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Mathematical Analysis of Solute Transport with Linear Sorption and Sedimentation in Porous Media via Olayiwola's Generalised Polynomial Approximation Method

[1*]ABUBAKAR, AD; [2]PETER, LJ; [3]YAHAYA, HM

*[1, 2]Department of Mathematics, Federal University Gashua, Nigeria.
[3]Darul-fikr College, Gashua, Yobe State, Nigeria

*Corresponding Author Email: ahmeddauda8@gmail.com
*ORCID ID: 0000-0002-4812-8148
*Tel:+2347068233938

Co-Authors email: luwajacob@gmail.com; yahayahabiba084@gmail.com

ABSTRACT: Solute transport plays a significant role in biogeochemical cycling and many other processes and can be dominated by advection, diffusion, or dispersion, or by any combination. Solutes are dissolved matters in an ecological media. Hence, the objective of this paper is to present a solution for Advection-dispersion equation (ADE) to analyse solute transport with linear sorption and sedimentation in porous media via the Olayiwola"s generalized polynomial approximation method (OGPAM). The results obtained were presented graphically and discussed. It was observed that increase in sedimentation decrease the solute concentration. Also, the dispersion coefficient indicates that solute concentration which is the amount of particle in the water decreases from the initial point and later becomes uniform a long distance and increases at later time. The profile pattern of the present work closely agreed with the analytical solution obtained from the existing literatures.

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Porous media is composed of a solid matrix with interconnected network of pores through which a fluid can flow. Over the decades, solute transport through porous media has been receiving attention by different researchers due to its socioeconomic activities that causes contamination to the environment. Therefore, it is very imperative to study the processes of solutes transport in porous media. Mathematically, these processes are described by the advection-dispersion equation (Barbour and Krahn, 1933). Analytical investigation into solute transport and sorption via intra-particle diffusion in the dual-porosity limit has been reported by (Auton *et al.*, 2024). Their investigation incorporated both physical and chemical adsorption mechanism. Wu and Hsieh, (2022) also presented an exact solutions to a one-dimensional ADE which examines the variation of solute concentration with and without the biodegradable effect in an unconfined aquifer of a finite domain by the generalized integral transform method. Their findings reveals that contaminated concentration decreases as the retardation factor, Peclect number and the first-order decaying coefficient increases for every location at a fixed duration. Tjock-Mbaga *et al.* (2022) investigated one-dimensional ADE for solute transport with two contaminant sources incorporating the source term. Their findings show that the concentration increases

*Corresponding Author Email: ahmeddauda8@gmail.com
*ORCID ID: 0000-0002-4812-8148
*Tel:+2347068233938

in the direction of flow, and decreases with distance because of the additional source function which decreases with position. Recently, a new analytical solution via generalized integral transform techniques was derived for a solute transport equation with variable dependent coefficient without source term by (Bharati *et al.,* 2018) and (Bharati *et al.,* 2019).

Radha *et al*. (2022) applied numerical method on contaminant transport analysis under non-linear sorption in a heterogeneous groundwater system for both finite and infinite domain. it was observed that the numerical solution obtained for the finite domain produces a marginal error than the numerical solution obtained for the semi-infinite domain. Additionally, several column experiments on the interactions between saturated/unsaturated porous media and the mechanisms of contaminant migration have been reviewed by (Shu *et al.,* 2023).

Physics-informed neural networks (PiNN) with periodic activation functions were employed by (Faroughi *et al.,* 2024) to study solute transport in heterogeneous and homogeneous porous media. Validation of PiNN with the existing solution showed a reduction in computational expenses in terms of inference time by three orders of magnitude compared to Finite Element Method for two dimensional cases. The effects of various flow rate conditions in saturated porous media such as hydrophobic nature of polyethylene and surfactants were studied by (Okutan *et al.,* 2024). A change was observed on sorption parameters due to the flow rate.

The higher values of distribution coefficient and first order mass transfer rate were determined at lower flow rates, which indicate that sorption rate increases under condition of low flow rates. Numerical simulation of solute transport in variably-saturated deformable soil during freeze-thaw cycles incorporating coupled water/vapor flow and heat transfer were investigated by (Huang and Rudolph, 2023). Their findings show that freezing front moves downward during the freezing stage. The quick dewatering before the freezing front results in local volume shrinkage, which is progressively restored when the thawed water fills the water loss zone during the thawing stage. Compared to freezing silt or clay, vapor flow in freezing sand is far more important. Solutes transport processes which include diffusion; consolidation and adsorption were investigated analytically in the proposed model of (Li *et al.,* 2023). Variable transformation and generalised integral transform technique (GITT) was adopted to solve the governing equation. Breakthrough time of solute transport before consolidation and after consolidation between large-strain aquitard and rigid porous medium were analysed and compared. The results show that greater Darcy velocity, stronger solute diffusivity, and weaker adsorption of soil particles result in shorter breakthrough time of solute transport in the large-strain aquitard, while the impacts of specific storage and void ratio on the breakthrough time are influenced by the adsorption of soil particles. Analytical model of contaminant advection, diffusion and degradation in capped sediments were also reported by (Chen *et al.,* 2024). Sediment cap was categorized into a stratified system consisting of a biologically active zone, cap protection layer, chemical isolation layer, and contaminated sediment layer. The key findings reveal that solutions obtained can be used for the design of a multi-layer containment system, verification of numerical models, and evaluation of experimental data. Effect of pumping rate on contaminant transport in riverbank filtration system (RBF) was investigated by (Abubakar *et al.,* 2021a). Findings from this work show that increase in pumping rate value enhance both the hydraulic head and concentration of colloids which slightly reduces the quality of pumped water from RBF.

Based on granular thermodynamics, Wu *et al.* (2024) presented a coupled transport model of pollutants-suspended particles in saturated porous media. Considerations included soil deformation, thermal conduction, and the adsorption and desorption of pollutants by suspended and soil particles. The findings unequivocally demonstrate that as initial porosity increases, coupled transport's penetration rate correspondingly increases, allowing for the quicker achievement of a stable concentration. In conclusion, changes in the initial pollutant diffusion coefficient have a significant impact on the transport process of pollutants. Paswan and Sharma, (2023) have used sensitivity and spatial moment analysis to examine the mobility and spreading behavior of contaminants when colloids are present.

The findings imply that the concentration of colloids and the physical and chemical interactions of contaminants with the suspended colloids and stationary solid matrix determine how well contaminants are transported when colloids are present. Also, detailed investigation on processes between interactions of solutes transport in saturated surface and subsurface water have been reported by (Delisle *et al.,* 2023) and (Abubakar *et al.,* 2021b).

The objective of this paper is to present a solution for Advection-dispersion equation (ADE) via the Olayiwola's generalized polynomial approximation method (OGPAM), incorporating sedimentation process.

MATERIALS AND METHODS

Mathematical formulation: Initially, the subsurface water is considered not solute free in this study. The solute transport is assumed to vary only in the horizontal direction of flow, which is described by unsteady one-dimensional advection-dispersion equation (ADE). A point source solute is assigned at the origin off semi-infinite aquifer from where the solute can disperse to the lower concentration from a higher concentration. In addition to Paul *et al.* (2022), sedimentation was incorporated into the model equation. Given these assumptions, the mathematical governing equations are as follows:

$$\phi\frac{\partial c}{\partial t'}+\rho k_d\frac{\partial c}{\partial t'}+\frac{\phi(\rho_s-\rho)gd_s}{l\,\&\,\mu}\frac{\partial c}{\partial x'}=\phi D_0\frac{\partial^2 c}{\partial x'^2}-\lambda\phi c+\phi\gamma \quad (1)$$

$$c(x',t)=c_{i;} \qquad x\le 0,\quad t=0 \quad (2)$$

$$c(x',t)=c_{0;} \qquad x=0,\quad t>0 \quad (3)$$

$$\frac{\partial c}{\partial x'}=0; \qquad x\to\infty,\quad t>0 \quad (4)$$

Dimensional analysis:

$$x=\frac{x'}{l},\quad \theta=\frac{c}{c_0},\quad t=\frac{u_0 t'}{l} \quad (5)$$

Using the dimensionless variables in (5), the model equation takes the form

$$R\frac{\partial\theta}{\partial t}+\alpha\frac{\partial\theta}{\partial x}=D\frac{\partial^2\theta}{\partial x^2}-\lambda_1\theta+\gamma_1 \quad (6)$$

$$\theta(x,0)=\phi_o,\qquad \theta(0,t)=1,\qquad \left.\frac{\partial\theta}{\partial x}\right|_{x=\infty}=0 \quad (7)$$

Nomenclature: D_o is longitudinal dispersion coefficient, c is the solute concentration, k_d is the linear equilibrium distribution coefficient of the solute between the aqueous phase and the solid phase, ρ is the density of the porous medium, λ is first-order decay constant in the liquid phase, γ is zero-order production-rate coefficient for solute production in the liquid Phase, ϕ is the water content, ρ_s is the dry bulk density of the solid matrix, g is the gravitational acceleration, μ is the dynamic viscosity, d_s is the cell diameter, s is the one-dimensional cell swimming speed,

Variable transformation:

$$\text{Let } z=l-le^{-x} \quad (8)$$

Using (8), equation (6) and (7) gives:

$$\left.\begin{array}{l} R\dfrac{\partial\theta}{\partial t}=D(l-z)^2\dfrac{\partial\theta}{\partial z^2}-\alpha_1(l-z)\dfrac{\partial\theta}{\partial z}-\lambda_1\theta+\gamma_1 \\ \theta(z,0)=\phi_0,\qquad \theta(0,t)=1,\qquad \left.\dfrac{\partial\theta}{\partial z}\right|_{z=1}=0 \end{array}\right\} \quad (9)$$

Where,

$$\left.\begin{array}{l} \alpha_1=\alpha+D \\ R=1+\dfrac{\rho k_d}{\phi} \end{array}\right\} \quad (10)$$

Solution using OGPAM:

Equation (9) was solved using OGPAM (Olayiwola, 2022), and obtained as follows:

$$\theta(z,t)=1+2\left(A_{10}e^{-A_8 t}+A_9-1\right)z+\left(1-A_{10}e^{-A_8 t}-A_9\right)z^2 \quad (11)$$

Where,

$$A_1=\frac{2D}{LR},\qquad A_2=\frac{2\alpha_1}{LR},\qquad A_3=\frac{\lambda_1}{R},\qquad A_4=\frac{\gamma_1}{R},\qquad A_5=\left(l^2-l+\frac{1}{3}\right)$$

$$A_6=\left(A_1A_5l+A_2l-A_3+A_4-\frac{A_2}{2}-\frac{A_2}{2}+\frac{A_3}{L}+\frac{A_2}{3L}-\frac{A_3}{3L^2}\right)$$

$$A_7=\left(-A_1A_5-A_2l+\frac{A_2}{2}+\frac{A_2}{2}-\frac{A_3}{L}-\frac{A_2}{3L}+\frac{A_3}{3L^2}\right),\ A_8=\frac{-3A_7}{2},\ A_9=\frac{2}{3}A_6\quad A_{10}=\phi_0-A_9$$

RESULTS AND DISCUSSION

In this work, the governing equation was solved using Olayiwola's Generalised Polynomial Approximation Method. The solutions are computed for the values of R = 2.4; D = 0.04; α = 0.04; λ_1 = 0.05; $l=1$; φ_1 = 1; t > 0; $x \geq \infty$, With the aid of Maple 17.

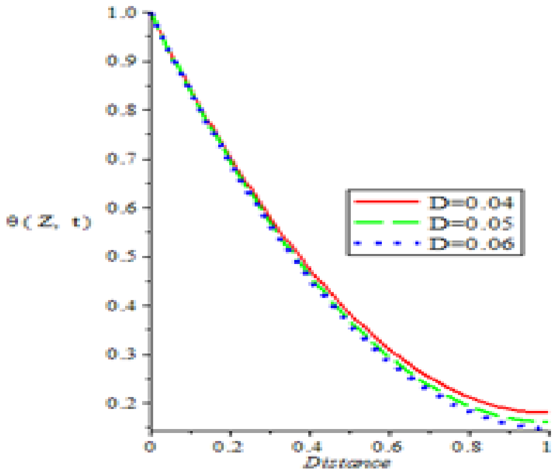

Fig. 1: Variation of solute concentration with dispersion coefficient D against distance.

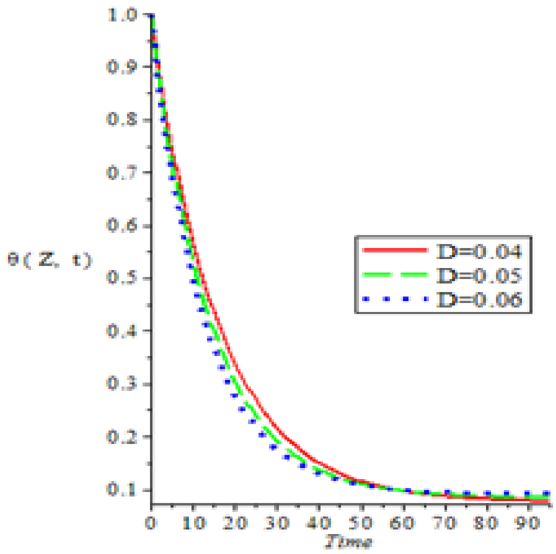

Fig. 2: Variation of solute concentration with dispersion coefficient D against time.

The impact of various physical parameters was presented in figures 1 to 3. Comparison between the present work and the work of (Paul *et al.*, 2022) are presented in Figure 4 and 5.

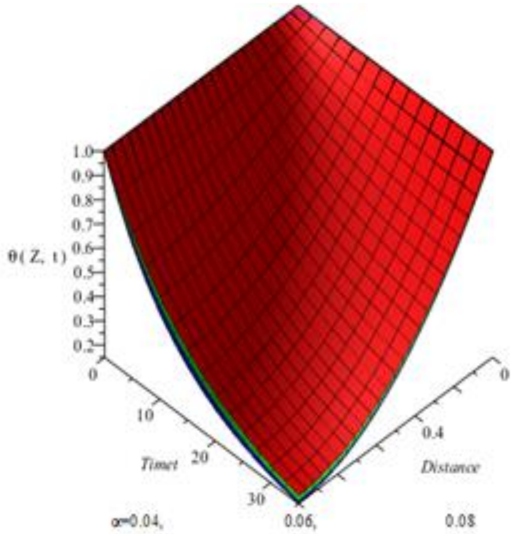

Fig. 3: Variation of solute concentration with sedimentation α against distance and time.

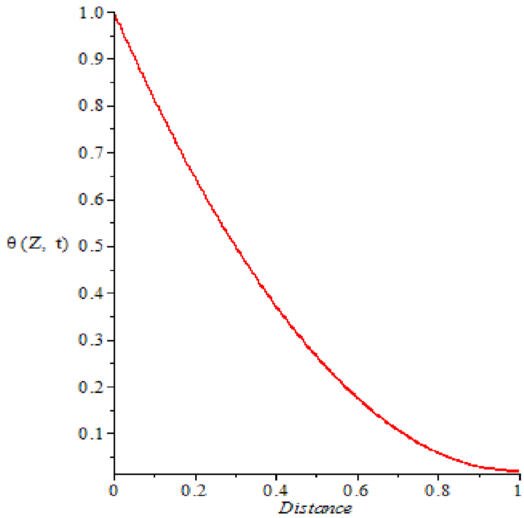

Fig. 4: Graphical solution profile of the present work against distance.

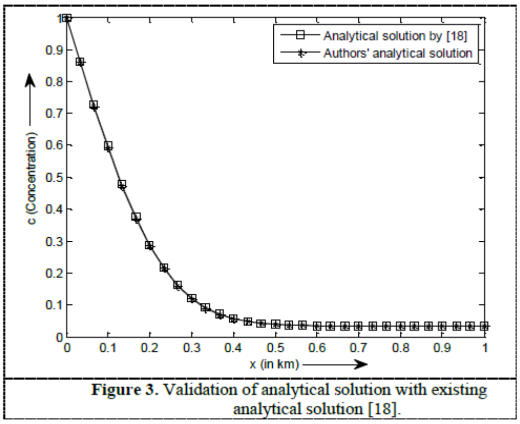

Fig. 5: Graphical solution profile of the work of (Paul *et al.*, 2022) and van (Genuchten and Alves, 1982) against distance.

Figures 1 and 2 Show the variation of solute concentration with dispersion coefficients D. It was observed from these figures that solute concentration which is the amount of particle in the water decreases from the initial point and later becomes uniform along distance and time. The dispersing coefficient did not show any appreciable changes on the concentration at the initial time. Figure 3 indicate variation of solute concentration with sedimentation α. It was observed that solute concentration attains maximum level at the initial point from where it continues on falling and becomes constant with space and time. These processes reduced the concentration of solutes which improves the quality of subsurface water. Figure 4 and 5 depicts the graphical solution of the present work and analytical solution of (Paul *et al.,* 2022), (van Genuchten and Alves, 1982) respectively. The profile pattern of the present work closely agreed with the analytical solution obtained from the existing literatures.

Conclusion: Sedimentation (transport parameter) which is one of the natural processes in an environmental media was incorporated in to one-dimensional unsteady solute transport equation under linear sorption. The model equation was solved analytically using OGPAM. More so, results presented graphically were achieved with the aid of Maple code. The improved mathematical model can be used to predict the rate of solute migration and also assist in decision-making processes for subsurface water management. For further research, interested researchers should consider other natural processes that affect the concentration of particles in an environmental media.

Declaration of Conflict of Interest: The authors declare no conflict of interest.

Data Availability: Data are available upon request from the corresponding author.

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