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Comparative Studies of Some Topological Properties on Hyperspaces of Convex Bodies and Wasserstein Hyperspace Associated with Riemannian Manifold

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ABSTRACT: The topological properties of some Gromov–Hausdorff hyperspaces of convex bodies associated with Riemannian manifold have been investigated, however, the objective of this paper is to provide a comparative studies of some topological properties on hyperspaces of convex bodies and Wasserstein hyperspace associated with Riemannian manifold, where some of these hyperspaces were proved to be AR, ANR, homogeneous, $\mathcal{M}_2 -$ universal and strongly $\mathcal{M}_2 -$ universal.

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The objective of this paper is to provide a comparative study of some topological properties on hyperspaces of convex bodies and Wasserstein hyperspace associated with Riemannian manifold. The Gromov – Hausdorff distance is a tool useful in studying topological properties of collection of Riemannian metrics. Given two compact metric spaces $X\ and\ Y$, the Gromov – Hausdorff distance between $X\ and\ Y$ is written as $d_{\mathcal{GH}}(X,Y)$ and is defined as the infimum of all Hausdorff distances $d_{\mathcal{H}}\big(i(X),j(Y)\big)$, for all metric spaces M and all isometric embeddings $i: X \to M\ and\ j: Y \to M$ that is $d_{\mathcal{GH}}(X,Y) = inf\, d_{\mathcal{H}}\big(i(X),j(Y)\big)$. (Antonyan, 2016). Clearly, If $X\ and\ Y$ are isometric, then the Gromov – Hausdorff distance between $X\ and\ Y$ is zero, that is $d_{\mathcal{GH}}(X,Y) = 0$; it is a metric on the family $\mathcal{GH}$ of isometry classes of compact metric spaces. The metric space $(\mathcal{GH}, d_{\mathcal{GH}})$ is called the Gromov – Hausdorff Hyperspace. When $d_{\mathcal{GH}}$ is well understood, we simply write $\mathcal{GH}$ as the Gromov – Hausdorff hyperspace. Where $d_{\mathcal{GH}}$ is the collection of isometric class of metric space and d_{GH} is metric induced in Gromov sense. A metric is intrinsic if the distance between any two points is the infimum of the length of curves joining the points, that is for any $x, y \in X, d(x,y) = inf\, l(\gamma)$, where l is the length of the curve γ . Any C^∞ Riemannian metric is intrinsic and this property is preserved under Gromov – Hausdorff limit. By [Jiwon, 2019], for $k \in \mathbb{R}$,

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$let\ {}_{curv\geq k}^{\mathcal{GH}}(M)$ be the Gromov – Hausdorff hyperspaces of intrinsic metric of curvature $\geq k\ on\ M$. Let ${}_{sec\geq k}^{GH}(M)$, ${}_{sec>k}^{\mathcal{GH}}(M)$ be the Gromov – Hausdorff hyperspaces of C^{∞} Riemannian metric on M of sectional curvatures $\geq k$, $> k$ respectively. Topological properties of these hyperspaces are largely a mystery which is why it is more common to give ${}_{sec>k}^{\mathcal{GH}}(M), the\ C^{\infty}$ topology resulting in a stratified space whose strata are Hilbert manifolds. In this paper, we compared those properties we have studied on hyperspaces of convex bodies and Wasserstein hyperspace and determine those properties that hold on both hyperspaces and those that are not satisfied on them. Lastly, we made used of Hessian matrix to evaluate principal curvature and degenerate critical point of sublinear function.

Definition of some relevant terms

Definition 1 (Burago, *et.al*, 2001)**:** A manifold M is said to be Reimannian manifold if the Reimannian metric is defined on it.

Definition 2 (Osipov and Oscar, 2017)**:** A subspace $A \subset X$ of topological space X is a subset of X with subspace (Induced) topology.

Definition 3 (Burago, *et.al*, 2001)**:** Given that $X\ and\ Y$ are topological spaces. Then Y is X-manifold if each point $y \in Y$ has a neighborhood N_y homeomorphic to an open subset $O \in X$.

Definition 4 **(**Fernandez and Unzueta , 2018 **):** A subspace A of metric space X is homotopy everywhere dense if there exist a homotopy $h: X \times \mathrm{I} \to \mathrm{X}$ with $h_o = id\ and\ h(X \times (0,1]) \subset A$. If X is an ANR, then $A \subset X$ is homotopy dense iff each map $\mathrm{I}^k \to X\ with\ k \in \omega\ and\ \partial \mathrm{I}^k \subset B$, can be uniformly approximated real boundary by map $\mathrm{I}^k \to B$, where B is a closed subset of X.

Definition 5 (Kiltho and Morawo, 2021): A closed subset $B \subset X$ is $Z - set$, if map $f: B \to \mathrm{X}$ be able to uniformly verge on by a map whose range misses B.

Definition 6 (Valov, 2020)**:** A $\sigma Z - set$ is a countable union of $Z - set$.

Definition 7 (Valov, 2020)**:** An embedding is a $Z - embedding$ if its image is a $Z - set$.

Definition 8 (Valov, 2020)**:** A topological space X is said to be $\sigma - compact$, if it is countable union of compact sets.

Definition 9 **(**Jiwon, 2019**):** A function $f: X \to \mathrm{Y}$ between topological spaces $X\ and\ Y$ is a homeomorphism if f is a bijection and continuous with continuous inverse.

Definition 10 (Memoli, 2012)**:** A hyperspace of Eucledean space $\mathbb{R}^n$ is a set of compacta of $\mathbb{R}^n$ prepared by the Hausdorff metric.

Definition 11 (Antonyan, 2016)**:** A topological space X is a Polish space if it admits a complete metric.

Definition 12 (Higueras and Montano, 2020)**:** A convex body D is a convex set with nonempty interior; i.e., if $int(D) \neq \emptyset$.

Definition 13 (Higueras and Montano, 2020)**:** Let $\mathcal{A}$ be a class of spaces $\mathcal{M}_o\ or\ \mathcal{M}_2$. A space $X\ is\ \mathcal{A} - universal$ if each space in $\mathcal{A}$ is homeomorphic to a closed subset of X.

Definition 14 (Antonyan, 2016)**:** A topological space X is said to be strongly $\mathcal{A} - universal$ if for every open cover $\mathcal{U}\ of\ X, each\ A \in \mathcal{A},$ every compact subset $A_1 \subset A$, and each map $f: A_1 \to X$ which restrict towards a $Z - embedding\ on\ A_1$, there exist a $Z - embedding\ f_1: A_1 \to X\ with\ f_1 \backslash A_1 = f \backslash A_1$ such that $f, f_1\ are\ \mathcal{U} - closed$.

MATERIALS AND METHODS

Here, we make use of Hessian Matrix and sublinear functional to determine some topological properties such as homeomorphism and diffeomorphism of hyperspaces of some convex bodies. Next, we follow the homotopy method to prove the homogeneity, Absolute neighborhood retract (ANR), strongly $\mathcal{M}_2 -$ universality, Polish and $\sigma - compactness$. **(**Victor M. P and Yoav, Z, 2020**)**.

The following are the materials used through out to get our results;

(1)B is a $C^{k,\alpha}$ convex body

(2) $\partial B$is boundary of convex body that has positive Gaussian curvature

(3)h_B is sublinear function

(4)∇^2 is Hessian Matrix (Belegradek, 2017)

(5)$W_p(X) = (W_p(X), d_{w,p})$ is Wasserstein hyperspace.

By Schneider, R (2020), let B be a nonempty compact convex set. The support function h_B is a linear functional $h_B: \mathbb{R}^n \to \mathbb{R}$, which is defined as $h_B(n) = \sup\{\langle t, n\rangle: t \in B\}$. Thus, $h_{B+k}(n) = h_B(n) + \langle n, k\rangle$, for every $k \in \mathbb{R}^n$. Therefore, $int(B) \neq \emptyset$ iff there is $k \in \mathbb{R}^n$ such that $h_B(n) + \langle n, k\rangle$ is positive, for every $n \neq 0$. Then the function h_O is sublinear function because for every positive number m, $h_B(mn) = mh_B(n)$ and $h_B(n + k) \leq h_B(n) + h_B(k)$. Contrariwise, Every sublinear real – valued function on Euclidean plane $\mathbb{R}^n$ is called the support function of single compact convex set in Euclidean plane $\mathbb{R}^n$.

If is a $C^{k,\alpha}$ convex body in Euclidean plane $\mathbb{R}^n$, where $k \geq 1$ and the function $v_B: \partial B \to \mathbb{S}^{n-1}$ designate a Gauss map given by the apparent unit normal, for $k \geq 2$. Then the Gaussian curvature of the $C^{k,\alpha}$ convex body is define as the determinant of the derivative of the Gauss map v_B. Since we know that the Gaussian curvature of the convex body B is ≥ 0 and Gauss map v_B is the set $C^{k-1,\alpha}$. With these character of Gauss map v_B, is a Lipchitz continuous diffeomorphism iff Gaussian curvature of the convex body

$B > 0$ and is a homeomorphism if the convex body B is strictly convex, that is, the boundary of convex body has no line segments. So, if the convex body B is strictly convex, then the restriction $\nabla^2 h_D \backslash_{\mathbb{S}^{n-1}} = {v^{-1}}_B$ so that sublinear function h_D is Lipchitz continuous.

The following are example of sublinear functions;

(1) $f(x_1, x_2, x_3) = 4x_1x_2 + x_3^2$,(2) $f(x_1, x_2, x_3) = x_1^2 + 2x_2x_3^5$ and

(3) $f(x_1, x_2, x_3) = x^3e^{2y} + 3x_3$.

Given that $f: \mathbb{R}^n \to \mathbb{R}$ is a linear functional taking as input a vector $x \in \mathbb{R}^n$ and outputting a scaler $f(x) \in \mathbb{R}$. If all partial order second derivatives of function f exists, then the Hessian matrix is a square $n \times n$ matrix which is usually written and arranged as follow, as in (Keith, O. 2023).

$$\nabla^2 = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix} \qquad (1)$$

RESULTS AND DISCUSSION

In this section, we are going to compare and contrast some topological properties which arise from infinite dimensional topology and convex geometry on hyperspaces of convex bodies and Wasserstein hyperspace. The results below are analogous to (Belegradek, 2018) where the author used convex geometry and infinite dimensional topology to determine the homeomorphism type of some convex compacta in $\mathbb{R}^n$, some convex bodies and also derived a number of properties of their $O(3) -$ quotients.

*Theorem 1***:** (Belegradek, 2019)**:** Given that $\mathfrak{D}$ is an arbitrary hyperspace of $\mathbb{R}^n$ such that $B_d \subset \mathfrak{D} \subset B^{2,\alpha}$. Then the arbitrary hyperspace $\mathfrak{D}$ has absolute retract (AR) structure.

Lemma 1 (Belegradek, 2017)**:** If $B_d \subset X \subset \mathcal{K}_s$, then metric space X is absolute retract (AR) that is homotopy everywhere dense in $\mathcal{K}_s$.

Theorem 2: Given that B_p is hyperspace of convex body with intrinsically infinitely dimensional positive smooth sectional curvature and $\mathcal{K}_s$ is a convex compacta in $\mathbb{R}^n$with steiner point s at the origin, then there exists Wasserstein hyperspace W_p such that $B_p \subset W_p \subset \mathcal{K}_s$, and the Wasserstein hyperspace W_p is homotopy everywhere dense absolute retract (AR) in $\mathcal{K}_s$.

Proof: By Schneider regularization, the hyperspace B_p is everywhere dense in $\mathcal{K}_s$ by Sandwitch theorem, so that $cl(W_p) = \mathcal{K}_s$. Then, the map s homeomorphically takes the hyperspace of convex body B_p and convex compacta $\mathcal{K}_s$ to every convex subset of $C(\mathbb{S}^{n-1})$. Due to the fact that any everywhere dense convex subset of a set in a linear metric space is homotopy everywhere dense. Thus, the hyperspace of convex body B_p is homotopy everywhere dense in $\mathcal{K}_s$ implies that B_p is also homotopy everywhere dense in the Wasserstein hyperspace W_p. Due to the fact that every homotopy dense subset of absolute retract (AR) is absolute retract (AR), this implies that the Wasserstein hyperspace W_p is absolute retract (AR).

The result below is analogous to (Monsuru, A M, *et.al*, 2023),where we proved that the hyperspace of convex body $B_p \in \mathcal{M}_2$.

Theorem 3: The Wasserstein hyperspace $W_p \in \mathcal{M}_2$ if the set $\{W_p - B_p\}$ lies in a subset $A \subset W_p$ such that $\in \mathcal{M}_2$.

Proof: In (Monsuru, AM, *et.al*, 2023, Theorem 3.5), we have shown that $B_p \in \mathcal{M}_2$. Hence,

$W_p = B_1 \cup B_2 \in \mathcal{M}_2$; i.e, their homeomorphic images in any metric space are $F_{\sigma\delta}$, which is true in the space of convex compacta $\mathcal{K}_s$. Provided that for $B_1, B_2 \in \mathcal{M}_2$, $B_1 \cup B_2 \in \mathcal{M}_2$, then, the hyperspace W_p is a $F_{\sigma\delta}$ in $\mathcal{K}_s$, which is complete. Therefore the Wasserstein hyperspace $W_p \in \mathcal{M}_2$.

Theorem 4 (Theorem of Inverse function)(William, S, 2023) **:** Let $f: X \to Y$ be a $C^1 - map$, where X and Y are open subsets of $\mathbb{R}^n$. If $df(x): \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism, then f is a local diffeomorphism near x, i.e, there exist open subset X_x such that $x \in X_x$ and $Y_{f(x)}$ such that $f(x) \in Y_{f(x)}$ such that the restriction $f \backslash_{X_x}: X_x \to Y_{f(x)}$ is a diffeomorphism.

From the *Theorem* 4.4 above, we formulated the following theorems.

*Theorem 5***:** Suppose $k \geq 2$ and $\alpha \in$ I, For any set $D \in W_p$, the following statements are equivalent;

(i) D is a $C^{k,\alpha}$ convex body and the boundary of convex body ∂D has positive Gaussian curvature (ii) $h_D \backslash_{\mathbb{S}^{n-1}}$

is a $C^{k,\alpha}$ and $\nabla^2 h_D\backslash_{\mathbb{S}^{n-1}}$ has no critical point.

Proof**:** $(i) \Rightarrow (ii)$. Since ∂D is a $C^{k,\alpha}$, the Gauss map v_D is a $C^{k-1,\alpha}$. Appearance of Gaussian curvature of the set ∂D means the Gauss map v_D remains a Lipchitz continuous diffeomorphic space C^1, and hence a $C^{k-1,\alpha}$ diffeomorphism by *Theorem 4* . Now, the Hessian gradient of $h_D\backslash_{\mathbb{S}^{n-1}}$ is given as $\nabla^2 h_D\backslash_{\mathbb{S}^{n-1}} = v^{-1}{}_D \Rightarrow h_D\backslash_{\mathbb{S}^{n-1}}$ is a $C^{k,\alpha}$ and there is no critical points in $\nabla^2 h_D\backslash_{\mathbb{S}^{n-1}}$

$(ii) \Rightarrow (i)$. By consider $h_D\backslash_{\mathbb{S}^{n-1}}$ as $C^{k,\alpha}$ and $h_D\backslash_{\mathbb{S}^{n-1}}$ is homogeneity of sublinear function h_D of convex body D, then ,the sublinear function h_D is $C^{k,\alpha}$ on $\mathbb{R}^n\backslash\{o\}$, and in particular, is differentiable in $\mathbb{R}^n\backslash\{o\}$. Then every support hyperplane intersect D in one point, and D is a convex body because, it has non-empty interior and the Gauss map v_D is a homeomorphism. As it was mentioned above, $\nabla^2 h_D\backslash_{\mathbb{S}^{n-1}} = v^{-1}{}_D$, by this, we concluded that $v^{-1}{}_D$ is a $C^{k-1,\alpha}$ diffeomorphism and the boundary of convex body ∂D is a $C^{k-1,\alpha}$ submanifold. This implies that ∂D is a $C^{k,\alpha}$.

Theorem 6: Suppose $r > 0$ and $D \in \mathcal{K}_s$ with $h_D\backslash_{\mathbb{S}^{n-1}} \in C^\infty$, then $D + r\mathbb{B}^n \in W_p$.

Proof: Let $C = r\mathbb{B}^n$. Since $h_{C+D} = h_C + h_D$, we have that h_{C+D} is C^∞. Then, $v^{-1}{}_{C+D} = \nabla^2 h_{C+D}\backslash_{\mathbb{S}^{n-1}}$, this implies that $v^{-1}{}_{C+D}$ is C^∞.

Next, to show that $C + D \in W_p$, we need to verify that v_{C+D} is a C^∞ diffeomorphism, which by *Theorem 4* above, it is equivalent to $v^{-1}{}_{C+D}$ without the critical points, that is $Hessh_{C+D} = (Hessh_C + Hessh_D) \geq 0$. Then the Hessian matrix of any convex function C^2 is nonnegative, so $Hessh_C \geq 0$, this implies that $Hessh_D \geq 0$, due to the fact that $\nabla^2 h_{r\mathbb{B}^n}(x) = rx/||x||$ is a diffeomorphism on boundary sphere $\mathbb{S}^{n-1}$. This proves that $C + r\mathbb{B}^n \in W_p$.

In the next result, we used method of (Arhangel'skill, A.V., and vann Mill, J. 2020) to explain the arbitrary hyperspace $\mathfrak{D}$. Further, we state condition for which this hyperspace $\mathfrak{D}$ is $\mathcal{M}_2-$ universal and topologically homogeneous. We make use of the following Lemma as tools which guides us to our result obtained.

Lemma 2 (Belegradek, 2017)**:** If $B_d \subset X \subset \mathcal{K}_s$, then metric X is an absolute retract (AR) that is homotopy everywhere dense in $\mathcal{K}_s$.

Lemma 3 (Banakh, T.,2021): Let C be a class of spaces , M be absolute neighborhood retract (ANR) and metric space X be a homotopy everywhere dense subset of M such that the metric space X has strong discrete approximation property (SDAP). If for some pair (K, C), the pair (M, X) is strongly (K, C) – universal, then the metric space X is strongly $C-$ universal.

Lemma 4 (Belegradek, I, 2017)**:** The hyperspace B_d is strongly $\mathcal{M}_2 -$ universal.

Theorem 7: Suppose W_p is hyperspace of convex bodies with intrinsically C^∞ boundary metrics and $\mathfrak{D}$ be an arbitrary hyperspace such that W_p is G_δ in $\mathfrak{D}$. Then (i) $W_p \in \mathcal{M}_2$ and is topologically homogeneous (ii) arbitrary hyperspace $\mathfrak{D}$ is strongly $\mathcal{M}_2-$ universal.

Proof: Let W_p^∞ designate the set W_p with C^∞ topology. In this topology, the Gaussian curvature of any convex body $\mathfrak{D} \in W_p^\infty$ are not the same continuously. Thus, the set W_p^∞ remains exactly the subset of hypersurfaces of positive Gaussian curvature in the space of all compact C^∞ hypersurfaces in the Eucledean space $\mathbb{R}^n$ prepared by infinite dimensional C^∞ topology. The later space is Polish space and any open subset of a Polish space is Polish space, hence the set W_p^∞ is a Polish space. Then, for $\sigma \in \{0, \infty\}$, let $\beta^\sigma(S^{n-1})$ denote the set $C^\infty(S^{n-1})$ equipped with the C^σ topology. Let $\mathbb{s}^\infty: W_p^\infty \to \beta^\sigma(S^{n-1})$ designate the map that links to a convex body its support function, i.e, $\mathbb{s}^\infty = \mathbb{s}$ as maps of sets. Similarly to $\mathbb{s}$, the map $\mathbb{s}^\infty$ is a topological embedding because the support function in the Wasserstein extensor W_p equivalents the distance to O from the support hyperplane, and both the tangent plane and the distance to the set O diverse in the infinite dimensional C^∞ topology, as set O lies in the interior of each set in the Wasserstein extensor W_p. Since W_p^∞ is Polish space, its homeomorphic $\mathbb{s}^\infty$ - image is G_δ, so that the unity map $\beta^\infty(\mathbb{s}^{n-1}) \to \beta^\sigma(\mathbb{s}^{n-1})$ takes every G_δ subset to a space in $\mathcal{M}_2$. Thus, $W_p \in \mathcal{M}_2$.

Next, since the hyperspace

$$W_p(X) = \left(P_p(X), d_{w,p}, convex\ bodies\ D \in \mathcal{K}_s\ with\ intrisically\ C^\infty\ boundary\ metrics\ \right).$$

By (Valov, 2020) , it shows that the map $\mathbb{s}^\infty: W_p^\infty \to \beta^\infty(\mathbb{s}^{n-1})$ is fixed point. Therefore, W_p is topologically homogeneous.

Also, *Lemma 3* above suggests that absolute neighborhood retract (ANR) together with strong discrete approximation property (SDAP) is strongly $\mathcal{M}_2 -$ universal if and only if it contains a strongly $\mathcal{M}_2 -$ universal homotopy dense G_δ subset. By assumption, W_p is G_δ in arbitrary hyperspace $\mathfrak{D}$. Then the space W_p is strongly $\mathcal{M}_2 -$ universal, due to *Lemma 4* above. By *Lemma 2*, the arbitrary hyperspace $\mathfrak{D}$ is an absolute neighborhood retract (ANR) with strong discrete approximation property (SDAP) and W_p is homotopy everywhere dense in X. Hence the arbitrary hyperspace $\mathfrak{D}$ is strongly $\mathcal{M}_2 -$ universal. This ends the proof.

Lemma 5 **(**Victor and Yoav, 2020**):** The Wasserstein hyperspace is not locally compact.
We are going to make use of *Lemma 5* above to prove that Wasserstein hyperspace W_p is not $\sigma - compact$, as shown in the *Theorem 8;*

Theorem 8: The Wasserstein hyperspace $W_p(X)$ of metric space X is not $\sigma - compact$.
Proof: If K is compact subset of $W_p(X)$, then, $intW_p(X) = \emptyset$ by *Lemma 5* above. A countable union of compact sets has an empty interior, that is $int(\bigcup K) = \emptyset$ (hence, cannot be equal to the entire Wasserstein space $W_p(X)$). By Baire property, the completeness of Wasserstein space $W_p(X)$ hold. This ends the proof.

Note that, the determinant of the Hessian matrix, when evaluated at a critical point of function f, is called the Gaussian curvature of the function considered as a manifold. The eigenvalues of the Hessian matrix at that point are the principal curvatures of the function while eigenvectors are the principal directions of curvature.
Next we are going to make use of above concepts, and equation (1) in the methodology above to evaluate examples below for principal curvatures and degenerate's critical points.

Example 1: Find the principal curvature of the scaler valued function $f(x_1, x_2, x_3) = 4x_1x_2 + x_3^2$
Solution: Given that $f(x_1, x_2, x_3) = 4x_1x_2 + x_3^2$.
Use the Hessian Matrix notation as illustrated in the methodology (equation 1) above, we have $\frac{\partial^2 f}{\partial x_1^2} = \frac{\partial}{\partial x_1}(4x_1) = 4$, $\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1}(4x_1) = 4$, $\frac{\partial^2 f}{\partial x_1 \partial x_3} = \frac{\partial}{\partial x_1}(2x_3) = 0$, $\frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{\partial}{\partial x_2}(4x_2) = 4$, $\frac{\partial^2 f}{\partial x_2^2} = \frac{\partial}{\partial x_2}(4x_1) = 0$, $\frac{\partial^2 f}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_2}(2x_3) = 0$, $\frac{\partial^2 f}{\partial x_3 \partial x_1} = \frac{\partial}{\partial x_3}(4x_2) = 0$, $\frac{\partial^2 f}{\partial x_3 \partial x_2} = \frac{\partial}{\partial x_3}(4x_1) = 0$ and $\frac{\partial^2 f}{\partial x_3^2} = \frac{\partial}{\partial x_3}(2x_3) = 2$.

So, $\nabla^2 = \begin{pmatrix} 4 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, $|\nabla^2| = \begin{vmatrix} 4 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 2 \end{vmatrix} =$
$4\begin{vmatrix} 0 & 0 \\ 0 & 2 \end{vmatrix} - 4\begin{vmatrix} 4 & 0 \\ 0 & 2 \end{vmatrix} = -64$
$\nabla^2 = \begin{pmatrix} 4-\lambda & 4 & 0 \\ 4 & -\lambda & 0 \\ 0 & 0 & 2-\lambda \end{pmatrix} =$
$(4-\lambda)\begin{vmatrix} -\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} - 4\begin{vmatrix} 0 & 0 \\ 0 & 2-\lambda \end{vmatrix} =$
$(4-\lambda)(\lambda^2 - 2\lambda) = 0$, which yield $\lambda = 0{,}0{,}2$.

Therefore, Principal curvatures are $0{,}0, and\ 2$.

Lemma 6 (Keith, 2023): If f is a sublinear function such that $|\nabla^2| = 0$, then the sublinear function f has degenerates critical points.

We are going to demonstrate application of *Lemma 6* above in the *Example 2* below;
Example 2: Given that $f(x_1, x_2, x_3) = x_1^3 e^{2x_2} + 3x_3$,
$\frac{\partial^2 f}{\partial x_1^2} = \frac{\partial}{\partial x_1}(3x_1^2 e^{2x_2}) = 6x_1 e^{2x_2}$, $\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1}(x_1^3 2e^{2x_2}) = 6x_1^2 e^{2x_2}$,

$\frac{\partial^2 f}{\partial x_1 \partial x_3} = \frac{\partial}{\partial x_1}(3) = 0$, $\frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{\partial}{\partial x_2}(3x_1^2 e^{2x_2}) = 6x_1^2 e^{2x_2}$, $\frac{\partial^2 f}{\partial x_2^2} = \frac{\partial}{\partial x_2}(x_1^3 2e^{2x_2}) = 4x_1^3 e^{2x_2}$

$\frac{\partial^2 f}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_2}(3) = 0$, $\frac{\partial^2 f}{\partial x_3 \partial x_1} = \frac{\partial}{\partial x_3}(3x_1^2 e^{2x_2}) = 0$, $\frac{\partial^2 f}{\partial x_3 \partial x_2} = \frac{\partial}{\partial x_3}(2x_1^3 e^{2x_2}) = 0$,

$$\frac{\partial^2 f}{\partial x_3^2} = \frac{\partial}{\partial x_2}(3) = 0$$
$$\nabla^2 = \begin{pmatrix} 6x_1 e^{2x_2} & 6x_1^2 e^{2x_2} & 0 \\ 6x_1^2 e^{2x_2} & 4x_1^3 e^{2x_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$|\nabla^2| = 6x_1 e^{2x_2} \begin{vmatrix} 4x_1^3 e^{2x_2} & 0 \\ 0 & 0 \end{vmatrix} - 6x_1^2 e^{2x_2} \begin{vmatrix} 6x_1^2 e^{2x_2} & 0 \\ 0 & 0 \end{vmatrix} = 0$$

Therefore, $|\nabla^2| = 0$, which shows that the function $f(x_1, x_2, x_3)$ has degenerate critical points

Conclusion: This paper compared some Topological properties on hyperspaces of convex bodies and Wasserstein hyperspace associated with Riemannian Manifold where we see that those spaces share same structures except $\sigma - compactness$ that failed on Wasserstein hyperspace. Also, we used Hessian matrix to determine the principal curvature and degenerate critical point of sublinear function.

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