The Ideal Structure Theorem for Dα

KWALAR Shadrach Nshanji
Department of Mathematics, University of Buea, Faculty of Science, P.O. Box 63, Buea, S.W. Province

ABSTRACT

In this paper we describe completely the closed ideals of the Banach algebra of functions f analytic in the unit disc such that their Taylor coefficients $\hat{f}$ satisfy the condition

$$\sum_{n \in IN} (1+n)^{\alpha} \,|\, \hat{f}(n) \,|^2 := \,||\, f \,||^2 < +\infty$$

for $2n+1 > \alpha > 2n-1$, ($n \in IN^*$ fixed), when the skeletons of the closed ideals under consideration are at most countable.

Keywords: Closed Ideals, Banach Algebras, K-algebra, Skeleton, Inner factors, Standard Ideals

RESUME

Dans cet article nous décrivons complètement les idéaux fermés de l'algebre de Banach D_α de fonctions f analytiques dans le disque unitaire dont les coefficients de Taylor $\hat{f}$ satisfient la condition

$$\sum_{n \in IN} (1+n)^{\alpha} \,|\, \hat{f}(n) \,|^2 := \,||\, f \,||^2 < +\infty$$

pour $2n+1>\alpha>2n-1$, ($n \in IN^*$ est fixe), quand les squelettes des idéaux considerés sont au plus denombrables.

Mots clés: Idéaux fermés, Algebres de Banach, K-algebre, Squelette, Facteurs intérieurs, Idéaux étandards.

Introduction:

For $\alpha \in \mathbb{R}$, let D_α be the set of all complex functions f analytic in the unit disc Δ of the complex plane such that their Taylor coefficients $\hat{f}$ satisfy the condition

$$\sum_{n \in \mathbb{N}} (1+n)^{\alpha} |\hat{f}(n)|^2 := \| f \|^2 < +\infty.$$

It is known that under pointwise operations each D_α is an algebra if and only if $\alpha > 1$ [1]. In 1972 B.I.Korenblum gave a complete description of the closed ideals of the Banach algebra D_2 [2] and later that same year, he, very sparsely, sketched the description of the closed ideals of the Banach algebras D_{2n} for $n \in \mathbb{N}^*$ [3].

In this paper we describe completely the closed ideals of D_α for $2n + 1 > \alpha > 2n - 1$, ($n \in \mathbb{N}^*$ fixed), when the skeletons of the ideals under consideration are at most countable.

Following V.M. Faivyshevskii, we shall say that the Banach algebra R with pointwise operations, of functions continuous on the unit circle, is called a K-algebra if the trigonometric polynomials are dense in R and $C^{\infty}(\partial\Delta) \subset R$.

Let R_+ be the analytic subalgebra of R consisting of those functions ϕ in R which can be extended to functions in the disc algebra Ao. Such a Banach algebra is called a K_+-algebra. The K_+-algebra R_+ will be said to be a KD-algebra of order n-1 (for some $n \in IN^*$) if

a) $R_+ \subset Ao^{n-1} := \{ f \in H(\Delta) \mid f^{(n-1)} \in Ao \}$ and

b) for any $f \in R_+$ such that

$f^{(j)}(e^{i\theta o}) = 0$ (for some fixed $e^{i\theta o}$ in $\partial\Delta$),

($j = 0,1,2,\ldots, n-1$), there is a sequence (g_k) in R_+

such that $g_k(e^{i\theta o}) = 0$ for all $k \in N$

and $\| g_k f - f \|_{R+} \longrightarrow 0$ as $k \longrightarrow \infty$.

For any f in Ao, Z(f) and E (f) will denote its zero set on $\overline{\Delta} = \Delta \cup \partial\Delta$ and $\partial\Delta$ respectively.

If I is a proper closed ideal of R_+, let $Z_I = \bigcap_{f \in I} Z(f)$ and $E_I = \bigcap_{f \in I} E(f)$.

E_I is called the skeleton of I.

Let I be a closed ideal in the K_+ -algebra $R_+ \subset A_o^{n-1}$, and G_I be the greatest common divisor of the inner factors of the functions in I. The "frame of order n-1" of the closed ideal $I \subset R_+$ is the collection R_{n-1} (I), consisting of

1) G_I;

2) $E_I = Z_I \cap \partial\Delta$

and $E_k (I) = \bigcap_{\substack{j \le k \\ f \in I}} Z(f^{(j)}) \cap \partial\Delta$, (k = 1,2,…, n-1).

It is known [4,3,5,6,7] that

a) Supp $\sigma \subseteq E_{n-1} (I) \subseteq \ldots\ E_1 (I) \subseteq E (I)$, where σ is the measure which determines the singular factor of the inner function G_I;

b) $E(I) \setminus E_{n-1}(I)$ is an isolated set.

Conversely, given the collection

$$R_{n-1} = \{G; E_0, E_1, \ldots, E_{n-1}\},$$

where G is an inner function and $E_0, E_1, \ldots E_{n-1}$ are closed subsets of $\partial\Delta$ satisfying a) and b) above, the set $\{f \in R_+ \mid G$ divides the inner factor of f and $E_k \subseteq Z(f^{(k)}) \cap \partial\Delta\}$ forms a closed ideal $I(R_{n-1})$ in R_+. Ideals of this type are said to be standard [6].

The statement of our main result, which we have called the Ideal Structure Theorem for D_α, is as follows:

Theorem:

Let I be a nonzero closed ideal in D_α for $2n+1>\alpha>2n-1$, where $n \in IN^*$ and α are fixed. Then if E_I is at most countable, I is a standard ideal.

We prove this $D\alpha$ Ideal Structure Theorem with the aid of the following result, which is due to V.M. Faivyshevskii [6, Theorem 5]:

Let the K_+-algebra R_+ be a KD –algebra of order n-1 for some fixed $n \in IN^*$. Then every closed ideal $I \subset R_+$, whose skeleton is at most countable, is standard.

To prove our theorem, it will thus suffice to show that for $2n+1>\alpha>2n-1$, D_α is a KD-algebra of order n-1.

To do this, let $R = L^2_\alpha$, where L^2_α is the set of all complex functions ψ on $\partial\Delta$ whose Fourier coefficients $\hat{\Psi}$ satisfy the condition:

$$\sum_{k \in Z} (1+|k|^{\alpha})|\hat{\Psi}(k)|^2\}^{1/2} := ||\psi||_{L^2_\alpha} < \infty,$$

for $1< \alpha< \infty$. One uses the fact that if $1< \alpha< +\infty$, then

a) D_α is a Banach algebra,

b) $L_\alpha^2 \subset C(\partial\Delta)$,

c) trigonometric polynomials are dense in L_α^2, and

d) $C^{\infty}(\partial\Delta) \subset L_{\alpha}^{2}$

to see that if $2n + 1 > \alpha > 2n-1$, ($n \in IN^*$), $R = L_{\alpha}^{2}$ is a K-algebra and $R_{+} = D_{\alpha}$. Now, for $\alpha > 2n-1$,

$D_{\alpha} \subset A_{o}^{n-1}$ since $f \in D_{\alpha}$ if and only if

$f^{(n-1)} \in D_{\alpha-2(n-1)} \subset A_{o}$.

It is thus left to prove

Lemma 1: Suppose that $n \in IN^*$ and $\alpha \in R$ are fixed and $2n+1>\alpha>2n-1$. For each $f \in D_{\alpha}$ such that

$$f^{(j)}(e^{i\theta_o}) = 0 \text{ for } j = 0,1, \ldots, n-1,$$

($e^{i\theta_o} \in \partial\Delta$ is fixed), there is a sequence (g_k) in D_{α} such that $g_k(e^{i\theta_o}) = 0$. ($k \in IN^*$), and $g_k f$ converges to f in D_{α}.

We remark that B.I. Korenblum and V.S. Korolevitch obtained this lemma for the particular case when $\alpha = 2n$ [8].

The proof of this lemma is a little lengthy and so we shall only sketch it here.

First of all, observe that one can take θ_o to be 0. Assuming then that $\theta_o = 0$, we consider h_k and g_k, ($k \in IN^*$), given on Δ by

$$h_k(z) = (z-1-k^{-1})^{-1}$$

and

$$g_k(z) = (z-1)h_k(z).$$

Furthermore, let $G_k = g_k f - f = k^{-1} f h_k$, ($k \in IN^*$).

Since $g_k(1) = 0$ for all $k \in IN^*$ and

$$g_k(z) = k(1+k)^{-1} - (1+k)^{-1} \sum_{m \in IN^*} (1+k^{-1})^{-m} z^m$$

($k \in IN^*$), we see that $g_k \in D_\beta$ for each real number β.

Hence we need show only that

$$||G_k||_{D\alpha} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \qquad (1)$$

To establish (1), we consider two cases .

Case 1: n=1 and so $3>\alpha>1$. We need two propositions

Proposition 1: If $g \in D_\beta$ for $\beta < 1$, then

$$|g(z)| = o\left((1-|z|)^{\frac{\beta-1}{2}}\right),\ |z| \longrightarrow 1^{-}.$$

Proposition 2: If $f \in D_\alpha$ for $3>\alpha>1$ and $f(1) = 0$, then

$$|f(z)| = o\left(|z-1|^{\frac{\alpha-1}{2}}\right),\ |z-1| \longrightarrow 0. \qquad (2)$$

Proposition 1 is an improvement of a result of Leon Brown and A.L. Shields [9]. Proposition 2 follows from Proposition 1 and the Hardy-Littlewood Theorem that if f is analytic in Δ, then

$|f'(z)| = o((1-|z|)^{\Upsilon-1})$, $(0<\Upsilon<1)$, if and only if f satisfies a "little o" Lipschitz condition (of order Υ) (page 429 of [10]).

Apart from the two propositions above, we also need the following lemma.

Lemma 2: If β is a real number and $h \in D_\beta$, there exist two positive numbers m (β) and M (β) such that

$$m(\beta)\{\,\|h\|^2_{D_\beta} - |h(o)|^2\} \leq \|h'\|^2_{D_{\beta-2}}$$

$$\leq M(\beta)\{\|h\|^2_{D_\beta} - |h(o)|^2\}. \quad (3)$$

The proof of this lemma is routine.

By virtue of (3), to establish (1), it is enough to show that

$$\|G_k''\|_{D_{\alpha-4}} \longrightarrow 0 \text{ as } k \longrightarrow \infty. \quad (4)$$

By a result of G.D. Taylor [1],

$\|G_k''\|D_{\alpha-4}$ is equivalent to

$$\{\int_0^{2\pi}\int_0^1 |G_k''(re^{i\theta})|^2 (1-r^2)^{3-\alpha} rdrd\theta\}^{1/2}$$

Since

$$|G_k''(z)|^2 \leq CK^{-2}\{|h_k(z)|^6|f(z)|^2 + |h_k(z)|^4|f'(z)|^2 + |h_k(z)|^2|f''(z)|^2\}, \quad (*)$$

where C is an absolute constant, to prove (4), it suffices to show that

$$k^{-2}\{\int_0^{2\pi}\int_0^1 |h_k(re^{i\theta})|^2|f''(re^{i\theta})|^2(1-r^2)^{3-\alpha} rdrd\theta\} \longrightarrow 0, \quad (5)$$

$$k^{-2}\{\int_0^{2\pi}\int_0^1 |h_k(re^{i\theta})|^4|f'(re^{i\theta})|^2(1-r^2)^{3-\alpha} rdrd\theta\} \longrightarrow 0, \quad (6)$$

and

$$k^{-2}\left\{\int_0^{2\pi}\int_0^1 |h_k(re^{i\theta})|^6\,|f(re^{i\theta})|^2(1-r^2)^{3-\alpha}\,r\,dr\,d\theta\right\}\longrightarrow 0, \quad (7)$$

as $k \longrightarrow \infty$. Statement (5) holds by the Lebesgue Dominated Convergence Theorem.

With the aid of Proposition 2, Lemma 2 and the fact that if $z-1=se^{i\phi}$ for $\frac{\pi}{2} \leq \phi \leq \frac{3\pi}{2}$, then $|z-1-k^{-1}|^4$ is comparable to $s^4 + k^{-4}$, one can show that each of (6) and (7) holds as $k \longrightarrow \infty$.

The proof of Lemma 1 is thus sketched for n = 1.

<u>Case 2:</u> $n \in \mathbb{N}^*$ is arbitrary and so $2n+1>\alpha>2n-1$.

We reason here as in Case 1, replacing Proposition 1, Proposition 2, Lemma 2 and (*) by Proposition 3, Proposition 4, Lemma 3 and (**), respectively.

<u>Proposition 3:</u> If $f \in D_\alpha$ and $\alpha<2n+1$, then

$$|f^{(n)}(z)| = o\left((1-|z|)^{\frac{\alpha-2n-1}{2}}\right), |z| \to 1^-.$$

This is clear by Proposition 1 since $f^{(n)} \in D_{\alpha-2n}$.

Proposition 4: If $f \in D_\alpha$ for $2n+1>\alpha>2n-1$ and

$$f^{(j)}(1) = 0 \text{ for } j = 0, 1, \ldots, n-1,$$

then for $\ell = 2,3,\ldots,n+1$, *we have*

$$\left|f^{(n+1-\ell)}(z)\right| = o\left(|z-1|^{\frac{\alpha+2\ell-2n-3}{2}}\right), |z-1| \to 0. \qquad (+)$$

Proof: By Proposition 3 and the Hardy –Littlewood theorem cited on page 5, (+) is valid for $\ell = 2$. Hence, for $\ell = 3,4,\ldots, n+1$, we have

$f^{(n+1-\ell)}(z) = \int^{z} f^{(n-\ell)}(w)\, dw$ and by finite induction we get the result.

Lemma 3: If $2n+1>\alpha>2n-1$, then as $k \longrightarrow \infty$,

$$\| G_k \|_{D_\alpha} \to 0$$

if and only if

$$\| G_k^{(n+1)} \|_{D_{\alpha-2n-2}} \to 0.$$

We now state (**).

If 2n+1>α>2n-1, then

$$|G_k^{(n+1)}(z)|^2 \leq k^{-2}\sum_{\ell=0}^{n+1} d_\ell |h_k(z)|^{2\ell+2} |f^{(n+1-\ell)}(z)|^2, (**),$$

where each d_ℓ, (ℓ = 0, 1 …., n+1), is an absolute constant.

<u>Proof:</u> Since α -2n –2 < 0, by Lemma 2 of [7],

$$|| G_k^{(n+1)} ||^2_{D\alpha-2n-2} \cong K,$$

where

$$K = \int_0^{2\pi}\int_0^1 \left|G_k^{(n+1)}(re^{i\theta})\right|^2 (1-r^2)^{(2n+1-\alpha)} r dr d\theta,$$

and so we need show only that the integral on the right tends to 0 as k ⟶ + ∞. By (**), it suffices to show that $\lim_{k \to +\infty} I_k(\ell) = 0$ (ℓ = 0,1,…, n+1), (++)

where $k^2 I_k(\ell)$ equals the integral

$$\int_0^{2\pi}\int_0^1 (\left|re^{i\theta} - 1 - k^{-1}\right|^{\ell+1})^{-2} \left|f^{(n+1-\ell)}(re^{i\theta})\right|^2 (1-r^2)^{(2n+1-\alpha)} r dr d\theta.$$

Since $f^{(n+1)} \in D_{\alpha-2n-2}$, ($I_k$ (0)) is a null sequence by the Lebesgue Dominated Convergence Theorem (since

$|h_k| \leq k$ and $h_k(z)/k \longrightarrow 0$ in Δ). By virtue of the fact that

$|f^{(n)}(re^{i\theta})| = o((1-r)^{(\alpha-2n-1)/2}), r \longrightarrow 1^-$,

$I_k(1) \longrightarrow 0$ as $k \longrightarrow +\infty$. To prove (++) for $\ell = 2,3,\ldots, n+1$, we use Lemma 4.

The proof of Lemma 1 is thus sketched.

Remark: If g_k is defined as in the proof of lemma 1, then there exists a function f in D_{2n+1} such that

$$f^{(j)}(1) = 0, (j = 0,1,\ldots, n-1),$$

but $g_k f$ does not tend to f in D_{2n+1}.

We have two corollaries to our main result.

Corollary 1: If $2n+1>\alpha>2n-1$, I is a nonzero closed ideal in D_α, and its skeleton E_I is at most countable, then I is a principal ideal.

One way to see this is to use three results of B.I. Korenblum, V.M. Faivyshevskii and Leon Brown – A.L. Shields [11,6,9].

Corollary 2: If $1< \alpha<\infty$ and $f\in D_\alpha$ is an outer function with $E(f)$ at most countable, then f is cyclic in D_1.

This is a generalization of a result of Leon Brown – A.L. Shields [9, Theorem 3].

Acknowledgement: I am grateful to Professor Leon Brown who introduced me to Ideal Structures of Banach algebras.

BIBLIOGRAPHY

1) G.D. Taylor, "Multipliers on D_α," *Trans. Amer. Math. Soc.*, **(1966)**, 123, 229-240.

2) B.I. Korenblum, "Invariant Subspaces of the Shift Operator in a Weighted Hilbert Space", *Mat. Sbornik,* **(1972)**, 89(131), 110-137, English Transl. *–Maths. USSR Sbornik,* **(1972)**, 18, 111 – 138.

3) B.I. Korenblum, "Invariant Subspaces of the Shift Operator in some Weighted Hilbert Spaces", *DAN SSSR,* **(1972)**, 202, n° 6, 1258-1260; English Transl.- *Soviet Math. Dockl.,* **(1972)**, 13 n° 1, 272 – 275.

4) B.I. Korenblum, "The closed Ideals of A_0^n," *Functional Anal. Appl,.* **(1972)**, 6 (3), 2003 – 2014.

5) V.M. Faivyshevskii, "Construction of Ideals of some Alge-bras of Analytic Functions", *DAN SSR* **(1973)**, 211, n° 3, 537-539; English Transl.- *Societ Math. Dok1.* **(1973)**, 14 n° 4, 1067-1070.

6) V.M. Faivyshevskii, "Math Programming and Related Questions ....", *Central Ekonom – Mat. Inst. Akad. Nauk. SSSR, Moscow,* **(1976)**, 150-161; English Transl. – *Transl. Amer. Math. Soc.* **(1980)**, 115 (2), 93 – 102.

7) F.A. Shamoyan, "Closed Ideals in Algebras of Anal. Functions ....," *Izv. Akad. Nauk. Armen SSR,* **(1981)**, XVI, n° 3, 74.

B.I. Korenblum, V.S. Korolevitch, "On Anal. Functions Regular in the Circle and Smooth ...," *Math. Zametki,* **(1970)**, 7, 165-172 (Lemma 5).

9) Leon Brown – A.L. Shields, "Cyclic Vectors in the Dirichlet Space", *Trans. Amer. Math. Soc.* **(1984)**, 285 n°1, 269-309 (10(a)).

10) G.H. Hardy – J.E. Littlewood, "Some Properties of Fractional Integrals, II" *Math. Zeitschr.* **(1932),** 34, 403 – 439.

11) B.I. Korenblum, "Functions Hol. In a dis ...," *DAN SSSR,* **(1971)**, 200, 24-27; English Transl.- *Soviet Math. Dokl.*, **(1971)**, 12, 1312 – 1315.

Received: 09/01/2004
Accepted: 31/12/2004