I\!E[\ln^+ \| M_0 \| + \ln^+ \| M_0^{-1} \|] < \infty$, where $M := (M_i)$ is some sequence in $\mathbf{G}(d, m)$, associated with R_n and $V \in M(m, k)$ for some k. Let $Z := (Z_i)_{i\in I\!N_0}$ be a sequence of i.i.d. random matrices in GL(d) with common distribution μ_Z on GL(d). Assume that $I\!E \ln^+ \| Z_0 \| < \infty$. Then by the Furstenberg Kesten theorem(see [5]), the almost sure limit $\lim\limits_{n\to\infty} \frac{1}{n} \ln \| Z_n \cdots Z_0 \| =: \lambda_{\mu_Z}$ exists and is a constant. The number λ_{μ_Z} is called the upper Lyapunov exponent associated with μ_Z. Given a sequence $(A, B, C) \in GL(d) \times M(d, m) \times GL(m)$ the following sequence $M \subset \mathbf{G}(d, m)$, is associated with it: $M := (M_i)_{i\in I\!N_0}$, $M_i := \begin{bmatrix} A_i & B_{i+1} \\ 0_{m\times d} & C_{i+1} \end{bmatrix}$.

The main objectives of this paper are to prove theorem 3.9 which characterises the different values that $\lim\limits_{n\to\infty} \frac{1}{n} \ln \| R_n V \|$ takes and theorem 5.5, which gives conditions ensuring that it takes a single value independent of V(V beeing non-zero).

The growth rates of the autoregressive processes $(P_n(A, B))$ and $(Q_n(B, C))$ are also studied. These processes are defined by $P_n(A, B) := A_n R_{n-1}(A, B, \tilde{I}_m)$, $Q_n(B, C) := R_n(\tilde{I}_d, B, C)$. They obey $P_n(A, B) = A_n(P_{n-1}(A, B) + B_n)$ and $Q_n(B, C) = Q_{n-1}(B, C) + B_{n+1} C_n \cdots C_1$ (henceforth we write P_n for $P_n(A, B)$ and Q_n for $Q_n(B, C)$). Several questions remain unanswered in relation to the growth rates of these processes and are taken up elsewhere.

The paper is organised as fiollows: In section 2 it is shown that if $M := (M_i)$ is a sequence of i.i.d. random matrices in GL(k) for which $I\!E[\ln^+ \| M_0 \| + \ln^+ \| M_0^{-1} \|] < \infty$ and Σ is a bounded subset of $I\!R^k$, then $\lim\limits_{n\to\infty} \frac{1}{n} \ln(\sup\limits_{x\in\Sigma} \| M_n \cdots M_0 x \|)$ exists almost surly and is a constant. The major tool here is a theorem of Furstenbeg and Kifer ([4]). By defining $\Sigma(V) := \{Vx : x \in S_{p-1}\}$($S_{p-1}$ is the unit sphere in $I\!R^p$) for $V \in I\!M(k, p)$, $p \in I\!N$, it is then shown that $\lim\limits_{n\to\infty} \frac{1}{n} \ln \| M_n \cdots M_0 V \|$ exists almost surely for all $V \in I\!M(k, p)$ under the condition above and the limit depends on the position of V in a filtration of $\mathbb{M}(k,p)$. Using a suitable choice of the sequence M, it is then shown in section 3 that under additional regularity conditions $\lim_{n\to\infty}\frac{1}{n}\ln\parallel R_nV\parallel$ exists almost surely for all $V\in\mathbb{M}(m,p)$, $p\in\mathbb{N}$ and the limit depends on the position of V in a filtration of $\mathbb{M}(m,p)$. Section 4 contains statements on P_n and Q_n which are corrolaries to statements proven in section 3. Section 5 gives conditions under which $\lim_{n\to\infty}\frac{1}{n}\ln\parallel R_nV\parallel$ exists almost surely and has a single value for all non-zero V for which the process (R_nV) is well defined.

2 Random matrices and filtrations of matrix spaces

The main result here relies on the following theorem:

Theorem 2.1. *Let $(\Omega, F, \mathbb{P})$ be a probability space and $(M_i)_{i\in\mathbb{N}_0}$ be a sequence of i.i.d. random matrices in* GL(d) *defined on* Ω. *Assume that* $\mathbb{E}\left[\ln^+\parallel M_0\parallel+\ln^+\parallel M_0^{-1}\parallel\right]<\infty$. *Then there exists a constant* r, *a filtration* $\{0_{d\times1}\}=\mathbb{L}_{r+1}\subset\mathbb{L}_r\subset\cdots\subset\mathbb{L}_0=\mathbb{R}^d$ *and real constants* $-\infty<\lambda_r<\lambda_{r-1}<\cdots<\lambda_0=\lambda_{\mu_M}$, *such that for* $x\in\mathbb{R}^d\backslash\{0_{d\times1}\}$, *if* $x\in\mathbb{L}_i\backslash\mathbb{L}_{i+1}$, *then* $\lim_{n\to\infty}\frac{1}{n}\ln\parallel M_n\cdots M_0x\parallel=\lambda_i$ *$\mathbb{P}$-a.s. and* $\lim_{n\to\infty}\frac{1}{n}\ln\parallel M_n\cdots M_0\parallel=\lambda_{\mu_M}$ *$\mathbb{P}$-a.s.*

Proof : See [4]

Remark 2.2. For the subspaces $\mathbb{L}_i$, $\mathbb{P}$-a.s. $\lim_{n\to\infty}\frac{1}{n}\ln\parallel M_n\cdots M_0|_{\mathbb{L}_i}\parallel=\lambda_i$ (see [4]).

Definition 2.3. *Let $\mathbb{T}=\mathbb{N}$ or Z, $f:\mathbb{T}\to\mathbb{R}^d$. The number $\lambda(f):=\limsup_{t\to\infty}\frac{1}{t}\ln\parallel f(t)\parallel$ is called the Lyapunov index of f*

Lemma 2.4. *The Lyapunov index has the following properties:*

(i) *If $c\neq0$ is the constant function, then $\lambda(c)=0$ ($\lambda(0)=-\infty$).*

(ii) If $\alpha\in\mathbb{R}\backslash\{0\}$, then $\lambda(\alpha f)=\lambda(f)$.

(iii) $\lambda(f+g)\leq\max\{\lambda(f),\lambda(g)\}$ with equality, if $\lambda(f)\neq\lambda(g)$.

Proof : see [3]

Lemma 2.5. *Let $(M_t)_{t\in\mathbb{R}^+}$ be* d × m *matrices and $\mathbb{L}$ an* n*-dimensional linear subspace of $\mathbb{R}^m$ with basis $\{v_1,\cdots,v_n\}$, $n\leq m$. Then* $\limsup_{t\to\infty}\frac{1}{t}\ln\parallel M_t|_{\mathbb{L}}\parallel=\max_{\{v_1,\ldots,v_n\}}\limsup_{t\to\infty}\frac{1}{t}\ln\parallel M_tv_j\parallel$.

Proof : See [3], [11]

For $\Sigma\subset\mathbb{R}^m$, write $\mathbb{L}(\Sigma)$ for the linear subspace of $\mathbb{R}^m$ generated by Σ and if B is a matrix for which Bx is defined for all $x\in\Sigma$, set $\parallel B\parallel_\Sigma:=\sup_{\{x\in\Sigma\}}\parallel Bx\parallel$.

Lemma 2.6. *Let $(M_t)_{t\in\mathbb{R}^+}$ be real* d × m *matrices. Suppose that Σ_1, Σ_2 are non-empty bounded sets in $\mathbb{R}^m$ and $\mathbb{L}(\Sigma_1)=\mathbb{L}(\Sigma_2)$. Then* $\limsup_{t\to\infty}\frac{1}{t}\ln\parallel M_t\parallel_{\Sigma_1}=\limsup_{t\to\infty}\frac{1}{t}\ln\parallel M_t\parallel_{\Sigma_2}$.

Proof : $\Sigma_1=0_{m\times1}$ if and only if $\Sigma_2=0_{m\times1}$ since $\mathbb{L}(\Sigma_1)=\mathbb{L}(\Sigma_2)$. In this case the assertion of the lemma is trivially true. Assume therefore that $\Sigma_1\neq0_{m\times1}$ and let

$\{e_1, \ldots, e_k\} \subset \Sigma_1$, be linearly independent vectors such that $\mathbb{L}(\Sigma_1) = \mathbb{L}(\{e_1, \ldots, e_k\})$. For each $i \in \{1, \ldots, k\}$, $\limsup_{t\to\infty} \frac{1}{t} \ln \| M_t e_i \| \le \limsup_{t\to\infty} \frac{1}{t} \ln \| M_t \|_{\Sigma_1}$.

Therefore
$$\max_{i\in\{1,\ldots,k\}} \limsup_{t\to\infty} \frac{1}{t} \ln \| M_t e_i \| \le \limsup_{t\to\infty} \frac{1}{t} \ln \| M_t \|_{\Sigma_1} . \tag{3}$$

Since $\mathbb{L}(\Sigma_1) = \mathbb{L}(\{e_1, \ldots, e_k\})$, it follows from lemma 2.5 and (3), that
$\limsup_{t\to\infty} \frac{1}{t} \ln \| M_t|_{\mathbb{L}(\Sigma_1)} \| = \max_{\{e_1,\ldots,e_k\}} \limsup_{t\to\infty} \frac{1}{t} \ln \| M_t e_i \| \le \limsup_{t\to\infty} \frac{1}{t} \ln \| M_t \|_{\Sigma_1}$.

Thus
$$\limsup_{t\to\infty} \frac{1}{t} \ln \| M_t|_{\mathbb{L}(\Sigma_1)} \| \le \limsup_{t\to\infty} \frac{1}{t} \ln \| M_t \|_{\Sigma_1} . \tag{4}$$

On the other hand, $\| M_t \|_{\Sigma_1} = \sup_{\{x\in\Sigma_1\}} \| M_t x \| = \sup_{\{x\in\Sigma_1\}} \| M_t \sum_{i=1}^{k} r_i(x) e_i \|$
$= \sup_{\{x\in\Sigma_1\}} \| \sum_{i=1}^{k} r_i(x) M_t e_i \| \le \sup_{\{x\in\Sigma_1\}} \left(\max_{i\in\{1,\ldots,k\}} |r_i(x)| \right) \sum_{i=1}^{k} \| M_t e_i \|$. Now $\Sigma_1 \neq 0_{m\times 1}$ and is bounded.
Thus $0 < \sup_{\{x\in\Sigma_1\}} \left(\sup_{i\in\{1,\ldots,k\}} |r_i(x)| \right) < \infty$ and so $\lim_{t\to\infty} \frac{1}{t} \ln \left[\sup_{\{x\in\Sigma_1\}} \left(\sup_{i\in\{1,\ldots,k\}} |r_i(x)| \right) \right]$
$= 0$. From this, lemma 2.4(iii), lemma 2.5 and the equality $\mathbb{L}(\Sigma_1) = \mathbb{L}(\{e_1, \ldots e_k\})$,
$\limsup_{t\to\infty} \frac{1}{t} \ln \| M_t \|_{\Sigma_1} \le \limsup_{t\to\infty} \frac{1}{t} \ln \sum_{i=1}^{k} \| M_t e_i \| \le \max_{i\in\{1,\ldots,k\}} \limsup_{t\to\infty} \frac{1}{t} \ln \| M_t e_i \|$
$= \limsup_{t\to\infty} \frac{1}{t} \ln \| M_t|_{\mathbb{L}(\{e_1,\ldots,e_k\})} \| = \limsup_{t\to\infty} \frac{1}{t} \ln \| M_t|_{\mathbb{L}(\Sigma_1)} \|$. Therefore

$$\limsup_{t\to\infty} \frac{1}{t} \ln \| M_t \|_{\Sigma_1} \le \limsup_{t\to\infty} \frac{1}{t} \ln \| M_t|_{\mathbb{L}(\Sigma_1)} \| . \tag{5}$$

(4) and (5) together imply that $\limsup_{t\to\infty} \frac{1}{t} \ln \| M_t \|_{\Sigma_1} = \limsup_{t\to\infty} \frac{1}{t} \ln \| M_t|_{\mathbb{L}(\Sigma_1)} \|$. Similarly, $\limsup_{t\to\infty} \frac{1}{t} \ln \| M_t \|_{\Sigma_2} = \limsup_{t\to\infty} \frac{1}{t} \ln \| M_t|_{\mathbb{L}(\Sigma_2)} \|$. Since $\mathbb{L}(\Sigma_1) = \mathbb{L}(\Sigma_2)$, the assertion now follows.

For Σ a non-empty bounded set in $\mathbb{R}^m$, define $\overline{\Sigma} := \{\Lambda \subset \mathbb{R}^m : \mathbb{L}(\Lambda) = \mathbb{L}(\Sigma)\}$.

Theorem 2.7. *Let* $M = (M_i)_{i\in\mathbb{N}_0}$ *be a sequence of i.i.d. random matrices in* GL(k) *defined on a probability space* $(\Omega, F, \mathbb{P})$ *with common distribution* μ.
Assume that $\mathbb{E}\left[\ln^+ \| M_0 \| + \ln^+ \| M_0^{-1} \|\right] < \infty$ *and let* $\hat{\lambda}(.)$ *be defined as follows; for non-empty bounded* $\Sigma \subset \mathbb{R}^k$ *set* $\hat{\lambda}(\Sigma) := \mathbb{P}$*-a.s-*$\lim_{n\to\infty} \frac{1}{n} \ln \| M_n \cdots M_0 \|_{\Sigma}$.
(i)*If* $\{0_{k\times 1}\} = \mathbb{L}_{r+1} \subset \mathbb{L}_r \subset \cdots \subset \mathbb{L}_0 = \mathbb{R}^k$ *are the subspaces given by theorem 2.1 associated with* μ *and* $-\infty < \lambda_r < \cdots < \lambda_0 = \lambda_\mu$ *are the associated constants then for* $0_{k\times 1} \neq \Sigma \subset \mathbb{R}^k$, *it holds that* $\Sigma \subseteq \mathbb{L}_j$ and $\Sigma \not\subset \mathbb{L}_{j+1} \Longleftrightarrow \hat{\lambda}(\Sigma) = \lambda_j$.
(ii) $\hat{\lambda}(.)$ *is constant on* $\overline{\Sigma}$ *for each non-empty bounded* $\Sigma \subset \mathbb{R}^k$.

Proof : If $0_{k\times 1} \neq \Sigma \subset \mathbb{R}^k$, then there exists some $j \in \{r, \ldots, 0\}$ for which $\{0_{k\times 1}\} = \mathbb{L}_{r+1} \subset \ldots \subset \mathbb{L}_j$; $\Sigma \subseteq \mathbb{L}_j$, $\Sigma \not\subset \mathbb{L}_{j+1}$. If $\bar{y} \in (\mathbb{L}_j \backslash \mathbb{L}_{j+1}) \cap \Sigma$, then $\frac{1}{n} \ln \| M_n \cdots M_0 \bar{y} \| \le \frac{1}{n} \ln \| M_n \cdots M_0 \|_{\Sigma}$ for all $n \in \mathbb{N}$. Theorem 2.1 implies

$$\lambda_j = \liminf_{n\to\infty} \frac{1}{n} \ln \| M_n \cdots M_0 \bar{y} \| \le \liminf_{n\to\infty} \frac{1}{n} \ln \| M_n \cdots M_0 \|_{\Sigma} \ \mathbb{P}\text{–a.s.} \tag{6}$$

It will now be shown that $\lambda_j \geq \limsup_{n\to\infty} \frac{1}{n} \ln \| M_n \cdots M_0 \|_\Sigma$ $\mathbb{P}$-a.s. Let us recall that j has been fixed such that $\Sigma \subseteq \mathbb{L}_j$. Thus $\| M_n \cdots M_0 \|_\Sigma \leq \sup_{x\in\Sigma} \| M_n \cdots M_0|_{\mathbb{L}_j} \| \| x \|$
$= \| M_n \cdots M_0|_{\mathbb{L}_j} \| \sup_{x\in\Sigma} \| x \|$. Since Σ is bounded and satisfies $\Sigma \neq 0_{k\times 1}$,
$\lim_{n\to\infty} \frac{1}{n} \ln \sup_{x\in\Sigma} \| x \| = 0$. By remark 2.2, $\lim_{n\to\infty} \frac{1}{n} \ln \| M_n \cdots M_0|_{\mathbb{L}_j} \| = \lambda_j$ $\mathbb{P}$-a.s. Therefore $\limsup_{n\to\infty} \frac{1}{n} \ln \| M_n \cdots M_0 \|_\Sigma \leq \lambda_j$ $\mathbb{P}$-a.s. This together with (6) imply that
$\lim_{n\to\infty} \frac{1}{n} \ln \| M_n \cdots M_0 \|_\Sigma = \lambda_j$ $\mathbb{P}$-a.s. Thus $\Sigma \subseteq \mathbb{L}_j,\ \Sigma \not\subseteq \mathbb{L}_{j+1} \Longrightarrow \hat{\lambda}(\Sigma) = \lambda_j$.
For the other direction of the equivalence, assume that $\lim_{n\to\infty} \frac{1}{n} \ln \| M_n \cdots M_0 \|_\Sigma = \lambda_j$ $\mathbb{P}$-a.s. for some λ_j and $\Sigma \subseteq \mathbb{L}_j$, $\Sigma \not\subseteq \mathbb{L}_{j+1}$ is false. Since $\{0_{k\times 1}\} = \mathbb{L}_{r+1} \subset \mathbb{L}_r \subset \cdots \subset \mathbb{L}_0 = \mathbb{R}^k$, there must exist some $i \neq j$ for which $\Sigma \subseteq \mathbb{L}_i$, $\Sigma \not\subseteq \mathbb{L}_{i+1}$. This implies however that $\lim_{n\to\infty} \frac{1}{n} \ln \| M_n \cdots M_0 \|_\Sigma = \lambda_i$ $\mathbb{P}$-a.s. This contradicts $\lim_{n\to\infty} \frac{1}{n} \ln \| M_n \cdots M_0 \|_\Sigma = \lambda_j$ $\mathbb{P}$-a.s., since $\lambda_i \neq \lambda_j$.

(ii) That $\hat{\lambda}(\cdot)$ is constant on $\overline{\Sigma}$ follows if lemma 2.6 is applied to the elements ω of the set of measure 1 on which (i) holds.

Definition 2.8. *Let* V *be a vector space. A filtration of* V *is understood to be a sequence of subspaces* $0 = \mathbb{L}_{r+1} \subset \mathbb{L}_r \subset \cdots \subset \mathbb{L}_1 \subset \mathbb{L}_0 = V$ *for some* $r \in \mathbb{N}_0$, *where each* $\mathbb{L}_j$ *is a vector space.*

Lemma 2.9. *Let* $\mathbf{V}$ *be a real vector space of dimension* d *and suppose that* $\gamma : \mathbf{V} \to \mathbb{R}\cup\{-\infty\}$ *is a map satisfying the following conditions:* $\gamma(0) = -\infty$, $\gamma(tx) = \gamma(x)$, *for every* $t \in \mathbb{R}\backslash\{0\}$ *and* $x \in \mathbf{V}$ $\gamma(x_1 + x_2) \leq \max\{\gamma(x_1), \gamma(x_2)\}$ *for every* $x_1,\ x_2 \in \mathbf{V}$. *Then* γ *can take at most* d *distinct values on* $\mathbf{V}\backslash\{0\}$. *The sets* $\mathbf{V}_\mu := \{x \in \mathbf{V} : \gamma(x) \leq \mu\}$ *with* $\mu \in \mathbb{R}$ *are linear subspaces of* $\mathbf{V}$. *Let* $-\infty \leq \gamma_p < \cdots < \gamma_1 < \infty$ *be the different values* γ *takes. The sets* $\mathbf{V}_i := \mathbf{V}_{\gamma_i}$ *form a filtration of* $\mathbf{V}$ *and it holds that* $\gamma(x) = \gamma_i \Longleftrightarrow x \in \mathbf{V}_i\backslash\mathbf{V}_{i+1}$.

Proof : See [3].

Definition 2.10. *A map* γ *defined on a real vector space with values in* $\mathbb{R} \cup \{-\infty\}$ *fullfilling the asssumptions of lemma 2.9 and* $\gamma(x + y) = \max\{\gamma(x), \gamma(y)\}$ *for* $\gamma(x) \neq \gamma(y)$ *is called a characteristic exponent.*

For $V \in M(k,p)$, let $\Sigma(V) := \{Vx : x \in S_{p-1}\}$, $\overline{V} := \{W \in M(k,p) : \mathbb{L}(\Sigma(W)) = \mathbb{L}(\Sigma(V))\}$. From the preceeding theorem and lemma the following corollary is obtained:

Corollary 2.11. *Let the sequence* M *satisfy the assumptions of theorem 2.7 and* $p \in \mathbb{N}$. *Then, for every matrix* $V \in M(k,p)$, *the limit* $\lambda(V) := \mathbb{P}$-a.s.- $\lim_{n\to\infty} \frac{1}{n} \ln \| M_n \cdots M_0 V \|$ *exists and is constant on* $\overline{V}$. *If* $k = p$ *and* $V \in GL(k)$, *then* $\lambda(V) = \lambda_\mu$. *Moreover the map*
$\lambda : \mathbb{M}(k,p) \longrightarrow \mathbb{R} \cup \{-\infty\},\ V \longmapsto \lambda(V)$ *defines a characteristic exponent.*

Proof : If $V = 0_{k\times p}$, then it is clear that $\lambda(V) = -\infty$. Let $V \neq 0_{k\times p}$, then $\| M_n \cdots M_0 V \|$
$= \sup_{\{x\in S_{p-1}\}} \| M_n \cdots M_0 V x \| = \sup_{\{y=Vx:x\in S_{p-1}\}} \| M_n \cdots M_0 y \| = \| M_n \cdots M_0 \|_{\Sigma(V)}$. Since $\Sigma(V)$ is bounded, there exists some j such that $\Sigma(V) \subseteq \mathbb{L}_j$ and $\Sigma(V) \not\subseteq \mathbb{L}_{j+1}$. Theorem 2.7 now states that $\lim_{n\to\infty} \frac{1}{n} \ln \| M_n \cdots M_0 \|_{\Sigma(V)} = \lambda_j$ $\mathbb{P}$-a.s.
Therefore $\lambda(V) = \lim_{n\to\infty} \frac{1}{n} \ln \| M_n \cdots M_0 V \| = \lambda_j$ $\mathbb{P}$-a.s. It's now shown that if $W \in \overline{V}$ then $\lambda(W) =$

$\lambda(V)$. Let $W \in \overline{V}$ then by the definition of $\overline{V}$, $\mathbb{L}(\Sigma(W)) = \mathbb{L}(\Sigma(V))$. Therefore $\Sigma(W) \in \overline{\Sigma(V)}$. By theorem 2.7, $\hat{\lambda}(\Sigma(W)) = \hat{\lambda}(\Sigma(V))$.
Now $\| M_n \cdots M_0 W \| = \| M_n \cdots M_0 \|_{\Sigma(W)}$. Therefore $\lambda(W) = \hat{\lambda}(\Sigma(W)) = \hat{\lambda}(\Sigma(V)) = \lambda(V)$, showing that λ is constant on $\overline{V}$. If $k = p$ and $V = I_k$, then $\lambda(I_k) = \lambda_\mu$. It is easy to see that $\bar{I}_k = GL(k)$. Therefore $\lambda(V) = \lambda_\mu$ for all $V \in GL(k)$ since λ is constant on $\bar{I}_k$. Showing that λ defines a characteristic exponent, involves a straightforward computation.

The following is the main theorem of this section:

Theorem 2.12. *Let $(\Omega, F, \mathbb{P})$ be a probability space and $M := (M_i)_{i \in \mathbb{N}_0}$, a sequence of i.i.d. random matrices in $GL(k)$ defined on Ω. Assume that $\mathbb{E}\left[\ln^+ \| M_0 \| + \ln^+ \| M_0^{-1} \|\right] < \infty$ and let $-\infty < \lambda_r < \cdots < \lambda_0 = \lambda_\mu$ and $\{0_{k\times p}\} = \mathbb{L}_{r+1} \subset \mathbb{L}_r \subset \cdots \subset \mathbb{L}_0 = \mathbb{R}^k$ be the subspaces and constants in theorem 2.7. For each $j \in \{0, \dots, r\}$ and $p \in \mathbb{N}$ define*
$\mathbf{L}_j := \{V \in M(k,p) : \Sigma(V) \subseteq \mathbb{L}_j\}$ and $\lambda(V) := \mathbb{P}\text{-a.s-}\lim_{n\to\infty} \frac{1}{n} \ln \| M_n \cdots M_0 V \|$, $V \in \mathbb{M}(k,p)$. Then λ defines a characteristic exponent and

(i) *for each $j \in \{0, \dots, r+1\}$, $\mathbf{L}_j$ is a linear subspace of $M(k,p)$.*

(ii) $\{0_{k\times p}\} = \mathbf{L}_{r+1} \subset \mathbf{L}_r \subset \cdots \subset \mathbf{L}_0 = M(k,p)$.

(iii) *if $V \in M(k,p) \backslash \{0_{k\times p}\}$ then $V \in \mathbf{L}_j \backslash \mathbf{L}_{j+1} \Longleftrightarrow \lambda(V) = \lambda_j$.*

(iv) *if $k = p$ then $GL(k) \subset \mathbf{L}_0 \backslash \mathbf{L}_1$.*

Proof : That λ defines a characteristic exponent follows from corollary 2.11.
(i) By lemma 2.9 and the fact that λ defines a characteristic exponent(corollary 2.11), for each $j \in \{0, \dots r\}$ the set $\{V \in M(k,p) : \lambda(V) \leq \lambda_j\}$ is a linear subspace of $M(k,p)$. Therefore $\mathbf{L}_j$ is linear since $\mathbf{L}_j = \{V \in M(k,p) : \Sigma(V) \subseteq \mathbb{L}_j\} = \{V \in M(k,p) : \lambda(V) \leq \lambda_j\}$.
(ii) Since $\lambda_j < \lambda_{j-1}$, $\mathbf{L}_j \subset \mathbf{L}_{j-1}$.
Further, $\lambda_0 = \lambda_\mu$ implies that $\mathbf{L}_0 = \{V \in M(k,p) : \lambda(V) \leq \lambda_\mu\} = M(k,p)$.
$\mathbf{L}_{r+1} = \{V \in \mathbb{M}(k,p) : \Sigma(V) \subseteq \mathbb{L}_{r+1}\} = 0_{k\times p}$. That λ takes precisely the values $-\infty < \lambda_r < \lambda_{r-1} < \cdots < \lambda_0$ follows from the following proof of (iii):
(iii) Let $V \in M(k,p) \backslash \{0_{k\times p}\}$. Then $V \in \mathbf{L}_j \backslash \mathbf{L}_{j+1}$ for some j and this is equivalent to $\Sigma(V) \subseteq \mathbb{L}_j$ and $\Sigma(V) \not\subset \mathbb{L}_{j+1} \Longleftrightarrow \hat{\lambda}(\Sigma(V)) = \lambda_j \Longleftrightarrow \lambda(V) = \lambda_j$. That for $k = p$, $GL(k) \subset \mathbf{L}_0 \backslash \mathbf{L}_1$ follows from the fact that $\bar{I}_k = GL(k)$, $\lambda(I_k) = \lambda_0 \Longleftrightarrow I_k \in \mathbf{L}_0 \backslash \mathbf{L}_1$. Therefore $GL(k) \subseteq \mathbf{L}_0 \backslash \mathbf{L}_1$. Also, for $V := \left[\ x, 0_{k\times(p-1)}\ \right]$ $x \in \mathbb{L}_0 \backslash \mathbb{L}_1$, $\lambda(V) = \lambda_0$, showing that $V \in \mathbf{L}_0 \backslash \mathbf{L}_1$. Therefore $GL(k) \subset \mathbf{L}_0 \backslash \mathbf{L}_1$.

3 The growth rate of R_n

The growth rate of (R_n) is now studied. We first create the framework in which to argue. This is done in lemmas 3.1 and 3.2. The arguments begin proposition 3.3.

$M(d,m)$ is a fnite dimensional real vector space, hence all norms on $M(d,m)$ are equivalent. Let $\mathcal{F}(M(d,m))$ be the Borel σ-field generated by the open sets with respect to the metric induced by the matrix norm associated with the standard Euclidean norm $\| \,.\, \|$ in $\mathbb{R}^{d+m}$. $(M(d,m), \mathcal{F}(M(d,m)))$ is a measurable space.
Set $\mathcal{G}(d,m) := \left\{ \begin{bmatrix} A & B \\ 0_{m\times d} & C \end{bmatrix} : A \in M(d,d), C \in M(m,m) \right\}$. $\mathcal{G}(d,m)$ is a subspace of $M(d+m, d+m)$. $\mathcal{F}(\mathcal{G}(d,m)) := \{G \cap \mathcal{G}(d,m) : G \in \mathcal{F}(M(d+m,d+m))\}$ is a σ-field and $(\mathcal{G}(d,m), \mathcal{F}(\mathcal{G}(d,m)))$ is a measurable space.

Lemma 3.1. *Let* $(\Omega, \mathrm{F}, \mathbb{P})$ *be a probability space. A map* $\mathrm{M} : \Omega \longrightarrow \mathcal{G}(\mathrm{d}, \mathrm{m})$,
$\omega \longmapsto \mathrm{M}(\omega) := \begin{bmatrix} \mathrm{A} & \mathrm{B} \\ 0_{\mathrm{m}\times\mathrm{d}} & \mathrm{C} \end{bmatrix}(\omega)$ *is* $(\mathrm{F}, \mathcal{F}(\mathcal{G}(\mathrm{d}, \mathrm{m})))$*-measurable if and only if* A *is* $(\mathrm{F}, \mathcal{F}(\mathrm{M}(\mathrm{d}, \mathrm{d})))$*-measurable,* B *is* $(\mathrm{F}, \mathcal{F}(\mathrm{M}(\mathrm{d}, \mathrm{m})))$*-measurable and* C *is* $(\mathrm{F}, \mathcal{F}(\mathrm{M}(\mathrm{m}, \mathrm{m})))$*- measurable.*

Proof : See [11]
Define $\mathbf{G}(\mathrm{d}, \mathrm{m}) := \mathcal{G}(\mathrm{d}, \mathrm{m}) \cap \mathrm{GL}(\mathrm{d}+\mathrm{m})$, then by lemma 3.1, from the point of view of measurability, considering some random $(\mathrm{A}, \mathrm{B}, \mathrm{C})$ in $\mathrm{GL}(\mathrm{d}) \times \mathrm{M}(\mathrm{d}, \mathrm{m}) \times \mathrm{GL}(\mathrm{m})$ is equivalent to considering a random matrix M in $\mathbf{G}(\mathrm{d}, \mathrm{m})$. The following lemma is usefull.

Lemma 3.2. *Let* $\mathrm{d}, \mathrm{m} \in \mathbb{N}$ *and* $\mathrm{M} = \begin{bmatrix} \mathrm{A} & \mathrm{B} \\ 0_{\mathrm{m}\times\mathrm{d}} & \mathrm{C} \end{bmatrix}$ *be a random matrix in* $\mathbf{G}(\mathrm{d}, \mathrm{m})$*. Then*

(a) $\max\{\mathbb{E}\ln^+ \| \mathrm{M} \|, \mathbb{E}\ln^+ \| \mathrm{M}^{-1} \|\} < \infty$, *if and only if*

(b) $\max\left\{\mathbb{E}\left|\ln \| \mathrm{A} \|\right|, \mathbb{E}\left|\ln \| \mathrm{C} \|\right|, \mathbb{E}\ln^+ \| \mathrm{B} \|, \mathbb{E}\ln^- \left|\det \mathrm{C}\right|, \mathbb{E}\ln^- \left|\det \mathrm{A}\right|\right\} < \infty.$

Proof : It is first shown that (a) $\Longrightarrow$ (b). Notice that $\max\{\| \mathrm{A} \|, \| \mathrm{B} \|, \| \mathrm{C} \|\} \leq \| \mathrm{M} \|$. hence $\max\{\ln^+ \| \mathrm{A} \|, \ln^+ \| \mathrm{B} \|, \ln^+ \| \mathrm{C} \|\} \leq \ln^+ \| \mathrm{M} \|$. From this and the assumption (a) of the

$$\text{lemma,}\quad \max\{\mathbb{E}\ln^+ \| \mathrm{A} \|, \mathbb{E}\ln^+ \| \mathrm{B} \|, \mathbb{E}\ln^+ \| \mathrm{C} \|\} \leq \mathbb{E}\ln^+ \| \mathrm{M} \| < \infty. \tag{7}$$

M is invertible with $\mathrm{M}^{-1} = \begin{bmatrix} \mathrm{A}^{-1} & -\mathrm{A}^{-1}\mathrm{B}\mathrm{C}^{-1} \\ 0_{\mathrm{m}\times\mathrm{d}} & \mathrm{C}^{-1} \end{bmatrix}$. Since $\mathbb{E}\ln^+ \| \mathrm{M}^{-1} \| < \infty$ it follows similar to (7) that

$$\max\{\mathbb{E}\ln^+ \| \mathrm{A}^{-1} \|, \mathbb{E}\ln^+ \| \mathrm{A}^{-1}\mathrm{B}\mathrm{C}^{-1} \|, \mathbb{E}\ln^+ \| \mathrm{C}^{-1} \|\} \leq \mathbb{E}\ln^+ \| \mathrm{M}^{-1} \| < \infty. \tag{8}$$

Since $1 \leq \| \mathrm{A} \| \| \mathrm{A}^{-1} \|$, $\ln^- \| \mathrm{A} \| \leq \ln^+ \| \mathrm{A}^{-1} \|$, hence $\mathbb{E}\ln^- \| \mathrm{A} \| \leq \mathbb{E}\ln^+ \| \mathrm{A}^{-1} \|$. This and (8) imply $\mathbb{E}\ln^- \| \mathrm{A} \| < \infty$. Similar reasoning applied to C shows that $\mathbb{E}\ln^- \| \mathrm{C} \| < \infty$.

$$\text{Thus}\quad \max\{\mathbb{E}\ln^- \| \mathrm{A} \|, \mathbb{E}\ln^- \| \mathrm{C} \|\} < \infty. \tag{9}$$

This and (7) imply

$$\max\{\mathbb{E}|\ln \| \mathrm{A} \| |, \mathbb{E}\ln^+ \| \mathrm{B} \|, \mathbb{E}|\ln \| \mathrm{C} \| |\} < \infty. \tag{10}$$

It is now shown that $\max\{\mathbb{E}\ln^- |\det \mathrm{A}|, \mathbb{E}\ln^- |\det \mathrm{C}|\} < \infty$, to complete the proof of $(a) \Longrightarrow (b)$.
Let $\delta_1(\mathrm{A}) \geq \cdots \geq \delta_{\mathrm{d}}(\mathrm{A}) > 0$ be the singular values of A.
Then $\| \mathrm{A} \| = \delta_1(\mathrm{A})$, $|\det \mathrm{A}| = \delta_1(\mathrm{A}) \cdots \delta_{\mathrm{d}}(\mathrm{A})$ and $\| \mathrm{A}^{-1} \| = \delta_{\mathrm{d}}(\mathrm{A})^{-1}$, hence
$-\ln|\det \mathrm{A}| = -\sum_{\mathrm{i}=1}^{\mathrm{d}} \ln \delta_{\mathrm{i}}(\mathrm{A}) \leq -\mathrm{d}\ln \delta_{\mathrm{d}}(\mathrm{A}) = \mathrm{d}\ln \| \mathrm{A}^{-1} \|$. Therefore
$\mathbb{E}\ln^- |\det \mathrm{A}| \leq \mathrm{d}\mathbb{E}\ln^+ \| \mathrm{A}^{-1} \|$. This and (8) show that $\mathbb{E}\ln^- |\det \mathrm{A}| < \infty$. Similarly $\mathbb{E}\ln^- |\det \mathrm{C}| < \infty$. Thus $\max\{\mathbb{E}\ln^- |\det \mathrm{A}|, \mathbb{E}\ln^- |\det \mathrm{A}|\} < \infty$. $(a) \Longrightarrow (b)$ has thus been proven. We now prove $(b) \Longrightarrow (a)$. Evidently,
$3\ \ln^- |\det \mathrm{M}| = \max\{0, -(\ln|\det \mathrm{A}| + \ln|\det \mathrm{C}|)\} \leq \ln^- |\det \mathrm{A}| + \ln^- |\det \mathrm{C}|$. Therefore

$$\mathbb{E}\ln^- |\det \mathrm{M}| \leq \mathbb{E}\ln^- |\det \mathrm{A}| + \mathbb{E}\ln^- |\det \mathrm{C}|. \tag{11}$$

By (b), $\mathbb{E}\ln^- |\det \mathrm{A}| + \mathbb{E}\ln^- |\det \mathrm{C}| < \infty$. From (11),

$$\mathbb{E}\ln^- |\det \mathrm{M}| < \infty. \tag{12}$$

Let $\delta_1(M) \geq \cdots \geq \delta_{d+m}(M) > 0$ be the singular values of M, then $|\det M| = \delta_1(M) \cdots \delta_{d+m}(M)$ and $\| M^{-1} \| = \delta_{d+m}(M)^{-1}$. Therefore $\| M^{-1} \| \, |\det M| \leq \| M \|^{d+m-1}$, hence $\ln^+ \| M^{-1} \| \leq \ln^- |\det M| + (d+m-1) \ln^+ \| M \|$. If it is that $\mathbb{E} \ln^+ \| M \| < \infty$, then by (12), the assertion follows. But
$\| M \| \leq \| A \| + \| B \| + \| C \| \leq 3 \max\{\| A \|, \| B \|, \| C \|\}$. Hence
$\ln^+ \| M \| \leq \ln 3 + \ln^+ \| A \| + \ln^+ \| B \| + \ln^+ \| C \|$.
Therefore $\mathbb{E} \ln^+ \| M \| \leq \ln 3 + \mathbb{E}(\ln^+ \| A \| + \ln^+ \| B \| + \ln^+ \| C \|) < \infty$.

Let m and d be given, d, m $\in \mathbb{N}$. Define $\hat{x}_d := (0^*_{d\times 1}, x^*)^*$, $x \in \mathbb{R}^m$, $*$ denoting the transpose. If $\Sigma \subset \mathbb{R}^m$ then define $\hat{\Sigma}_d := \{\hat{x}_d : x \in \Sigma\}$. "A sequence M in $\mathbf{G}(d,m)$" shall be talked of, and "a sequence $(M_i)_{i\in \mathbb{N}_0}$ of i.i.d. random matrices in $\mathbf{G}(d,m)$ with $M_i := \begin{bmatrix} A_i & B_{i+1} \\ 0_{m\times d} & C_{i+1} \end{bmatrix}$" shall be meant. If $Z := (Z_i)$ is a sequence of square matrices, write $\Phi_n(Z) := Z_n \cdots Z_0$. Let $(A,B,C) := (A_i, B_{i+1}, C_{i+1})$ be a sequence of i.i.d. random elements of $GL(d) \times M(d,m) \times GL(m)$. By lemma 3.1 the sequence M with $M_i := \begin{bmatrix} A_i & B_{i+1} \\ 0_{m\times d} & C_{i+1} \end{bmatrix}$ is a sequence of i.i.d. random matrices in $\mathbf{G}(d,m)$. If

$$\max\{\mathbb{E}|\ln \| A_0 \||, \mathbb{E}|\ln \| C_1 \||, \mathbb{E}\ln^+ \| B_1 \|, \mathbb{E}\ln^- |\det A_0|, \mathbb{E}\ln^- |\det C_1|\} < \infty \tag{13}$$

then by lemma 3.2, this is equivalent to

$$\mathbb{E}\left[\ln^+ \| M_0 \| + \ln^+ \| M_0^{-1} \|\right] < \infty. \tag{14}$$

In the sequel, the objects of interest are sequences (A, B, C) of random elements of $GL(d) \times M(d,m) \times GL(m)$ which satisfy (13). However the associated sequences M in $\mathbf{G}(d,m)$ which satisfy (14) are considered since they fit in our set up in a natural way.

Proposition 3.3. *Let* M *be a sequence in* $\mathbf{G}(d,m)$ *such that* $\mathbb{E}\left[\ln^+ \| M_0 \| + \ln^+ \| M_0^{-1} \|\right] < \infty$ *and* $\Sigma \subset \mathbb{R}^m$ *be bounded. If* $\limsup_{n\to\infty} \frac{\| C_{n+1} \cdots C_1 \|_\Sigma}{\| \Phi_n(M) \|_{\hat{\Sigma}_d}} < 1$ $\mathbb{P}$*-a.s, then* $\lim_{n\to\infty} \frac{1}{n} \ln \| R_n \|_\Sigma = \lim_{n\to\infty} \frac{1}{n} \ln \| \Phi_n(M) \|_{\hat{\Sigma}_d}$ $\mathbb{P}$*-a.s.*

Proof : $\| \Phi_n(M) \|_{\hat{\Sigma}_d} = \sup_{x\in\Sigma} \left[\| R_n x \|^2 + \| C_{n+1} \cdots C_1 x \|^2\right]^{\frac{1}{2}}$. Therefore

$$\| \Phi_n(M) \|_{\hat{\Sigma}_d} \geq \| R_n \|_\Sigma . \quad \text{and} \quad \text{thus} \tag{15}$$

$$\limsup_{n\to\infty} \frac{1}{n} \ln \| \Phi_n(M) \|_{\hat{\Sigma}_d} \geq \limsup_{n\to\infty} \frac{1}{n} \ln \| R_n \|_\Sigma \tag{16}$$

$\mathbb{P}$-a.s. Also, $\| \Phi_n(M) \|^2_{\hat{\Sigma}_d} = \sup_{\{x\in\Sigma\}} \left[\| R_n x \|^2 + \| C_n \cdots C_1 x \|^2\right]$. Consequently
$\| \Phi_n(M) \|^2_{\hat{\Sigma}_d} \leq \| R_n \|^2_\Sigma + \| C_n \cdots C_1 \|^2_\Sigma$ and thus
$\| \Phi_n(M) \|^2_{\hat{\Sigma}_d} - \| C_n \cdots C_1 \|^2_\Sigma \leq \| R_n \|^2_\Sigma$. From this

$$\frac{2}{n} \ln \| \Phi_n(M) \|_{\hat{\Sigma}_d} + \frac{1}{n} \ln \left[1 - \frac{\| C_{n+1} \cdots C_1 \|^2_\Sigma}{\| \Phi_n(M) \|^2_{\hat{\Sigma}_d}}\right] \leq \frac{2}{n} \ln \| R_n \|_\Sigma .$$

By assumption $\limsup_{n\to\infty} \frac{1}{n} \ln \frac{\| C_n \cdots C_1 \|_\Sigma}{\| \Phi_n(M) \|_{\hat{\Sigma}_d}} < 1$ $I\!P$-a.s. Therefore

$\lim_{n\to\infty} \frac{1}{n} \ln \left[1 - \frac{\| C_{n+1} \cdots C_1 \|_\Sigma^2}{\| \Phi_n(M) \|_{\hat{\Sigma}_d}^2} \right] = 0$ $I\!P$-a.s. Thus

$\liminf_{n\to\infty} \frac{1}{n} \ln \| \Phi_n(M) \|_{\hat{\Sigma}_d} \leq \liminf_{n\to\infty} \frac{1}{n} \ln \| R_n \|_\Sigma^2$ $I\!P$-a.s. This and (16) complete the proof.

Let μ be a probability measure on GL(k), $k \in I\!N$ such that

$\int_{GL(k)} \ln^+ \| g \| + \ln^+ \| g^{-1} \| \, d\mu(g) < \infty$. By theorem 2.1, a constant $r(\mu)$, constants $-\infty < \lambda_{r(\mu)}(\mu) < \cdots < \lambda_0(\mu) = \lambda_\mu$ and subspaces $\{0_{k\times 1}\} = I\!L_{r(\mu)+1}(\mu) \subset I\!L_{r(\mu)}(\mu) \subset \cdots \subset I\!L_0(\mu) = I\!R^k$ are associated with μ. Set $\Psi_\mu := \{\lambda_j(\mu),\ j = 0, \ldots, r(\mu)\}$. If μ is a probability measure on GL(p), $p \in I\!N$ and ν a probability measure on GL(k), $k \in I\!N$ (k and p not necessarily different). $\Psi_\mu \geq \Psi_\nu$ shall be written if $\lambda_1 \geq \lambda_2$ whenever $\lambda_1 \in \Psi_\mu$ and $\lambda_2 \in \Psi_\nu$ and $\Psi_\mu < \Psi_\nu$ if $\lambda_0(\mu) < \lambda_{r(\nu)}(\nu)$.

Remark 3.4. Let M be a sequence in $\mathbf{G}(d, m)$. By lemma 3.1 the sequece C is a sequence of i.i.d. random matrices in GL(m). If $I\!E\left[\ln^+ \| M_0 \| + \ln^+ \| M_0^{-1} \|\right] < \infty$, the inequalities (7) and (8) imply $I\!E\left[\ln^+ \| C_1 \| + \ln^+ \| C_1^{-1} \|\right] < \infty$. By theorem 2.1, there exists a constant $r(\mu_C)$, constants $-\infty < \lambda_{r(\mu_C)}(\mu_C) < \cdots < \lambda_0(\mu_C) = \lambda_{\mu_C}$ and subspaces $\{0_{m\times 1}\} = I\!L_{r(\mu_C)+1}(\mu_C) \subset I\!L_{r(\mu_C)}(\mu_C) \subset \cdots \subset I\!L_0(\mu_C) = I\!R^m$ such that for $x \in I\!R^m$, $x \in I\!L_i(\mu_C) \backslash I\!L_{i+1}(\mu_C) \iff \lim_{n\to\infty} \frac{1}{n} \ln \| \Phi_n(C)x \| = \lambda_i(\mu_C)$ $I\!P$-a.s. In this case, if $\Sigma \subset I\!R^m$, write $C(i, j, \Sigma)$ for the statement: $\Sigma \subseteq I\!L_i(\mu_C)$ and $\Sigma \not\subseteq I\!L_{i+1}(\mu_C), \hat{\Sigma}_d \subseteq I$

Proposition 3.5. *Let* M *be a sequence in* $\mathbf{G}(d, m)$.
(i) *If* $I\!E\left[\ln^+ \| M_0 \| + \ln^+ \| M_0^{-1} \|\right] < \infty$, *then* Ψ_{μ_C} *is well defined.*
(ii) *If* (i) *holds and* Σ *is a bounded subset of* $I\!R^m$ *which satisfies* $C(i, j, \Sigma)$, *where* $\lambda_i(\mu_C) < \lambda_j(\mu_M)$ *then* $\lim_{n\to\infty} \frac{1}{n} \ln \| R_n \|_\Sigma = \lambda_j(\mu_M)$ $I\!P$*-a.s.*

Proof : (i) Since $I\!E\left[\ln^+ \| M_0 \| + \ln^+ \| M_0^{-1} \|\right] < \infty$, remark 3.4 implies that Ψ_{μ_C} is well defined.

(ii) Let $i, j \in I\!N_0$ and $\Sigma \subset I\!R^m$ be such that $\lambda_i(\mu_C) < \lambda_j(\mu_M)$ and $C(i, j, \Sigma)$ holds. By theorem 2.7, $\lim_{n\to\infty} \frac{1}{n} \ln \| C_{n+1} \cdots C_1 \|_\Sigma = \lambda_i(\mu_C) < \lambda_j(\mu_M) = \lim_{n\to\infty} \frac{1}{n} \ln \| \Phi_n(M) \|_{\hat{\Sigma}_d}$. $I\!P$-a.s. Hence, $\limsup_{n\to\infty} \frac{\| C_{n+1} \cdots C_1 \|_\Sigma}{\| \Phi_n(M) \|_{\hat{\Sigma}_d}} < 1$ $I\!P$-a.s. From proposition 3.3,

$\lim_{n\to\infty} \frac{1}{n} \ln \| R_n \|_\Sigma = \lim_{n\to\infty} \frac{1}{n} \ln \| \Phi_n(M) \|_{\hat{\Sigma}_d} = \lambda_j(\mu_M)$ $I\!P$-a.s.

For a bounded set $\Sigma \subset I\!R^m$, define $\lambda_R(\Sigma) := I\!P\text{-a.s.-}\lim_{n\to\infty} \frac{1}{n} \ln \| R_n \|_\Sigma$.

Theorem 3.6. *Let* M *be a sequence in* $\mathbf{G}(\mathrm{d},\mathrm{m})$. *Assume that*
(i)' $\mathbb{E}\left[\ln^+ \| M_0 \| + \ln^+ \| M_0^{-1} \|\right] < \infty$ (ii)' $\Psi_{\mu_C} < \Psi_{\mu_M}$.
Then for every bounded set $0_{m\times 1} \neq \Sigma \subset \mathbb{R}^m$,

(a) $\hat{\Sigma}_d \subseteq \mathbb{L}_j(\mu_M)$ and $\hat{\Sigma}_d \not\subset \mathbb{L}_{j+1}(\mu_M)$
$\Longleftrightarrow \lambda_R(\Sigma) = \lambda_j(\mu_M)$.

(b) $\lambda_R(.)$ *is constant on* $\overline{\Sigma}$.

Proof : (a) Assume that $\Sigma \neq 0_{m\times 1}$ is a bounded set in $\mathbb{R}^m$, Then $\hat{\Sigma}_d$ is bounded in $\mathbb{R}^{d+m}$. Hence there exist i and j for which $C(i,j,\Sigma)$ holds. Since $\Psi_{\mu_C} < \Psi_{\mu_M}$,
$\lambda_i(\mu_C) < \lambda_j(\mu_M)$. By proposition 3.5(ii),
$\lim_{n\to\infty} \frac{1}{n} \ln \| R_n \|_\Sigma = \lambda_j(\mu_M) = \lim_{n\to\infty} \frac{1}{n} \ln \| \Phi_n(M) \|_{\hat{\Sigma}_d}$ $\mathbb{P}$-a.s. By theorem 2.7, this is the case if and only if $\hat{\Sigma}_d \subseteq \mathbb{L}_j(\mu_M)$ and $\hat{\Sigma}_d \not\subset \mathbb{L}_{j+1}(\mu_M)$.

(b) is now proven. Remark that for a bounded set $\Sigma \subset \mathbb{R}^m$
$\widehat{\mathbb{L}(\Sigma)}_d = \mathbb{L}(\hat{\Sigma}_d)$. If $\Lambda \in \overline{\Sigma}$, then $\mathbb{L}(\Lambda) = \mathbb{L}(\Sigma) \Longleftrightarrow \widehat{\mathbb{L}(\Lambda)}_d = \widehat{\mathbb{L}(\Sigma)}_d \Longleftrightarrow \mathbb{L}(\widehat{\Lambda}_d) = \mathbb{L}(\hat{\Sigma}_d)$. From the fact that $\mathbb{L}(\widehat{\Lambda}_d) = \mathbb{L}(\hat{\Sigma}_d)$, $\hat{\Lambda}_d \in \overline{\hat{\Sigma}_d}$. By theorem 2.7, $\hat{\lambda}(\hat{\Sigma}_d) = \hat{\lambda}(\hat{\Lambda}_d)$. Thus $\lambda_R(\Sigma) = \lim_{n\to\infty} \frac{1}{n} \ln \| R_n \|_\Sigma$
$= \lim_{n\to\infty} \frac{1}{n} \ln \| \Phi_n(M) \|_{\hat{\Sigma}_d} = \hat{\lambda}(\hat{\Sigma}_d) = \hat{\lambda}(\hat{\Lambda}_d) = \lim_{n\to\infty} \frac{1}{n} \ln \| \Phi_n(M) \|_{\hat{\Lambda}_d} =$
$\lim_{n\to\infty} \frac{1}{n} \ln \| R_n \|_\Lambda = \lambda_R(\Lambda)$

Lemma 3.7. *If* $V \in M(m,k)$ *then* $\widehat{\Sigma(V)}_d = \Sigma(\hat{V}_d)$, *where* $\hat{V}_d := \begin{bmatrix} 0^*_{d\times k} & V^* \end{bmatrix}^*$.

Proof : Let $V \in M(m,k)$. Then by definition, $\Sigma(V) = \{Vx : x \in S_{k-1}\}$ and thus
$\widehat{\Sigma(V)}_d = \{\hat{x}_d : x \in \Sigma(V)\}$. Therefore $\widehat{\Sigma(V)}_d = \{\begin{bmatrix} 0^*_{d\times k} & V^* \end{bmatrix}^* x : x \in S_{k-1}\} = \Sigma(\hat{V}_d)$.
To simplify notation, $\mathbb{L}_j$ shall henceforth denote $\mathbb{L}_j(\mu_M)$ if nothing else is said.

Theorem 3.8. *Let* M *be a sequence in* $\mathbf{G}(\mathrm{d},\mathrm{m})$. *Assume that*
(i) $\mathbb{E}\left[\ln^+ \| M_0 \| + \ln^+ \| M_0^{-1} \|\right] < \infty$ (ii) $\Psi_{\mu_C} < \Psi_{\mu_M}$. *For any* $k \in \mathbb{N}$, *let* $\{0_{m\times k}\} = \mathbf{L}_{r+1} \subset \cdots \subset \mathbf{L}_0 = M(m,k)$ *and* $-\infty < \lambda_r < \cdots < \lambda_0 = \lambda_\mu$ *be the subspaces and constants given by theorem* 2.12. *For every matrix* $V \in M(m,k)$, *Define*
$\gamma_R(V) := \mathbb{P}$ *-a.s.-* $\lim_{n\to\infty} \frac{1}{n} \ln \| R_n V \|$. *Then*

(a) $\gamma_R(0_{m\times k}) = -\infty$ *and if* $V \in M(m,k)\backslash\{0_{m\times k}\}$, *then* $\hat{V}_d \in \mathbf{L}_j\backslash\mathbf{L}_{j+1} \Longleftrightarrow \gamma_R(V) = \lambda_j$.

(b) $\gamma_R(.)$ *is constant on* $\overline{V}$.

(c) *The map* $\gamma_R(.) : M(m,k) \longrightarrow \mathbb{R} \cup \{-\infty\}$, $V \mapsto \gamma_R(V)$ *defines a characteristic exponent.*

Proof : (a)That $\gamma_R(0_{m\times k}) = -\infty$ is clear. Let $0_{m\times k} \neq V \in M(m,k)$. Theorem 3.6 implies $\lambda_R(\Sigma(V)) = \lambda_j$ if and only if $\widehat{\Sigma(V)}_d \subseteq \mathbb{L}_j$ and $\widehat{\Sigma(V)}_d \not\subset \mathbb{L}_{j+1}$. By lemma 3.7 this is equivalent to $\Sigma(\hat{V}_d) \subseteq \mathbb{L}_j$ and $\Sigma(\hat{V}_d) \not\subset \mathbb{L}_{j+1}$, which in turn is equivalent to $\hat{V}_d \in \mathbf{L}_j\backslash\mathbf{L}_{j+1}$. Since $\| R_n V \| = \| R_n \|_{\Sigma(V)}$, $\gamma_R(V) = \lambda_R(\Sigma(V)) = \lambda_j \Longleftrightarrow \hat{V}_d \in \mathbf{L}_j\backslash\mathbf{L}_{j+1}$.

(b) Let $W \in \overline{V}$. Then $\mathbb{L}(\Sigma(W)) = \mathbb{L}(\Sigma(V))$. (For typographical reasons, $\mathbb{L}_\Sigma^W$ is written for $\mathbb{L}(\Sigma(W))$ and $\mathbb{L}_\Sigma^V$ for $\mathbb{L}(\Sigma(V))$ in the next equivalence).
Thus $\mathbb{L}(\Sigma(V)) = \mathbb{L}(\Sigma(W)) \Longleftrightarrow (\widehat{\mathbb{L}_\Sigma^W})_d = (\widehat{\mathbb{L}_\Sigma^V})_d \Longleftrightarrow \mathbb{L}\left(\widehat{\Sigma(W)}_d\right) = \mathbb{L}\left(\widehat{\Sigma(V)}_d\right)$
$\Longleftrightarrow \mathbb{L}(\Sigma(\hat{W}_d)) = \mathbb{L}(\Sigma(\hat{V}))$. For some j, $\gamma_R(V) = \lambda_j \Longleftrightarrow \hat{V}_d \in \mathbf{L}_j\backslash\mathbf{L}_{j+1}$
$\Longleftrightarrow \Sigma(\hat{V}_d) \subseteq \mathbb{L}_j,\ \Sigma(\hat{V}_d) \not\subset \mathbb{L}_{j+1} \Longleftrightarrow \mathbb{L}(\Sigma(\hat{V}_d)) \subseteq \mathbb{L}_j,\ \mathbb{L}(\Sigma(\hat{V}_d)) \not\subset \mathbb{L}_{j+1}$
$\Longleftrightarrow \mathbb{L}(\Sigma(\hat{W}_d)) \subseteq \mathbb{L}_j,\ \mathbb{L}(\Sigma(\hat{W}_d)) \not\subset \mathbb{L}_{j+1} \Longleftrightarrow \Sigma(\hat{W}_d) \subseteq \mathbb{L}_j,\ \Sigma(\hat{W}_d) \not\subset \mathbb{L}_{j+1}$

$\Longleftrightarrow \hat{W}_d \in \mathbf{L}_j \backslash \mathbf{L}_{j+1} \Longleftrightarrow \gamma_R(W) = \lambda_j$. Therefore $\gamma_R(W) = \gamma_R(V)$.

(c) $\gamma_R(V) = \lambda(\hat{V})$ where $\lambda(.)$ is defined in corollary 2.11. Since $\lambda(.)$ is a characteristic exponent, $\gamma_R(.)$ is a characteristic exponent.

Define $\gamma^R := I\!P\text{-a.s-}\lim_{n\to\infty} \frac{1}{n} \ln \| R_n \|$. The considerations so far sum up to the following theorem which is the first main theorem of this paper.

Theorem 3.9. *Let* M *be a sequence in* $\mathbf{G}(d,m)$ *with* (i) $I\!E\left[\ln^+ \| M_0 \| + \ln^+ \| M_0^{-1} \|\right] < \infty$, (ii) $\Psi_{\mu_C} < \Psi_{\mu_M}$. *Further let* $k \in I\!N$, *thenthere exists* $p \in I\!N$,

(a) *a sequence* $\{0_{m\times k}\} = \mathbf{S}_{p+1} \subset \mathbf{S}_p \subset \cdots \subset \mathbf{S}_0 = M(m,k)$ *and*

constants $-\infty < \gamma_p < \cdots < \gamma_0$ *such that, if* $V \in M(m,k)\backslash\{0_{m\times k}\}$, *then*

$V \in \mathbf{S}_i \backslash \mathbf{S}_{i+1} \Longleftrightarrow \lim_{n\to\infty} \frac{1}{n} \ln \| R_n V \| = \gamma_i$ *$I\!P$-a.s.*

(b) $\gamma_0 = \gamma^R$.

(c) *If* $m = k$, *then* $GL(m) \subseteq \mathbf{S}_0 \backslash \mathbf{S}_1$.

Proof : By theorem 3.8(c), $\gamma_R(.)$ defines a characteristic exponent. By lemma 2.9 there exist subspaces and constants satisfying the assertion (a). The subspaces are now constructed. Let $V \in M(m,k)\backslash\{0_{m\times k}\}$. Then by theorem 3.8,
$\hat{V}_d \in \mathbf{L}_i \backslash \mathbf{L}_{i+1} \Longleftrightarrow \lim_{n\to\infty} \frac{1}{n} \ln \| R_n V \| = \lambda_i$ $I\!P$-a.s. Let

$$\begin{aligned} \overline{E}_i &:= \mathbf{L}_i \backslash \mathbf{L}_{i+1},\ i \in \{r,\ldots,0\}, \\ W &:= \left\{ \hat{V}_d \in M(d+m,k) : V \in M(m,k) \right\}, \\ W_{r+1} &:= \emptyset,\ W_i := W \cap \overline{E}_i,\ i \in \{r,\ldots,0\}. \end{aligned}$$

Then $W = \cup_{k=r+1}^{0} W_k$, -a disjoint union and there exist indices $\{r \geq i_p > \cdots > i_0 \geq 0\} \subseteq \{r,\ldots,0\}$, such that $W_{i_j} \neq \emptyset$ for all $j \in \{p,\ldots,0\}$ (numbering is always started from i_0). Set

$$\begin{aligned} W_{i_p+1} &:= \{0_{(d+m)\times k}\},\ \widetilde{W}_{p+1} := \{0_{(d+m)\times k}\} \\ \widetilde{W}_j &:= \{W_{i_p+1}\} \cup \{\cup_{k=p}^{j} W_{i_k}\},\ j = p,\ldots,0. \end{aligned}$$

Further set

$$\begin{aligned} \mathbf{S}_j &:= \left\{ V \in M(m,k) : \hat{V}_d \in \widetilde{W}_j \right\},\ j = p,\ldots,0, \\ \mathbf{S}_{p+1} &:= \{0_{m\times k}\},\ \gamma_j := \lambda_{i_j},\ j = p,\ldots,0. \end{aligned}$$

Now $\{0_{m\times k}\} = \mathbf{S}_{p+1} \subset \mathbf{S}_p \subset \cdots \subset \mathbf{S}_0 = M(m,k)$ and
$-\infty < \gamma_p < \cdots < \gamma_0$. It is now shown that if $V \in M(m,k)\backslash\{0_{m\times k}\}$ Then
$V \in \mathbf{S}_j \backslash \mathbf{S}_{j+1} \Longleftrightarrow \gamma_R(V) = \gamma_j$. For $V \in M(m,k)$, $V \in \mathbf{S}_j \backslash \mathbf{S}_{j+1} \Longleftrightarrow \hat{V}_d \in W_{i_j}$
$\Longleftrightarrow \hat{V}_d \in \overline{E_{i_j}} \Longleftrightarrow \hat{V}_d \in \mathbf{L}_{i_j} \backslash \mathbf{L}_{i_j+1} \Longleftrightarrow \gamma_R(V) = \lambda_{i_j} \Longleftrightarrow \gamma_R(V) = \gamma_j$. It therefore only has to be shown that the sets $\mathbf{S}_j$ are indeed linear subspaces of $M(m,k)$. This however follows from the fact that $\gamma_R(.)$ defines a characteristic exponent and $\mathbf{S}_j = \{V \in M(k,p,) : \gamma_R(V) \leq \gamma_j\}$.
(b) $\gamma^R = I\!P\text{-a.s-}\lim_{n\to\infty} \frac{1}{n} \ln \| R_n \| = I\!P\text{-a.s-}\lim_{n\to\infty} \frac{1}{n} \ln \| R_n I_k \| \leq \gamma_0$
i.e. $\gamma^R \leq \gamma_0$. Also, $\| R_n V \| \leq \| R_n \| \| V \|$ implies that if $V \in \mathbf{S}_0 \backslash \mathbf{S}_1$ then
$\gamma_0 \leq \lim_{n\to\infty} \frac{1}{n} \ln \| R_n \| = \gamma^R$. Therefore $\gamma_0 = \gamma^R$.
(c) Let $m = k$. Then $\bar{I}_m = GL(m)$. Therefore since $\gamma_R(I_m) = \gamma_0$ and $\gamma_R(.)$ is constant on $\bar{I}_k$, $GL(m) \subseteq \mathbf{S}_0 \backslash \mathbf{S}_1$.

4 The growth rate of related autoregressive processes

The growth rates of the autoregressive processes (P_n) and (Q_n) defined in the introduction are now studied. It turns out that under the conditions
$I\!E\left[\ln^+ \| M_0 \| + \ln^+ \| M_0^{-1} \|\right] < \infty$ and $0 < \Psi_{\mu_M}$ the processes (P_n) and
$R_n(A, B, \tilde{I}_m)$ exhibit the same growth rates. The results of this section are corollaries to theorem 3.9. For the proof of corollary 4.3 two lemmata are needed:

Lemma 4.1. *Let $(\Omega, \mathcal{F}, I\!P)$ be a probability space and $B = (B_i)_{i \in I\!N_0}$ be a sequence of i.i.d. random matrices in $M(d,m)$ defined on Ω, for which $I\!E \ln^+ \| B_0 \| < \infty$ and Σ be a bounded subset of $I\!R^m$. Further, let $(\phi_i)_{i \in I\!N_0}$ be a sequence of random elements in $I\!R$ defined on Ω, for which $\limsup_{n\to\infty} \frac{1}{n} \ln |\phi_n| < 0$ $I\!P$-a.s. Then $\lim_{n\to\infty} |\phi_n| \| B_n \|_\Sigma = \lim_{n\to\infty} |\phi_n| \| B_n \| = 0$ $I\!P$-a.s.*

Proof : By assumption $\limsup_{n\to\infty} \frac{1}{n} \ln |\phi_n| < 0$ $I\!P$-a.s. Let Ω_1 be the set of measure 1 on which $\limsup_{n\to\infty} \frac{1}{n} \ln |\phi_n| < 0$. For each $\omega \in \Omega_1$ chosen arbitrarily, there exists $\varepsilon_1(\omega) > 0$ and $n_1(\varepsilon_1(\omega))$ such that for all $n \geq n_1(\varepsilon_1(\omega))$, $\frac{1}{n} \ln |\phi_n(\omega)| \leq \lambda(\omega) + \varepsilon_1(\omega) < 0$, where
$\lambda(\omega) := \limsup_{n\to\infty} \frac{1}{n} \ln |\phi_n(\omega)|$. Therefore for all $n \geq n_1(\varepsilon_1(\omega))$, $|\phi_n(\omega)| \leq e^{n(\lambda(\omega)+\varepsilon_1(\omega))}$. Now B is a sequence of i.i.d. random matrices. Therefore $(\ln^+ \| B_i \|)$ is a sequence of i.i.d. random variables. Since $I\!E \ln^+ \| B_0 \| < \infty$, Borel cantellis lemma implies $\limsup_{n\to\infty} \frac{1}{n} \ln^+ \| B_n \| \leq 0$ $I\!P$-a.s. Let Ω_2 be the set of measure 1 on which $\limsup_{n\to\infty} \frac{1}{n} \ln^+ \| B_n \| \leq 0$ and $\tilde{\Omega} := \Omega_1 \cap \Omega_2$. If $\omega \in \tilde{\Omega}$, $\gamma(\omega) := \limsup_{n\to\infty} \frac{1}{n} \ln^+ \| B_n(\omega) \|$ and $\varepsilon_2(\omega) > 0$ is chosen arbitrarily, then there exists $n_2(\varepsilon_2(\omega))$ such that for all $n \geq n_2(\varepsilon_2(\omega))$, $\frac{1}{n} \ln \| B_n(\omega) \| \leq \gamma(\omega) + \varepsilon_2(\omega)$. For each $\omega \in \tilde{\Omega}$ choose $\varepsilon_2(\omega)$ such that $\lambda(\omega)+\varepsilon_1(\omega)+\gamma(\omega)+\varepsilon_2(\omega) < 0$. Then for all $n \geq \max\{n_1(\varepsilon_1(\omega)), n_2(\varepsilon_2(\omega))\}$, $0 \leq |\phi_n(\omega)| \| B_n(\omega) \| \leq e^{n(\lambda(\omega)+\varepsilon_1(\omega)+\gamma(\omega)+\varepsilon_2(\omega))}$. Since $\lambda(\omega) + \varepsilon_1(\omega) + \gamma(\omega) + \varepsilon_2(\omega) < 0$, $\lim_{n\to\infty} e^{n(\lambda(\omega)+\varepsilon_1(\omega)+\gamma(\omega)+\varepsilon_2(\omega))} = 0$. Therefore $\lim_{n\to\infty} |\phi_n(\omega)| \| B_n(\omega) \| = 0$. These arguments may be carried out for every $\omega \in \tilde{\Omega}$. Since $I\!P(\tilde{\Omega}) = 1$, $I\!P\left(\lim_{n\to\infty} |\phi_n| \| B_n \| = 0\right) = 1$. Since $0 \leq |\phi_n| \| B_n \|_\Sigma \leq |\phi_n| \| B_n \|$ it holds that $I\!P\left(\lim_{n\to\infty} |\phi_n| \| B_n \|_\Sigma = 0\right) = 1$.
For any x, define $\tilde{x}$ to be the the constant sequence with $x_n = x$ for all n.

Lemma 4.2. *Let $(\Omega, F, I\!P)$ be a probability space and $(A_i, B_{i+1})_{i \in I\!N_0}$ be an i.i.d. sequence in $GL(d) \times M(d,m)$ defined on Ω satisfying $I\!E \ln^+ \| B_1 \| < \infty$. Assume that $\lambda > 0$ and $\Sigma \subset I\!R^m$ is bounded. Then in case of existence of either almost sure limit, $I\!P$-a.s.*
$\lim_{n\to\infty} \frac{1}{n} \ln \| R_n(A, B, \tilde{I}_m) \|_\Sigma = \lambda \iff \lim_{n\to\infty} \frac{1}{n} \ln \| P_n \|_\Sigma = \lambda$.

Proof : Assume that $I\!P$-a.s. $\lim_{n\to\infty} \frac{1}{n} \ln \| R_n(A, B, \tilde{I}_m) \|_\Sigma = \lambda$. We have: $P_n = A_n R_{n-1}(, A, B, \tilde{I}_m)$ and $R_n(A, B, \tilde{I}_m) = A_n R_{n-1}(A, B, \tilde{I}_m) + B_{n+1}$. Therefore
$P_n = R_n(A, B, \tilde{I}_m) - B_{n+1}$ and

$$\| P_n \|_\Sigma = \| R_n(A, B, \tilde{I}_m) \|_\Sigma \left\| \frac{R_n(A, B, \tilde{I}_m)}{\| R_n(A, B, \tilde{I}_m) \|_\Sigma} - \frac{B_{n+1}}{\| R_n(A, B, \tilde{I}_m) \|_\Sigma} \right\|_\Sigma . \tag{17}$$

Let $N(R_n, \Sigma) := \frac{1}{n} \ln \left\| \frac{R_n(A, B, \tilde{I}_m)}{\| R_n(A, B, \tilde{I}_m) \|_\Sigma} - \frac{B_{n+1}}{\| R_n(A, B, \tilde{I}_m) \|_\Sigma} \right\|_\Sigma .$

Then by (17), $\frac{1}{n}\ln \| P_n \|_\Sigma = \frac{1}{n}\ln \| R_n(A,B,\tilde{I}_m) \|_\Sigma + N(R_n,\Sigma)$. Further,

$$\frac{1}{n}\ln\left|1-\frac{\| B_{n+1} \|_\Sigma}{\| R_n(A,B,\tilde{I}_m) \|_\Sigma}\right| \leq N(R_n,\Sigma) \leq \frac{1}{n}\ln\left[1+\frac{\| B_{n+1} \|_\Sigma}{\| R_n(A,B,\tilde{I}_m) \|_\Sigma}\right] \tag{18}$$

Since $\lambda > 0$ and $I\!E \ln^+ \| B \| < \infty$, it follows from lemma 4.1 with $\phi_n := \| R_n(A,B,\tilde{I}_m) \|_\Sigma^{-1}$, that $I\!P$-a.s. $\lim\limits_{n\to\infty} \frac{\| B_{n+1} \|_\Sigma}{\| R_n(A,B,\tilde{I}_m) \|_\Sigma} = 0$.

Therefore $\lim\limits_{n\to\infty} N(R_n,\Sigma) = 0$, $I\!P$-a.s, by (18). Consequently $\lim\limits_{n\to\infty} \frac{1}{n}\ln \| P_n \|_\Sigma = \lim\limits_{n\to\infty} \frac{1}{n}\ln \| R_n(A,B,\tilde{I}_m) \|_\Sigma = \lambda$ $I\!P$-a.s. The proof of the other direction of the equivalence is similar, starting with $\| R_n(A,B,\tilde{I}_m) \|_\Sigma = \| P_n \|_\Sigma \left\| \frac{P_n}{\| P_n \|_\Sigma} + \frac{B_{n+1}}{\| P_n \|_\Sigma} \right\|_\Sigma$.

Let (A_i, B_{i+1}) be an i.i.d. sequence in $GL(d) \times M(d,m)$. With this sequence, a sequenece M^P in $\mathbf{G}(d,m)$, $M^P{}_i := \begin{bmatrix} A_i & B_{i+1} \\ 0_{m\times d} & I_m \end{bmatrix}$ is associated.

Corollary 4.3. *Let* M^P *be a sequence in* $\mathbf{G}(d,m)$, (i)″ $I\!E\left[\ln^+ \| M^P{}_0 \| + \ln^+ \| M^{P}{}_0^{-1} \|\right] < \infty$, (ii)″ $\{0\} < \Psi_{\mu_{M^P}}$.

For $k \in I\!N$ *and* $V \in M(m,k)$ *define* $\gamma_P(V) := I\!P$*-a.s-*$= \lim\limits_{n\to\infty} \frac{1}{n}\ln \| P_n V \|$ *and* $\gamma^P := I\!P$*-a.s-* $\lim\limits_{n\to\infty} \frac{1}{n}\ln \| P_n \|$. *Then*

(a) *There exists a sequence* $\{0_{m\times k}\} = \mathbf{S}_{p+1} \subset \mathbf{S}_p \subset \cdots \subset \mathbf{S}_0 = M(m,k)$ *and constants* $-\infty < \gamma_p < \cdots < \gamma_0$ *such that, if* $V \in M(m,k)\backslash\{0_{m\times k}\}$, *then* $V \in \mathbf{S}_i\backslash\mathbf{S}_{i+1} \Longleftrightarrow \lim\limits_{n\to\infty} \frac{1}{n}\ln \| P_n V \| = \gamma_i$ $I\!P$*-a.s.*

(b) $\gamma_0 = \gamma^P$.

(c) *If* $m = k$, *then* $\gamma_P(V) = \gamma^P$ *for all* $V \in GL(m)$.

Proof : If a generic matrix $\begin{bmatrix} A_i & B_{i+1} \\ 0_{m\times d} & I_m \end{bmatrix}$ of the sequence M^P is compared with a generic matrix $\begin{bmatrix} A_i & B_{i+1} \\ 0_{m\times d} & C_{i+1} \end{bmatrix}$ of the sequence M of theorem 3.9, $C := \tilde{I}_m$ may be set in that theorem and the process (R_n) of that theorem becomes $(R_n(A,B,\tilde{I}_m))$. Notice also that the condition (ii)″ in the present theorem is the condition (ii) of that theorem. Define

$\Phi_R(.) := I\!P$-a.s.- $\lim\limits_{n\to\infty} \frac{1}{n}\ln \| R_n(A,B,\tilde{I}_m)(.) \|$ on $M(m,k)$. Then: (i) there exists a sequence $\{0_{m\times k}\} = \mathbf{S}_{p+1} \subset \mathbf{S}_p \subset \cdots \subset \mathbf{S}_0 = M(m,k)$ and constants $-\infty < \gamma_p < \cdots < \gamma_0$ such that, if $V \in M(m,k)\backslash\{0_{m\times k}\}$, then $V \in \mathbf{S}_i\backslash\mathbf{S}_{i+1} \Longleftrightarrow \Phi_R(V) = \gamma_i$ $I\!P$-a.s.

(ii) $\gamma_0 = \Phi_R(I_k)$ (iii) If $m = k$ then $GL(m) \subseteq \mathbf{S}_0\backslash\mathbf{S}_1$. By assumption, $\gamma_j > 0$ for all $j \in \{1,\ldots,p\}$. By lemma 3.2, $I\!E \ln^+ \| B_0 \| < \infty$. By lemma 4.2 $\gamma_P(V) = \gamma_i \Longleftrightarrow \Phi_R(V) = \gamma_i \Longleftrightarrow V \in \mathbf{S}_i\backslash\mathbf{S}_{i+1}$. Therefore $\gamma_P(V) = \gamma_i \Longleftrightarrow V \in \mathbf{S}_i\backslash\mathbf{S}_{i+1}$.

(b) From (ii) above $\Phi_R(I_k) = \gamma_0 \Longleftrightarrow I_k \in \mathbf{S}_0\backslash\mathbf{S}_1$. Therefore $\gamma_0 = \lim\limits_{n\to\infty} \frac{1}{n}\ln \| P_n I_k \|$ $= \lim\limits_{n\to\infty} \frac{1}{n}\ln \| P_n \| = \gamma^P$ almost suely. Therefore $\gamma_0 = \gamma^P$.

(c) The assertion follows from (b) and the fact thhat $\bar{I}_k = GL(k)$.

Let (B_{i+1}, C_{i+1}) be an i.i.d. sequence in $M(d,m) \times GL(m)$. Associate with this sequence a sequenece

M^Q in $\mathbf{G}(d,m)$ with $M^Q{}_i := \begin{bmatrix} I_d & B_{i+1} \\ 0_{m\times d} & C_{i+1} \end{bmatrix}$.

Proposition 4.4. *Let* M^Q *be a sequence in* $\mathbf{G}(d,m)$, $\mathbb{E}\left[\ln^+ \| M^Q{}_0 \| + \ln^+ \| M^{Q\,-1}_0 \|\right] < \infty$.
(i) *Then* Ψ_{μ_C} *is well defined.* (ii) $\{\lambda_j(\mu_{M^Q}) \in \Psi_{\mu_{M^Q}} : \lambda_j(\mu_{M^Q}) \geq 0\} \neq \emptyset$.
(iii) *If* $\{\lambda_i(\mu_C) \in \Psi_{\mu_C} : \lambda_i(\mu_C) < 0\} \neq \emptyset$ *and* $\Sigma \subset \mathbb{R}^m$ *is such that* $C(i,j,\Sigma)$ *holds for some* i *and* j *with* $\lambda_i(\mu_C) < 0$ *and* $\lambda_j(\mu_M) \geq 0$, *then* $\lim_{n\to\infty} \frac{1}{n} \ln \| Q_n \|_\Sigma = 0$ $\mathbb{P}$*-a.s.*

Proof : The proof of (i) is that of the same statement in proposition 3.5.
(ii) By [4], the condition $\mathbb{E}\left[\ln^+ \| M^Q{}_0 \| + \ln^+ \| M^{Q\,-1}_0 \|\right] < \infty$ implies that either for all $x \in \mathbb{R}^{d+m}\backslash\{0_{(d+m)\times 1}\}$ and $\mathbb{P}$-a.s. $\lim_{n\to\infty} \frac{1}{n} \ln \| \Phi_n(M^Q)x \| = \lambda_0(\mu_{M^Q})$ or for some nontrivial μ_{M^Q}-invariant subspace $\mathbb{L}$ of $\mathbb{R}^{d+m}$, for all $x \in \mathbb{L}$, $x \neq 0_{(d+m)\times 1}$ and $\mathbb{P}$-a.s,
$\limsup_{n\to\infty} \frac{1}{n} \ln \| \Phi_n(M^Q)x \| \leq \alpha < \lambda_0(\mu_{M^Q})$, where $\alpha \in \mathbb{R}$. Now $\mathsf{Span}\{e_1,\ldots,e_d\}$ is μ_{M^Q}-invariant, where $\{e_1,\ldots,e_{d+m}\}$ denotes the standard basis in $\mathbb{R}^{d+m}$ and for every
$x \in \mathsf{Span}\{e_1,\ldots,e_d\}\backslash 0_{\{d+m\times 1\}}$, $\lim_{n\to\infty} \frac{1}{n} \ln \| \Phi_n(M^Q)x \| = 0$ $\mathbb{P}$-a.s. This shows that
$\lambda_0(\mu_{M^Q}) \geq 0$ and thus $\{\lambda_j(\mu_{M^Q}) \in \Psi_{\mu_{M^Q}} : \lambda_j(\mu_{M^Q}) \geq 0\} \neq \emptyset$.

(iii) is now proven. In (ii) it has been seen that $\{\lambda_j(\mu_{M^Q}) \in \Psi_{\mu_{M^Q}} : \lambda_j(\mu_{M^Q}) \geq 0\} \neq \emptyset$. By the assumption in (iii), $\{\lambda_i(\mu_C) \in \Psi_{\mu_C} : \lambda_i(\mu_C) < 0\} \neq \emptyset$. Choose j such that $\lambda_j(\mu_{M^Q}) \geq 0$ and i such that $\lambda_i(\mu_C) < 0$ and let $\Sigma \subset \mathbb{R}^m$ be such that $C(i,j,\Sigma)$ holds. Then by proposition 3.5, $\lim_{n\to\infty} \frac{1}{n} \ln \| R_n(\tilde{I}_d,B,C) \|_\Sigma = \lambda_j(\mu_{M^Q})$ $\mathbb{P}$-a.s. In particular $\lambda_j(\mu_{M^Q}) \geq 0$. It will now be shown that $\lambda_j(\mu_{M^Q}) \leq 0$. To do this, it is shown that $\limsup_{n\to\infty} \frac{1}{n} \ln \| R_n(\tilde{I}_d,B,C) \|_\Sigma \leq 0$. Since $C(i,j,\Sigma)$ holds, theorem 2.7 implies that $\lim_{n\to\infty} \frac{1}{n} \ln \| C_n \cdots C_1 \|_\Sigma = \lambda_i(\mu_C) < 0$ $\mathbb{P}$-a.s. For every ω in a set of measure 1, if $\varepsilon(\omega)$ is chosen small enough, there exists $k_0(\varepsilon(\omega))$ such that for $k \geq k_0(\varepsilon(\omega))$,

$$\| C_k \cdots C_1(\omega) \|_\Sigma \leq e^{-k\varepsilon(\omega)}. \tag{19}$$

Since $\mathbb{E}\left[\ln^+ \| M^Q{}_0 \| + \ln^+ \| M^{Q\,-1}_0 \|\right] < \infty$, lemma 3.2 implies $\mathbb{E}\ln^+ \| B_0 \| < \infty$. Since B is a sequence of i.i.d. random matrices, Borel Cantellis lemma implies
$\limsup_{n\to\infty} \frac{1}{n} \ln^+ \| B_n \| < \epsilon$ $\mathbb{P}$-a.s, $\epsilon > 0$ arbitrary. Choose $0 < \epsilon < \varepsilon(\omega)$. Then there exists $k_1(\epsilon)$ such that for $k \geq k_1(\epsilon)$, $\max\{1, \| B_{k+1}(\omega) \|\} \leq e^{(k+1)\epsilon}$. Taking (19) into consideration, $\| B_{k+1}C_k \cdots C_1(\omega) \|_\Sigma \leq \| B_{k+1} \| \| C_k \cdots C_1(\omega) \|_\Sigma \leq e^{(k+1)\epsilon} e^{-k\varepsilon(\omega)}$
$= e^{k(\epsilon-\varepsilon(\omega))}e^\epsilon$, $k \geq \max\{k_0(\varepsilon(\omega)), k_1(\epsilon)\}$.
Thus $\left\| \sum_{k=\max\{k_0(\varepsilon(\omega)),k_1(\epsilon)\}}^{\infty} B_{k+1}C_k \cdots C_1(\omega) \right\|_\Sigma < \infty$. This argument holds for aall ω in a set of measure 1. Now $\| R_n(\tilde{I}_d,B,C) \|_\Sigma = \| \sum_{k=0}^{n} B_{k+1}C_k \cdots C_1 \|_\Sigma$. Therefore the sequence $\left(\| R_n(\tilde{I}_d,B,C) \|_\Sigma\right)_{n\in\mathbb{N}}$ is $\mathbb{P}$-a.s bounded, implying
$\limsup_{n\to\infty} \frac{1}{n} \ln \| R_n(\tilde{I}_d,B,C) \|_\Sigma \leq 0$ $\mathbb{P}$-a.s. Thus $\lim_{n\to\infty} \frac{1}{n} \ln \| R_n(\tilde{I}_d,B,C) \|_\Sigma = 0$ $\mathbb{P}$-a.s. Let us notice that $Q_n = R_n(\tilde{I}_d,B,C)$. The assertion now follows.

From proposition 4.4 the following corollary is obtained:

Corollary 4.5. *Let* M^Q *be a sequence in* $\mathbf{G}(d,m)$, (i)''' $\mathbb{E}\left[\ln^+ \| M^Q{}_0 \| + \ln^+ \| M^{Q\,-1}_0 \|\right] < \infty$, (ii)''' $\Psi_{\mu_C} < \{0\}$. *Then for every bounded set* $\Sigma \subset \mathbb{R}^m$ *with* $\Sigma \neq \{0_{m\times 1}\}$,
$\lim_{n\to\infty} \frac{1}{n} \ln \| Q_n \|_\Sigma = 0$ $\mathbb{P}$*-a.s and for every* $V \in M(m,k)\backslash\{0_{m\times k}\}$,
$\lim_{n\to\infty} \frac{1}{n} \ln \| Q_n V \| = 0$ $\mathbb{P}$*-a.s.*

Proof : Let $0_{m\times 1} \neq \Sigma \subset I\!R^m$. Since $\Psi_{\mu_C} < \{0\}$, proposition 4.4 implies that if (i,j) is a pair of indices for which $C(i,j,\Sigma)$ is valid, then $\lim_{n\to\infty} \frac{1}{n} \ln \| Q_n \|_\Sigma = 0$ $I\!P$-a.s. Further if $V \in M(m,k)\backslash\{0_{m\times k}\}$ then $\lim_{n\to\infty} \frac{1}{n} \ln \| Q_n V \| = \lim_{n\to\infty} \frac{1}{n} \ln \| Q_n \|_{\Sigma(V)} = 0$ $I\!P$-a.s.

5 The upper Lyapunov exponent

In theorem 3.9 it was shown that $\gamma_R(.)$ takes values $-\infty < \gamma_p < \cdots < \gamma_0$. In particular, $\gamma_0 = \gamma^R$. It is easy to show that $\gamma_0 = \gamma^R \leq \lambda_{\mu_M}$. In [11] it is shown that there is a possibility that $\gamma_0 < \lambda_{\mu_M}$ occurs. In this section, we give conditions which ensure that $\gamma_0 = \lambda_{\mu_M}$. The proof of the theorem relies essentially on lemmas 5.3 and 5.4. Let $d \in I\!N$ and let M be a random matrix in $M(d,d)$. Define $\mu_M(\cdot) := I\!P(M \in \cdot)$.

Definition 5.1. *Let μ be a probability measure on $GL(d)$. A subspace $I\!L \subseteq I\!R^d$ is μ-invariant if $\mu(\{g \in GL(d) : gI\!L \subseteq I\!L\}) = 1$.*

Definition 5.2. *For $M \in GL(d)$, a subspace $I\!L \subseteq I\!R^d$ shall be called μ_M-invariant if $I\!P(\{\omega : M(\omega)I\!L \subseteq I\!L\}) = 1$.*

Let $M = \begin{bmatrix} A & B \\ 0_{m\times d} & C \end{bmatrix}$ be a random matrix in $\mathbf{G}(d,m)$. Associate with M the following random matrices: $M^A := \begin{bmatrix} A & 0_{d\times m} \\ 0_{m\times d} & 0_{m\times m} \end{bmatrix}$, $M^C := \begin{bmatrix} 0_{d\times d} & 0_{d\times m} \\ 0_{m\times d} & C \end{bmatrix}$ and
$M^B := \begin{bmatrix} 0_{d\times d} & B \\ 0_{m\times d} & 0_{m\times m} \end{bmatrix}$.

Lemma 5.3. *Let μ be a probability measure on $GL(n)$, $n \geq 2$. Then either* (a) *all μ-invariant subspaces of $I\!R^n$ are $\{0_{n\times 1}\}$ and $I\!R^n$ or* (b) *there exists a basis $\{e_1,\ldots,e_n\}$ of $I\!R^n$ with respect to which $\mu(\mathbf{G}(d,m)) = 1$, for some $d \in I\!N$ and $m \in I\!N$ with $d+m=n$.*
Assume that (b) *holds and let $I\!L = I\!L_0 \oplus I\!L_1$, with $I\!L \neq \{0_{(d+m)\times 1}\}$ be a subspace of $I\!R^n$, where $I\!L_0 \subseteq \mathsf{Span}\{e_1,\ldots,e_d\}$ and $I\!L_1 \subseteq \mathsf{Span}\{e_{d+1},\ldots,e_{d+m}\}$ are linear subspaces of $I\!R^{d+m}$. Let $M := \begin{bmatrix} A & B \\ 0_{m\times d} & C \end{bmatrix}$ be the representation in the basis $\{e_1,\ldots,e_{d+m}\}$, of a random matrix with distribution μ. Then*

(c) *$I\!L$ is μ_M-invariant if and only if*

(d) *(1) $I\!L_1 = \{0_{n\times 1}\}$, $I\!L$ is μ_{M^A}-invariant or*

(2) $I\!L_0 = \{0_{n\times 1}\}$, $I\!L$ is μ_{M^C}-invariant, and $I\!P(I\!L_1 \subseteq \mathsf{Ker}M^B(\omega)) = 1$ or

(3) $I\!L_0 \neq \{0_{n\times 1}\}$, $I\!L_1 \neq \{0_{n\times 1}\}$ and

(i) *$I\!L_0$ is μ_M-invariant and μ_{M^A}-invariant,*

(ii) *$I\!L_1$ is μ_{M^C}-invariant,*

(iii) *$I\!P(\mathsf{Im}M^B(\omega)|_{I\!L_1} \subseteq I\!L_0) = 1$.*

Proof : The conditions (a) and (b) are mutually exclusive and one of them always holds. Assume that (a) is false. Let $\mathbb{L}$ be a μ_M-invariant subspace of $\mathbb{R}^{d+m}$. Also note that d (1) (2) and (3) are mutually exclusive. Assume that (d)(1) and (d)(2) are false and $\mathbb{L} = \mathbb{L}_0 \oplus \mathbb{L}_1$, where $\mathbb{L}_0 \neq \{0_{n\times 1}\}$, $\mathbb{L}_1 \neq \{0_{n\times 1}\}$, $\mathbb{L}_0 \subseteq \mathsf{Span}\{e_1, \ldots, e_d\}$ and $\mathbb{L}_1 \subseteq \mathsf{Span}\{e_{d+1} \cdots e_{d+m}\}$. By the μ_M-invariance of $\mathbb{L}$, $\mathbb{P}(M(\omega)\mathbb{L} \subseteq \mathbb{L}) = 1$. Consequently it is assumed henceforth, that for all ω, $M(\omega)\mathbb{L} \subseteq \mathbb{L}$. Now $\mathbb{P}(\{\omega : M(\omega)$ acts on $\mathsf{Span}\{e_1, \ldots, e_d\}\}) = 1$ and $\mathbb{P}(M(\omega)|_{\mathsf{Span}\{e_1,\ldots,e_d\}} = M^A(\omega)|_{\mathsf{Span}\{e_1,\ldots,e_d\}}) = 1$. Therefore the μ_M-invariance of a subspace $\mathbb{L}_0 \subseteq \mathsf{Span}\{e_1, \cdots, e_d\}$ is equivalent to its being μ_{M^A}-invariant.
Since $\mathbb{P}(M(\omega)$ acts on $\mathsf{Span}\{e_1, \ldots, e_d\}) = 1$ and $\mathbb{L}_0 \subseteq \mathsf{Span}\{e_1, \ldots, e_d\}$, $\mathbb{P}(\{\omega : M(\omega)\mathbb{L}_0 \subseteq \mathsf{Span}\{e_1, \ldots, e_d\}\}) = 1$. Let $v \in \mathbb{L}$ be chosen arbitrarily. Then v can be written in a unique manner as $v = v_1 + v_2$, where $v_1 \in \mathbb{L}_0$ and $v_2 \in \mathbb{L}_1$. Now

$$M(\omega)v = M^A(\omega)v_1 + M^B(\omega)v_2 + M^C(\omega)v_2 \in \mathbb{L}. \tag{20}$$

Also $\mathbb{P}(M^C(\omega)$ acts on $\mathsf{Span}\{e_{d+1}, \ldots, e_{d+m}\}) = 1$. Now $M^C(\omega)v_2 \in \mathsf{Span}\{e_{d+1}, \ldots, e_{d+m}\}$ and $M^A(\omega)v_1 + M^B(\omega)v_2 \in \mathsf{Span}\{e_1, \ldots, e_d\}$. Therefore $M^C(\omega)v_2 \in \mathbb{L}_1$ and

$$M^A(\omega)v_1 + M^B(\omega)v_2 \in \mathbb{L}_0. \tag{21}$$

Now $\mathbb{L}_1$ is a vector space. Therefore $-v_2 \in \mathbb{L}_1$. Consequently $\overline{v} := v_1 - v_2 \in \mathbb{L}$. For the vector $\overline{v}$, $M(\omega)\overline{v} = M^A(\omega)v_1 - M^B(\omega)v_2 - M^C(\omega)v_2 \in \mathbb{L}$. Also, $-M^C(\omega)v_2 \in \mathbb{L}_1$ and

$$M^A(\omega)v_1 - M^B(\omega)v_2 \in \mathbb{L}_0. \tag{22}$$

Since $\mathbb{L}_0$ is a vector space, (21) and (22) imply that $M^A(\omega)v_1 \in \mathbb{L}_0$. From this and (21), $M^B(\omega)v_2 \in \mathbb{L}_0$. Since v was chosen arbitrarily and the argument holds for all ω, $\mathbb{L}_0$ is μ_{M^A}-invariant, $\mathbb{L}_1$ is μ_{M^C}-invariant and $\mathbb{P}(\mathsf{Im}M^B(\omega)|_{\mathbb{L}_1} \subseteq \mathbb{L}_0\}) = 1$. Since $\mathbb{L}_0$ is μ_{M^A}-invariant, it is also μ_M-invariant. Therefore (d)(3) (i) – (iii) hold.
Assume that (d)(3) is false and (d)(1) is false, then $\mathbb{L} = \mathbb{L}_1$ and $\mathbb{L}_0 = \{0_{n\times 1}\}$. Since M acts on $\mathsf{Span}\{e_1, \ldots, e_d\}$ with probability 1, (20) implies that for $v = v_2 \in \mathbb{L}_1$, $M(\omega)v = M^C(\omega)v_2 + M^B(\omega)v_2 \in \mathbb{L}_1$. Since $\mathbb{P}(M^B(\omega)v_2 \in \mathbb{L}_0) = 1$ and $\mathbb{L}_0 = \{0_{n\times 1}\}$, $\mathbb{P}(\mathbb{L}_1 \subseteq \mathsf{Ker}M^B(\omega)) = 1$. Therefore $M^B(\omega)v_2 = 0_{(d+m)\times 1}$ and thus $M^C(\omega)v_2 \in \mathbb{L}_1$. Since v_2 was arbitrary and the argument holds for all ω, it follows that $\mathbb{L}_1$ is μ_{M^C}-invariant. The remaining case is trivial and (c) $\Longrightarrow$ (d) has been shown. For the other direction of the equivalence, it only has to be shown that (d)(3) or (d)(2) implies (c), since the assertion is clear in the remaining case. Thus assume that $\mathbb{L} = \mathbb{L}_0 \oplus \mathbb{L}_1$ for which (d)(3) holds. Then again, for every $v_1 + v_2 = v \in \mathbb{L}$, (20) is valid. By (d)(3)(i), $\mathbb{L}_0 \subseteq \mathsf{Span}\{e_1, \ldots, e_d\}$ is μ_{M^A}-invariant. By (d)(3)(iii), $M^A(\omega)v_1 + M^B(\omega)v_2 \in \mathbb{L}_0$. By (d)(3)(ii), $M^C(\omega)v_2 \in \mathbb{L}_1$. From this, $M(\omega)v \in \mathbb{L}_0 \oplus \mathbb{L}_1 = \mathbb{L}$. Therefore $\mathbb{L}$ is μ_M-invariant. If (d)(2) holds, then by a similar argument as the preceeding one, $\mathbb{L}$ is μ_M-invariant.

Lemma 5.4. *Let* $H = \begin{bmatrix} A & B \\ 0_{m\times d} & C \end{bmatrix}$ *be a random matrix in* $\mathbf{G}(d, m)$. *Assume that* $\mathbb{P}(\mathsf{Im}H^B = 0_{(d+m)\times 1}) < 1$. *If one of the following conditions* (a) *or* (b) *holds, then there exists no proper* μ_H*-invariant subspace* $\mathbb{L}$ *of* $\mathbb{R}^{d+m}$ *for which* *span*$\{e_{d+1}, \ldots, e_{d+m}\} \subseteq \mathbb{L}$.
(a) *The only proper* μ_H*-invariant subspaces* $\mathbb{L}$ *of* $\mathbb{R}^{d+m}$ *for which* $\mathbb{L} \subseteq \mathsf{Span}\{e_1, \ldots e_d\}$ *are* $\{0_{(d+m)\times 1}\}$ *and* $\mathsf{Span}\{e_1, \ldots, e_d\}$,
(b) *For every proper* μ_H*-invariant subspace* $\mathbb{L} = \mathbb{L}_0 \oplus \mathbb{L}_1$ *with* $\mathbb{L}_1 \subseteq \mathsf{Span}\{e_{d+1}, \ldots, e_{d+m}\}$, $\mathbb{L}_1 \neq 0_{(d+m)\times 1}$, $\mathbb{P}(\mathsf{Im}H^B(\omega)|_{\mathbb{L}_1} \subseteq \mathbb{L}_0) < 1$.

Proof : Assume that there exists a μ_H-invariant subspace $\mathbb{L}$ for which $\mathsf{Span}\{e_{d+1},\dots,e_{d+m}\}\subseteq\mathbb{L}$. Then $\mathbb{L}=\mathbb{L}_0\oplus\mathsf{Span}\{e_{d+1},\dots,e_{d+m}\}$ with $\mathbb{L}_0\subseteq\mathsf{Span}\{e_1,\dots,e_d\}$. Note that by lemma 5.3 $\mathbb{L}_0$ in this representation is μ_H-invariant. By the condition (a) either $\mathbb{L}_0=\{0_{(d+m)\times 1}\}$ or $\mathbb{L}_0=\mathsf{Span}\{e_1,\dots,e_d\}$.

If $\mathbb{L}_0=\mathsf{Span}\{e_1,\dots,e_d\}$, then $\mathbb{L}=\mathbb{R}^{d+m}$ and is not proper. If $\mathbb{L}_0=\{0_{(d+m)\times 1}\}$, then by lemma 5.3(d)(2), $\mathbb{P}(\mathsf{Im}H^B(\omega)|_{\mathsf{Span}\{e_{d+1},\dots,e_{d+m}\}}\subseteq 0_{(d+m)\times 1})=1$. This contradicts the assumption that $\mathbb{P}(\mathsf{Im}H^B(\omega)=0_{(d+m)\times 1})<1$. Thus under the condition (a) there exists no proper μ_H-invariant subspace $\mathbb{L}$ for which $\mathsf{Span}\{e_{d+1},\dots,e_{d+m}\}\subseteq\mathbb{L}$.

If the assumption (b) holds then the existence of a μ_H-invariant subspace $\mathbb{L}$ for which $\mathsf{Span}\{e_{d+1},\dots,e_{d+m}\}\subseteq\mathbb{L}$ will lead to a contradiction, since in that case by lemma 5.3(d)(3)(iii), it must hold that $\mathbb{P}(\mathsf{Im}H^B(\omega)|_{\mathsf{Span}\{e_{d+1},\dots,e_{d+m}\}}\subseteq\mathbb{L}_0)=1$ which contradicts $\mathbb{P}(\mathsf{Im}H^B(\omega)|_{\mathsf{Span}\{e_{d+1},\dots,e_{d+m}\}}\subseteq\mathbb{L}_0)<1$.

The second main theorem of this paper can now be proven.

Theorem 5.5. *Let* M *be a sequence in* $\mathbf{G}(d,m)$. *Assume that* $\mathbb{E}\left[\ln^+\|M_0\|+\ln^+\|M_0^{-1}\|\right]<\infty$ *and* $\Psi_{\mu_C}<\Psi_{\mu_M}$. *Then the following statements hold:*

(i) $\lambda_{\mu_M}=\lambda_{\mu_A}$.

(ii) $\mathbb{P}(\mathsf{Im}M_0^{B_1}=0)<1$.

In addition to this

(iii) *if* M_0 *satisfies the condition* (a) *or* (b) *of lemma 5.4, then for every* $V\in GL(m)$ *and* $\mathbb{P}$*-a.s.* $\lim_{n\to\infty}\frac{1}{n}\ln\|R_nV\|=\lambda_{\mu_A}=\gamma^R=\gamma_0=\lambda_{\mu_M}$.

(iv) *if* M_0 *satisfies condition* (a) *of lemma 5.4 and* $\mathsf{Span}\{e_{d+1},\dots,e_{d+m}\}$ *and* $0_{(d+m)\times 1}$ *are the only* μ_{MC}*-invariant subspaces of* $\mathbb{R}^{d+m}$, *then for every bounded set* $\Sigma\in\mathbb{R}^m$ *and* $\mathbb{P}$*-a.s.* $\lim_{n\to\infty}\frac{1}{n}\ln\|R_n\|_\Sigma=\lambda_{\mu_A}=\gamma^R=\gamma_0=\lambda_{\mu_M}$.

(v) *under the conditions of* (iv), *for every* $k\in\mathbb{N}$, *for every* $V\in M(m,k)\backslash\{0_{m\times k}\}$, $\lim_{n\to\infty}\frac{1}{n}\ln\|R_nV\|=\lambda_{\mu_A}=\gamma^R=\gamma_0=\lambda_{\mu_M}$ $\mathbb{P}$*-a.s.*

Proof : (i) By [4] $\lambda_{\mu_M}=\max\{\lambda_{\mu_A},\lambda_{\mu_C}\}$. Also by the assumption of the theorem ($\Psi_{\mu_C}<\Psi_{\mu_M}$), $\lambda_0(\mu_C)<\lambda_{r(\mu_M)}(\mu_M)$.
Therefore $\lambda_{\mu_C}=\lambda_0(\mu_C)<\lambda_{r(\mu_M)}(\mu_M)\le\lambda_0(\mu_M)=\lambda_{\mu_M}=\max\{\lambda_{\mu_C},\lambda_{\mu_A}\}$. Therefore $\lambda_{\mu_M}=\lambda_{\mu_A}$.

(ii) Assume that $\mathbb{P}(\mathsf{Im}M_0^{B_1}=0)=1$. Then $\Psi_{\mu_M}\subseteq\Psi_{\mu_C}\cup\Psi_{\mu_A}$. Therefore in that case $\Psi_{\mu_C}\cap\Psi_{\mu_M}\neq\emptyset$. This contradicts $\Psi_{\mu_C}<\Psi_{\mu_M}$.

(iii) Suppose that the condition (a) or (b) of lemma 5.4 holds. Then there exists no proper μ_M-invariant subspace $\mathbb{L}$ of $\mathbb{R}^{d+m}$ such that $\mathsf{Span}\{e_{d+1},\dots,e_{d+m}\}\subseteq\mathbb{L}$. By theorem 3.8(b), $\lambda_R(.)$ is constant on $\overline{V}$ and in addition to this, $\gamma_R(V)=\lambda_j\Longleftrightarrow\hat{V}_d\in\mathbf{L}_j\backslash\mathbf{L}_{j+1}$. If $V\in GL(m)$, then $\mathbb{L}(\Sigma(V))=\mathbb{L}(\Sigma(I_m))$. This is equivalent to saying that $V\in\bar{I}_m$. Hence $\gamma_R(V)=\gamma_R(I_m)$. From the definition of the subspaces $\mathbf{L}_j$,
$\gamma_R(I_m)=\lambda_j\Longleftrightarrow(\hat{I}_m)_d\in\mathbf{L}_j\backslash\mathbf{L}_{j+1}\Longleftrightarrow\Sigma((\hat{I}_m)_d)\subseteq\mathbb{L}_j$ and $\Sigma((\hat{I}_m)_d)\not\subset\mathbb{L}_{j+1}$. But

$\Sigma((\hat{I}_m))_d = \mathsf{Span}\{e_{d+1}, \ldots, e_{d+m}\}$. and there exists no proper μ_M-invariant subspace $\mathbb{L} \subset \mathbb{R}^{d+m}$ such that $\mathsf{Span}\{e_{d+1}, \ldots, e_{d+m}\} \subseteq \mathbb{L}$. Therefore there exists no proper μ_M-invariant subspace $\mathbb{L} \subseteq \mathbb{R}^{d+m}$ such that $\Sigma((\hat{I}_m))_d \subseteq \mathbb{L}$. $\Sigma(\hat{I}_m) \subseteq \mathbb{L}$. As a result $\mathbb{L}_j = \mathbb{L}_0$. This implies that $\mathbf{L}_j = \mathbf{L}_0$ and $\lambda_j = \lambda_{\mu_M}$. Thus $\lambda_R(V) = \lambda_{\mu_M} = \lambda_{\mu_A}$. By theorem 3.9(c), it holds that $GL(m) \subseteq \mathbf{S}_0 \backslash \mathbf{S}_1$. This implies that $\lambda_R(V) = \gamma_0$. Thus $\lambda_R(V) = \gamma_0 \leq \gamma^R \leq \lambda_{\mu_M} = \lambda_R(V)$. Therefore $\lambda_R(V) = \gamma_0 = \gamma^R = \lambda_{\mu_M} = \lambda_{\mu_A}$.

(iv) If the condition (a) of lemma 5.4 holds and $\mathsf{Span}\{e_{d+1}, \ldots, e_{d+m}\}$ and $0_{(d+m)\times 1}$ are the only μ_{M^C}-invariant subspaces of $\mathbb{R}^{d+m}$, then all μ_M-invariant subspaces $\mathbb{L} = \mathbb{L}_0 \oplus \mathbb{L}_1$ with $\mathbb{L}_0 \subseteq \mathsf{Span}\{e_1, \ldots, e_d\}$ and $\mathbb{L}_1 \subseteq \mathsf{Span}\{e_{d+1}, \ldots, e_{d+m}\}$ are (1) $0_{(d+m)\times 1} \oplus \mathsf{Span}\{e_{d+1}, \ldots, e_{d+m}\}$ or (2) $\mathsf{Span}\{e_1, \ldots, e_d\} \oplus \mathsf{Span}\{e_{d+1}, \ldots, e_{d+m}\}$ or (3) $\mathsf{Span}\{e_1, \ldots, e_d\} \oplus 0_{(d+m)\times 1}$ (4) $0_{(d+m)\times 1}$. Now the condition (ii) and lemma 5.3(d)(3)(iii) prohibit (1) from occuring. Therefore all μ_M-invariant subspaces of $\mathbb{R}^{d+m}$ are $0_{(d+m)\times 1}$, $\mathsf{Span}\{e_1, \ldots, e_d\}$ and $\mathbb{R}^{d+m}$. Consequently there exist no proper μ_M-invariant subspaces of $\mathbb{R}^{d+m}$, in which non-zero vectors $\hat{x}$ lie. As a result, for non-zero bounded subsets $\Sigma \subset \mathbb{R}^m$ there exist no proper μ_M-invariant subspaces $\mathbb{L} \subset \mathbb{R}^{d+m}$ for which $\hat{\Sigma}_d \subseteq \mathbb{L}$. By theorem 3.6, $\lim_{n\to\infty} \frac{1}{n} \ln \| R_n \|_\Sigma = \lambda_{\mu_M} = \lambda_{\mu_A} = \gamma^R = \gamma_0$ $\mathbb{P}$-a.s., for every non-zero bounded subset $\Sigma \subset \mathbb{R}^m$. From this and theorem 3.8, for every $k \in \mathbb{N}$, for every $V \in M(m,k) \backslash \{0_{m\times k}\}$, $\lim_{n\to\infty} \frac{1}{n} \ln \| R_n V \| = \lambda_{\mu_A} = \gamma^R = \gamma_0 = \lambda_{\mu_M}$, $\mathbb{P}$-a.s.

So far, the almost sure asymptotic behaviour of $(\frac{1}{n} \ln \| (R_n)V \|)$ has been fully described when the conditions $\mathbb{E}[\ln^+ \| M_0 \| + \ln^+ \| M_0 - 1 \|] < \infty$ and $\Psi_{\mu_C} < \Psi_{\mu_M}$ are satisfied, by a sequence $M \subset \mathbf{G}(d,m)$. Sufficient conditions have also been given under which $\lim_{n\to\infty} \frac{1}{n} \ln \| R_n V \| = \lambda_{\mu_M} = \lim_{n\to\infty} \frac{1}{n} \ln \| R_n \|$ almost surely for every non-zero vector V for which the process $(R_n V)$ is well defined. Simple examples show that when the condition $\Psi_{\mu_C} < \Psi_{\mu_M}$ fails, then the existence of the almost sure limit $\lim_{n\to\infty} \frac{1}{n} \ln \| R_n \|$ can no longer be guaranteed, even if $\mathbb{E}[\ln^+ \| M_0 \| + \ln^+ \| M_0^{-1} \|] < \infty$. Note that under the latter condition $\lim_{n\to\infty} \frac{1}{n} \ln \| M_n \cdots M_0 \|$ exists almost surely.

Apart from applications in stochastic modelling , numerous examples of which are given in [12], the results obtained here may be used in the study of the behaviour of sample paths of solutions of stochastic differential equations. Notice that with a slight change of index $X_n = A_n X_{n-1} + B_{n+1}$ can be written as as $X_n = A_n X_{n-1} + B_n$.
Therefore $X_n - X_{n-1} = (A_n - I)X_{n-1} + B_n$. In other words, $\triangle X_n = (A_n - I)X_{n-1} + \triangle W_n$, where $\triangle W_n := B_n$, i.e. B_n is the increment of the process W at time n. But then $\triangle X_n = (A_n - I)X_{n-1} + \triangle W_n$ is the discrete time approximation of the continuous time stochastic differential equation $dX_t = ((A(t) - I)X_t)dt + dW_t$. The latter equation is well known when W is brownian motion.

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