PRINT ISSN 1119-8362
Electronic ISSN 2659-1499

Full-Text Available Online At
https://www.ajol.info/index.php/jasem
https://www.bioline.org.br/ja

J. Appl. Sci. Environ. Manage.
Vol. 28 (11B Supplementary) 3815-3822 Nov. 2024

Gromov - Hausdorff Convergence and Topological Stability for Actions on Wasserstein Hyperspaces

***[1]MORAWO, MA; [2]DANYARO, M L; [3]BAILEY, AS; [4]AZEEZ, KY**

*[*1]Bells University of Technology, Ota, Ogun State, Nigeria*
[2]College of Agriculture Science and Technology, Gujba, Yobe State, Nigeria
[3]Karl Kumm University, Vom, Plateau State, Nigeria
[4]Obafemi Awolowo University, Ile – Ife, Osun State, Nigeria

**Corresponding Author Email: morawomonsuruajibola@gmail.com*
***ORCID: https://orcid.org/0009- 0008-7928-7468*
**Tel: +2348053616625*

Co-Authors Email: *lawandanyaro@gmail.com; abbasbailey1@gmail.com; Azeezyetunde20@gmail.com*

ABSTRACT: In this paper, we make use of topological stability from Gromov – Hausdorff view point to establish the Gromov – Hausdorff convergence and stability of induced actions on Wasserstein hyperspaces between maps.

DOI: https://dx.doi.org/10.4314/jasem.v28i11.40

Cite this Article as: MORAWO, M. A; DANYARO, M. L; SALEH, A. B; AZEEZ, K. Y. (2024). Gromov-Hausdorff Convergence and Topological Stability for Actions on Wasserstein Hyperspaces. *J. Appl. Sci. Environ. Manage.* 28 (11B Supplementary) 3815-3822

Dates: Received: 21 September 2024; Revised: 27 October 2024; Accepted: 04 November 2024 Published: 30 November 2024

Keywords: topological stability; Hausdorff convergence; Hausdorff distance; Wasserstein distance; Metric Measure space.

We know that, when a metric is induced with Gromov–Hausdorff distance $d_{\mathcal{GH}}$, the family of all isometry classes of compact metric spaces X *and* Y, say, is a complete and separable metric space. Gromov-Hausdorff distance on metric spaces was formally presented in (Gromov, 1981). Since then, his work has been applied in studying convergence and collapsing theory in a field called Riemannian geometry, see (Cheeger and Tobias, 1996). Particularly, (Cheeger, *et.al*, 1992) introduced the notion of equivalent Gromov-Hausdorff convergence for isometric actions of topological groups on Riemannian manifolds to study collapsing of Riemannian manifold under bounded curvature and diameter, and fundamental groups of almost negatively curved manifolds.

Conversely, from the concepts of Perelman's stability results in geometry, Alexandrov, we understand that for every $k \in \mathbb{R}, n \in \mathbb{N}$, every compact Alexandrov n-space Y of curvature $\geq k$ with $d_{\mathcal{GH}}(X, Y) < \varepsilon$ is certainly homeomorphic to metric space X, and every ε-Gromov-Hausorff approximation can be estimated by a homeomorphism map (Nhan – Phu and Chung, 2019). (Rong and Xu, 2012) explored this idea to study stability of exponential Lipschitz and co-Lipschitz maps in Gromov-Hausdorff topology. Recently, (Arbieto and Morales, 2017) established stability under Gromov-Hausdorff topology for expansive maps having pseudo-orbit tracing property, see (Dong, 2021). Combining the ideas of (Dong, 2018) and (Nhan – Phu and Keonhee, 2018); (Arbieto and Morales, 2017) result has been stretched for such

**Corresponding Author Email: morawomonsuruajibola@gmail.com*
***ORCID: https://orcid.org/0009- 0008-7928-7468*
**Tel: +2348053616625*

actions of a finitely generated group G in (Metzgeretal, 2020) and (Dong, 2021). In another dimension, several stability results of Gromov-Hausdorff topology were also proved for certain isometric actions, see (Inavov and Tuizhilin, 2019), (Sturm, 2020) and (Victor et al, 2021,). In this paper, results of stability under Gromov-Hausdorff distance for compact metric spaces X, Y and Z were established and idea of (Facundo, M and Zhengchao, X 2021) were used to establish Gromov – Hausdorff Convergence results for isometric actions on Wasserstein hyperspaces of topological groups associated with compact metric spaces.

Definition of Some Important Terms

Definition 1 (Donjuan *et.al*, 2021)**:** Suppose X is a metric space with metric d and let $\varepsilon > 0$. A subset $S \subseteq X$ is called an ε-net if $B'_{\varepsilon}(S) = X$, i.e for every $x \in X$, $\exists s \in S$ such that $d(x, s) \leq \varepsilon$.

Definition 2 (Jiwon, 2019)**:** Given that Z is a metric space with metric C and let X, Y be subsets of Z. The Hausdorff distance between X and Y, denoted by $d_{\mathcal{H}} = \inf(\varepsilon) > 0$ such that $X \subset B'_{\varepsilon}(Y) \subset B'_{\varepsilon}(X)$.

Definition 3 (Atonyan, 2020)**:** Let X and Y be metric spaces. The Gromov-Hausdorff ($\mathcal{GH}$) distance between X and Y, denoted by $d_{\mathcal{GH}}(X, Y)$, is the $\inf(r) > 0$ such that $\exists$ a metric space Z with metric d and its subspaces X' and Y' being isometric to X and Y respectively such that $d_{\mathcal{H}}(X', Y') < r$.

Definition 4 (Atonyan, 2020)**:** The Gromov - Hausdorff distance, $d_{\mathcal{GH}}$ is a metric on the set of all isometry classes of compact metric spaces.

Definition 5 (Inavov and Tuzhilin, 2019)**:** Let (X, d_X) and (Y, d_Y) be metric spaces and let $\varepsilon > 0$. An ε-isometric map between X and Y is a map $f: X \to Y$ satisfying $\left|d_Y\left(f(x_1), f(x_2)\right) - d_Y(x_1, x_2)\right| \leq \varepsilon$ for every $x_1, x_2 \in X$. We call a map $f: X \to Y$ an ε-isometry if it is an ε-isometric map and $Y = B'_{\varepsilon}\left(f(X)\right)$. In this case, the map f is also called an ε-$\mathcal{GH}$ approximation from X to Y.

Definition 6 (Nhan-phu, 2019)**:** For any ε-$\mathcal{GH}$ approximation $f: X \to Y$, there is an approximation inverse $f': Y \to X$ made, for example; Let $y \in Y$, and $x \in X$ such that $d_Y(f(x), y) \leq \varepsilon$ and approximate inverse of the function $f: X \to Y$ is defined as $f'(y) \coloneqq x$. Then, the approximate inverse $f': Y \to X$ is a 3ε-$\mathcal{GH}$ approximation. From the construction of approximate inverse f', it is clear that

$sup_{x \in X} d_X\left(x, (f' o\ f)(x)\right) \leq 2\varepsilon$ and
$sup_{y \in Y} d_Y\left(y, (f\ o\ f')(y)\right) \leq \varepsilon$.

For every $\varepsilon > 0$, if $d_{\mathcal{GH}}(X, Y) < \varepsilon$ then $\exists 2\varepsilon$-$\mathcal{GH}$ approximation from X to Y; and $\exists$ an ε-$\mathcal{GH}$ approximation $f: X \to Y$ so that $d_{\mathcal{GH}}(X, Y) < 2\varepsilon$.

Definition 7 (Nhan-phu, 2019)**:** Let X and Y be metric spaces with metric d . We define an alternative $\mathcal{GH}$-distance between X and $Y d_{\mathcal{GH}}(X, Y)$, as follows; $d_{\mathcal{GH}}(X, Y) \coloneqq inf\{\varepsilon > 0 : there\ are\ \varepsilon - \mathcal{GH}\ approximations\ f: X \to Y, g: Y \to X\}$ if the infimum exists, and $d_{\mathcal{GH}}(X, Y) = \infty$, if the infimum does not exist.

Definition 8 (Nhan-phu, 2019)**:** Suppose X and Y are metric spaces and $\varepsilon, \delta > 0$. Then, X and Y are (ε, δ)-approximations of each other if there exists an ε-net $\{x_1, .., x_m\}$ in X and an ε-net $\{y_1, \dots, y_m\}$ in Y satisfying;

$\left|d_X\left(x_i, x_j\right) - d_Y\left(y_i, y_j\right)\right| < \delta$, for every $1 \leq i, j \leq m$.

Definition 9 (Funcondo, M and Zhengchao, X, 2021)**:** Let X a metric space and $W_p(X) = \left(P_p(X), d_{w,p}\right)$. Then $W_p(X)$ is l^p – Wasserstein hyperspace of X. Note that when X is compact, $P_p(X) = P(X)$ for any $p \in [1, \infty)$, where $P(X)$ is the family of all Borel probability measures on metric space X.

Definition 10 (Panareto, 2020): Optimal transport is a process of transferring one probability distribution to another.

MATERIALS AND METHODS

According to Dong (2021). The Gromov – Hausdorff distance between two maps $g: X \to X$ and $h: Y \to Y$ of metric spaces X and Y is defined and designate by

$$d_{\mathcal{GH}}(g, h) = inf\left\{\begin{matrix}\epsilon > 0 : \exists\ \epsilon - isometrics\ i: X \to Y\ and \\ j: Y \to X\ such\ that\ d(hoi, iog) \\ < \epsilon\ and\ d(joh, goj) < \epsilon\end{matrix}\right\} \dots (1)$$

From equation (1), we get a notion of convergence for maps: If X_L is a sequence of metric spaces, we say that a sequence of maps $h_L: X_L \to X_L$ converges to map $g: Y \to Y$ of a metric space X, shortly designated as $h_L \to g$, if $\lim_{L \to \infty} d_{\mathcal{GH}}(g, h_L) = 0$. But, $h_L \to g$ does not indicate that map h_L converges to g in the sence of Gromov. In fact, the constant sequence $h_L = g$ always satisfies $h_L \to g$ while h_L converges to g in the Gromov sense only if g is continuous. By (Donjuan *et.al*, 2021) let (X, d) be a metric space. For any $p \in [1, \infty)$, the set of probability Borel measures μ is designated by $P_p(X)$, which satisies that there exists some $x_n \in X$ such that $\int_X d^p(x, x_n) d\mu(x) < \infty$. Note that if X is bounded, then $P_p(X)$corresponds with $P(X)$, the set of all probability Borel measures of X. For every probability Borel measures μ, v on X, the set of all probability Borel measures on $X \times X$ with marginal μ and v is designated by$\Pi(\mu, v)$. This means that $\pi \in \Pi(\mu, v)$ if and only if π is a Borel subsets $A, B \in X$.

By (Facundo, M and Zhengchao, X, 2021), for every $p > 0$, every $\mu, v \in P_p(X)$, and $\pi \in \Pi(\mu, v)$, $M_p(\pi) = \int_{X\times X} d^P(x_1, x_2) d\pi(x_1, x_2)$ and the map W^p is defined on product $P_p(X) \times P_p(X)$ by $W_p(\mu, v) = \inf_{\pi\in\Pi(\mu,v)} M_p(\pi)$, where $\mu, v \in P_p(X)$ map W_p defined a metric on $P_p(X)$ (Xu, *et.al*, 2019). Note that if X is compact, then $P_p(X)$ is also compact (Facundo M and Zhengchao, X, 2021), then, for every $\mu, v \in P_p(X)$, the set of all $\pi \in \Pi(\mu, v)$ such that $M_p(\pi_o) = \inf_{\pi\in\Pi(\mu,v)} M_p(\pi)$could be designated by $Opt_p(\mu, v)$. So, if X is a polish space endowed with a metric d, then $Opt_p(\mu, v) \neq \emptyset$ for every $\mu, v \in P_p(X)$.

By (Demetci *et.al*, 2020), let (X, d_X) and (Y, d_Y) be compact metric spaces and $\phi: X \to Y$ be a Borel map. Then, we have induced (pushforward) map $\phi_*: P(X) \to P(Y), \mu \to \phi_*\mu$, where $\phi_*\mu(A) = \mu(\phi^{-1}(A))$, for every Borel set $A \subset Y$.
We are going to consider the following properties as in (Alexander, *et.al*, 2018)

For every map $f: X \to X$ and $g: Y \to Y$ of the metric spaces X and Y respectively to achieve some of our results:
(i) If $X = Y$, then $d_{\mathcal{GH}}(g, h) \leq d(g, h)$.
(ii) $d_{\mathcal{GH}}(X, Y) \leq d(f, g)$ and $d_{\mathcal{GH}}(X, Y) = d_{\mathcal{GH}}(Id_X, Id_Y)$ where Id_X and Id_Y are identity of X and Y respectively.
(iii) If X and Y are compact and h is isometry, then $d_{\mathcal{GH}}(g, h) = 0$.
(iv) Symmetry property hold, that is, $d_{\mathcal{GH}}(g, h) = d_{\mathcal{GH}}(h, g)$.
(v) For any map $t: T \to T$, on any metric space T, one has the triangle inequality $d_{\mathcal{GH}}(g, h) \leq d_{\mathcal{GH}}(g, t) + d_{\mathcal{GH}}(t, h)$.
(vi) Definite property hold, $d_{\mathcal{GH}}(g, h) \geq 0$ and if X and Y are bounded, then $d_{\mathcal{GH}}(g, h) < \infty$.
(vii) If X is compact and there is a sequence of isometrics $h_n: Y_n \to Y_n$ such that $\lim_{n\to\infty} d_{\mathcal{GH}}(g, h_n) = 0$ then g is also an isometry.

RESULTS AND DISCUSSION

Suppose X is a metric space and Wasserstein hyperspace, $W_p(X) = (W_p(X), d_{w,p})$. Then, the space $W_p(X)$ is known as the l^p- Wasserstein hyperspace of metric space X. If $P_p(X) = P(X)$, it implies that X is compact. Then, for every $p \geq 1$, the set $P_p(X)$ is the collection of all Borel probability measures on X. The following two Theorems 1 and 2 are existing results concerning Wasserstein hyperspaces, see (Villani, 2018) for proofs.

Theorem 1: For $p \geq 1$, if X is Polish , then the Wasserstein space $W_p(X)$ is also a Polish space.

Theorem 2: For $p \geq 1$, if X is compact, then the Wasserstein space $W_p(X)$ is also a compact space.

Theorem 3 (Mikhailov, 2018)**:** Suppose$X$ and Y are compact metric spaces such that there exists a (non - necessarily compact) metric space Z and isometric embedding $\varphi_X: X \to Z$ and $\varphi_Y: Y \to Z$. Then, we have the following equation of Hausdorff distance with respect to embeddings$\varphi_X: X \to Z$ and $\varphi_Y: Y \to Z$,$d_{\mathcal{H}}^{H(Z)}((\varphi_X)_*(\mathrm{H(X)}), (\varphi_Y)_*(\mathrm{H(Y)}) = d_{\mathcal{H}}^{Z}(\varphi_X(X), \varphi_Y(Y))$

Lemma 1 (Nhan, 2019)**:** Let (X, d_1) and (Y, d_2) be two compact metric spaces and the maps $f, g: X \to Y$ are measurable. Then for every $p \in [1, \infty)$, $\mu \in P(X)$, we have $W_p^p(f_*\mu, g_*\mu) \leq \int_X d_Y^p(f(x_1), g(x_1))\, d\mu(x_1)$
Since the Wasserstein extensor W_p sends the metric space $X \in \mathcal{M}$into Wasserstein hyperspace $W_p(X) \in \mathcal{M}$, then this Wasserstein extensor defines a map from $\mathcal{M}\ to\ \mathcal{M}$ analogously to the case of Gromov – Hausdorff such that $W_p: \mathcal{M} \to \mathcal{M}$. Moreover, the map W_p sending x to Dirac measure $\delta_x \in P(X)$ is a isometric embedding from X into $P(X)$. Therefore, the map $W_p: \mathcal{M} \to \mathcal{M}$ is a metric extensor, which we call the l^p- Wasserstein extensor in the consequence. Inspired by Theorem 3 and Lemma 1, we obtained the following results:

Theorem 1: Given that α and β are actions of a topological group G on a compact metric spaces X and Y, respectively. If $f: X \to Y$ is an ε-measurable $\mathcal{GH}$ − approximation, then for any $p \in [1, \infty)$, the pushforward map $f_*: W_p(X) \to W_p(Y)$ is

$\varepsilon^{\sim}$-measurable $\mathcal{GH}-$ approximation, where

$$\varepsilon^{\sim}=8\,\varepsilon+\sqrt[p]{\{9p(D(X)^{p-1}+D(Y)^{p-1})\varepsilon\}}\ldots\ldots\ (1)$$

Proof: Let $\emptyset_1, \emptyset_2 \in P(X)$ and $\mu \in OPt(\emptyset_1, \emptyset_2)$ be any coupling. Then the pushforward map $(f \times f)_*\mu \in \Pi(f_*\emptyset_1, f_*\emptyset_2)$ and hence

$$d_{w,p}^{Y}(f_*\emptyset_1, f_*\emptyset_2) \le \left(\int_{Y\times Y} d_Y^p(y_1,y_2)d((f \times f)_*\mu)(x_2,y_2)\right) =$$

$$\int_{X\times X} d_Y^p\left(f(x_1), f(y_2)\right)d\mu(x_1,y_1)\ldots\ldots\ldots..(2)$$

As the function

$$t(x)=x^p, x\ge 0\ldots\ldots\ldots\ (3)$$

and by differentiating equation (3), we have

$$t^1(x)=px^{p-1}\ldots\ldots\ldots\ (4)$$

Then, by equation (3) and (4), we have for every $x, y \ge 0$:

$$|x^p-y^p| \le P|x-y|max\{x^{p-1}, y^{p-1}\} \le P|x-y|(x^{p-1}+y^{p-1})\ldots\ldots..(5)$$

Therefore, for every $x_1, y_1 \in X$,Lemma 1 and equation(2) yields

$$\left|d_{w,p}^{Y}\left(f(x_1), f(y_1)\right) - d_{w,p}^{X}(x_1,y_1)\right| \le P\left|d_{w,p}^{Y}\left(f(x_1), f(y_1)\right) - d_{w,p}^{X}(x_1,y_1)\right| \le (d_{w,p-1}^{Y}\left(f(x_1), f(y_1)\right) + d_{w,p-1}^{X}(x_1,y_1))\ldots.(6)$$

Hence,

$$\left|d_{w,p}^{Y}\left(f(x_1), f(y_1)\right) - d_{w,p}^{X}(x_1,y_1)\right| \le PD\left|d_{w,p}^{Y}\left(f(x_1), f(y_1)\right) - d_{w,p}^{X}(x_1,y_1)\right| \le PD\varepsilon\ldots\ldots\ (7)$$

Where D is diameter of $x^{p-1}+y^{p-1}$.

It follows that

$$W_p(f_*\phi_1, f_*\phi_2) \le W_p(\phi_1,\phi_2) + pD\varepsilon\ldots\quad (8)$$

Hence, by equation (8), l^p- Wasserstein of Pushforward of ϕ_1 and ϕ_2 is given as

$$W_p(f_*\phi_1, f_*\phi_2) \le \sqrt[p]{\left(W_p(\phi_1,\phi_2)+pD\varepsilon\right)} \le W_p(\phi_1,\phi_2) + \sqrt[p]{(PD\varepsilon)}\ldots\ldots\ldots\quad (9)$$

Therefore, from equation (9), we have

$$W_p(f_*\phi_1, f_*\phi_2) \le \sqrt[p]{\left(W_p(\phi_1,\phi_2)+pD\varepsilon\right)} \le W_p(\phi_1,\phi_2) + \sqrt[p]{(PD\varepsilon)}\ldots\ldots\ldots...(10)$$

Next, assume g is the inverse of f. Then, in the Gromov sense, let $g: Y \to X$ be the measurable $\mathcal{GH}-$ approximate inverse of f. Then g is a $9\varepsilon-\mathcal{GH}$ approximation from Y to X and if

$$\sup_{x\in X} d^X\left(x, f^1 of(x)\right) \le 4\varepsilon \text{ and } \sup_{y\in Y} d^Y\left(y, fog(y)\right) \le 4\varepsilon.\ \ldots\ldots\ldots(11)$$

If we Apply the same process as stated above, from (10) and (11), we get the following for the l^p-Wasserstein hyperspace for Pushforward of g_* and f_*,

$$W_p\left(g_*(f_*\phi_1), g_*(f_*\phi_2)\right) \le W_p(f_*\phi_1, f_*\phi_2) + \sqrt[p]{(qPD\varepsilon)}\ldots\ldots\ldots..\ (12)$$

As stated in equation above, since $\sup_{x\in X} d^X\left(x, gof(x)\right) \le 4\varepsilon$, by considering *Lemma 1* above, we get

$$W_p(f_*\phi, g_*\phi) \le \int_X d_Y^p\left(f(x), g(x)\right)d\phi(x)\ldots\ldots\ (13)$$

for the l^p- Wasserstein hyperspace for Pushforward of g_* and $f_*$and also, we have the following for the l^p- Wasserstein hyperspace for Pushforward of composition of g_* and f with respect to ϕ_1, ϕ_2,

$$W_p(g_*o\,f)_{*\phi_1,\phi_2} \le 4\varepsilon \text{ and } W_p((gof)_*\phi_1,\phi_2) \le 4\varepsilon\ldots\ldots\ldots.\quad .(14)$$

Therefore, by equations(12), (13), and (14), the l^p- Wasserstein hyperspace between $\phi_1\ and\ \phi_2$ is thus;

$$W_p(\phi_1,\phi_2) \le W_p\left(\phi_1, g_*(f_*\phi_1)\right) + W_p\left(g_*(f_*\phi_1), g_*(f_*\phi_2)\right) + W_p(g_*(f_*\phi_2),\phi_2) \le 8\varepsilon + Wp(f_*\phi_1, f_*\phi_2) + \sqrt[p]{(qPD\varepsilon)}\ldots\ldots\ldots\ (15)$$

Finally, for every $\phi \in p(Y), g \in G$, let $\mu \in \Pi((\beta, g\ o\ f)_*\phi_1(f\ o\alpha, g)_*\phi_1)$. Since for every $g \in G$, $d_{sup}(fo\alpha, g, \beta, g\ of) \le \varepsilon$, Then, by *Lemma 1* with equation (15) above, we have

$$W_p\left((\beta)_*, g\ of_*(\phi_1), f_*o(\alpha)_*, g(\phi_1)\right) = W_p((\beta, gof)_*\phi_1, (fo\alpha, g)_*\phi_1) \le \int_{YXY} d_Y^p(x_2,y_2)d\mu\,(x_2,y_2)$$

$= \int_X d_X^p\big(\beta, gof(x), fo\alpha, g(x_1)\big)d\phi_1(x_1) \le \varepsilon^p < \varepsilon^{\sim p}$, by using equation (1). This ends the proof.

Theorem 2: Let α and β be isometrically conjugated actions of a finitely generated topological group G on compact metric spaces X and Y. For $p \in [1, \infty)$, let $\alpha \in W_p(X)$ and $\beta \in W_p(Y)$, where $W_p(X)$ is $l^p -$ Wasserstein hyperspace of X and $W_p(Y)$ is $l^p -$ Wasserstein hyperspace of Y. Then, α is topologically $\mathcal{GH}$ – stable if and only if β is topologically $\mathcal{GH}$ – stable.

Proof: Let X and Y be compact metric spaces and G be finitely generated group. Let $\alpha \in W_p(X)$ and $\beta \in W_p(Y)$ be isometrically conjugated actions. Here we need to prove that if α is topologically $\mathcal{GH}$ – stable, then, β is topologically $\mathcal{GH}$ – stable.

Since G generated by a finite set F. Then, for a finite generator F of G Let $\varepsilon > 0$ and $\delta > 0$ be given by the definition of the topological $\mathcal{GH}$ – stability of α with respect to F. Provided that α and β are isometrically conjugated, since α and β are isometric, then, $d_{\mathcal{GH},F}W_p(\alpha, \beta) = 0$ in a Gromov sense. Now, choose$\gamma \in W_p(Z)$ (where $W_p(Z)$ is the $l^p -$ Wasserstein hyperspace of metric space Z such that $d_{\mathcal{GH},F}W_p(\beta, \gamma) < \frac{\delta}{2}$. ……………………………..(16)

From equation (16), the Gromov – Hausdorff distance of $l^p -$ Wasserstein hyperspace between the isometric conjugated actions α and γ under the finite generator set F is

$$d_{\mathcal{GH},F}W_p(\alpha, \gamma) \le 2\left(d_{\mathcal{GH},F}W_p(\alpha, \beta) - d_{\mathcal{GH},F}W_p(\beta, \gamma)\right) < \delta \quad \text{………………………….. (17)}$$

Thus, by the definition of topological stability of action α we have a continuous ε – isometry $i: Z \to X$ such that the composition $\alpha_g oi = io\gamma_g$……….. (18)

for every $g \in G$. Since α and β are isometrically conjugated, we have an isometry $j: Y \to X$ such that $\alpha_g = jo\beta_g oj^{-1}$ for every $g \in G$. By replacing composition in equation (18), we have $jo\beta_g oj^{-1}oi = io\gamma_g, \forall g \in G$.

Let us define $m = j^{-1}oi: Z \to Y$, we have $jo\beta_g om = jomo\gamma_g$, so, $\beta_g om = mo\gamma_g$, for every $g \in G$. Provided that j is an isometry and i is a continuous ε – isometry. Therefore, β is topologically $\mathcal{GH}$ – stable with respect to F. Hence, from equation(17) , we get equation of Gromov – Hausdorff distance of $l^p -$ Wasserstein hyperspace between the isometric conjugated actions α, β and γ under the finite generator set F as$d_{\mathcal{GH},F}^{W(Y)}(\alpha, \beta) \le 2\left(d_{\mathcal{GH},F}^{W(X)}(\alpha, \beta) - d_{\mathcal{GH},F}^{W(Y)}(\alpha, \gamma)\right) < \delta$.

Theorem 3: Given that X, Y and Z are compact metric spaces and $i: X \to Z$ and $j: Y \to Z$ are isometric embeddings. Then, for $p \ge 1$, there is a Borel measurable map $t: X \to Y$suchthat for any $x \in X$, $d_Z\big(x, t(x)\big) \le \delta + \varepsilon$. Then, the Hausdorff distance between isometric embeddings is

$$d_{\mathcal{H}}^{W_p(Z)}\left((i)_*\left(W_p(X)\right), (j)_*\left(W_p(Y)\right)\right) \ge d_{\mathcal{H}}^Z\big(i(X), j(Y)\big).$$

Proof: Let $\{x_n\}_{n\in N}$ be an ε – net of compact metric space X. Suppose $\{x_m\}_{m\in M}$ of X is a Voronoi cell defined as

$$X_m = \{x \in X: d_X(x, x_m)\} = \min_{i \le x \le n} d_X(x, x_i) \quad \text{………………………………..(19)}$$

We can adjust the Voronoi cell (equation 19) in such a way that they will be disjoint. For example, let $Y_1 = X_1$ and $Y_m = X_m - \bigcup_{i=1}^{m-1} X_i$ for $m \ge 1$…….........(20) From equation (20), let $\{X_m\}_{1\le m\le n}$ denote the cell after our adjustment. Since the map $t: X \to Y$ is Borel measurable, the cell is Borel measurable. Then, for each x_m, we let $y_m \in Y$ such that

$$d_z(x_m, y_m) \le \delta \quad \text{……………………………….....(21)}$$

Then, we define the map $\delta: X \to Y$ by mapping x to y_m if $x \in X_m$ which implies that the map $\delta: X \to Y$ is measurable. Moreover, for such $x \in X_m$, there is an ε – net $\{x_k\}_{1\le m\le n}$, and one has that

$$d_Z(x, x_m) = d_X(x, x_m) \le \varepsilon \quad \text{………………… (22)}$$

Therefore, by triangle inequality, using equations (21) and (22), we have $d_Z\big(x, t(x)\big) \le d_Z(x, x_m) + d_Z(x_m, y_m) \le \delta + \varepsilon$, which proves the first statement.

To provide proof for the second statement, recall that from the definition of Hausdorff distance, we have;

$$d_{\mathcal{H}}^Z(X, Y) = \max\{\sup_{x\in X} \inf_{y\in Y} d_Z(x, y), \sup_{y\in Y} \inf_{x\in X} d_Z(x, y)\} \quad \text{…….(23)}$$

Then, without loss of generality, we assume that the Hausdorff distance

$$d_{\mathcal{H}}^{Z}(X,Y) = \sup_{y\in Y}\inf_{x\in X} d_Z(x,y)\ldots\ldots\ldots\ldots\ldots(24)$$

Suppose $x \in X, and\ y \in Y$ such that the distance between x_1 and y_1 is defined as

$$d_Z(x_1,y_1) = \sup_{y\in Y}\inf_{x\in X} d_Z(x,y)\ldots\ldots\ldots\ldots\ldots (25)$$

Since metric spaces X and Y are compact, this guarantee the validity of existence of the point (x_1, y_1), for $x_1 \in X$ and $y_1 \in Y$. Then, consider Dirac delta measure $\delta_{y_1} \in p(Y)$ and any $\mu X \in X$. Then, we can identify the Dirac delta measure δ_{y_1} with $(\phi_Y)_*\delta_{y_1} \in p(Z)$ and μX with $(\phi_Y)_*\mu X \in p(Z)$. Then, for any $p \geq 1$, we have Wasserstein distance between Dirac delta measure δ_{y_1} and measure $\mu X \in X$ as;

$$d_{W,p}^{Z}(\delta_{y_1}, \mu X) = \int_X d_Z(x,y_1)^p d\mu X(x) \geq d_Z(x_1,y_1)\ldots\ldots\ldots\ldots\ldots\ldots (26)$$

Then, for $p > 1$, we have $d_{W,p>1}^{Z}(\delta_{y_1}, \mu X) = \sup_{x\in sup(\mu X)} d_Z(x_1,y_1)\ldots\ldots\ldots\ldots (27)$

Therefore, from equation (24), (25),(26)and (27), we have, for $p \geq 1$,

$$d_{\mathcal{H}}^{Wp(Z)}(Wp(X), Wp(Y)) \geq \inf_{\mu X\in p(X)} d_{W,p}^{Z}(\delta_{y_1}, \mu X) \geq d_Z(x_1,y_1) = d_{\mathcal{H}}^{Z}(x_1,y_1).$$

This proves the theorem.

Theorem 4: For any $X \in \mathcal{M}$ and a metric space Y which is not necessarily compact. For any $p \geq 1$, if $f: X \to Y$ is an isometric embedding, then the map $f_*: W_p(X) \to W_p(Y)$ is isometric embedding and the homeomorphism of X which is isometric to a topologically $\mathcal{GH}$ – stable is itself topologically $\mathcal{GH}$ – stable.

Proof: For $\alpha_1, \alpha_2 \in p(X)$, let $\mu \in C(\alpha_1, \alpha_2)$ be any coupling, consider pushforward map $\mu_* = (f \times f)_*\mu$, where $f \times f: X \times X \to Y \times Y$ takes (x, x_1) to $(f(x), f(x_1)) \Rightarrow f_*\alpha_1, f_*\alpha_2 \in P_p(Y)$ and the pushforward $\mu_* \in C(f_*\alpha_1, f_*\alpha_2)$. Then, for $p \geq 1$, we have the following:

$$d_{w,p}^{Y}(f_*\alpha_1, f_*\alpha_2) \leq \sqrt[p]{\int_{Y\times Y} d_Y^p(y_1,y_2)d\mu_*(y_1,y_2)}$$

$$= \sqrt[p]{\int_{Y\times Y} d_Y^p f(x_1), f(x_2) d\mu_*(x_1,x_2)}$$

$$= \sqrt[p]{\int_{X\times X} d_X^p(x_1,x_2)d\mu_*(x_1,x_2)}\ldots\ldots\ldots\ldots (28)$$

The equality in equation (28) holds due to the fact that f is an isometric embedding. Now, for $p > 1$, $(f \times f)(sup(\mu)) = sup(\mu_*)$, provided that f is an isometric embedding. Then,

$$d_{w,p>1}^{Y}(f_*\alpha_1, f_*\alpha_2) \leq \sup_{(y_1,y_2)\in \sup(\mu_*)} d_Y(y_1,y_2)$$
$$= \sup_{(y_1,y_2)\in f\times f\sup(\mu)} d_Y(y_1,y_2)$$
$$= \sup_{(x_1,x_2)\in f\times f\sup(\mu)} d_Y(y_1,y_2)$$
$$.= \sup_{(x_1,x_2)\in \sup(\mu)} d_X(x_1,x_2)\ldots\ldots (29)$$

By taking the infimum over $\mu \in C(\alpha_1, \alpha_2)$ in equation (29), we have , for

$$p \geq 1,\ d_{w,p}^{Y}(f_*\alpha_1, f_*\alpha_2) \leq d_{w,p}^{X}(\alpha_1,\alpha_2)\ldots\ldots\ldots(30)$$

Provided that f is continuous, $f(X)$ is compact in Y and hence closed, we have

$$d_{w,p}^{Y}(f_*\alpha_1, f_*\alpha_2) = d_{w,p>1}^{f(X)}(f_*\alpha_1, f_*\alpha_2) \leq d_{w,p}^{X}(\alpha_1,\alpha_2)\ldots\ldots\ldots\ldots\ldots(31)$$

Provided that $f^{-1}: f(X) \to X$ is also an isometric embedding, we have

$$d_{w,p}^{X}(\alpha_1,\alpha_2) = d_{w,p}^{X}(f_*^{-1} o f_*\alpha_1, f_*^{-1} o f_*\alpha_2) \leq d_{w,p}^{f(X)}(f_*\alpha_1, f_*\alpha_2).\ \ldots\ldots\ldots\ldots\ldots (32)$$

Therefore, from equation (31) and (32), we have

$$d_{w,p}^{X}(\alpha_1,\alpha_2) = d_{w,p}^{Y}(f_*\alpha_1, f_*\alpha_2)\ldots\ldots\ldots\ldots (33)$$

Thus, f_* is an isometric embedding.

Next, in the case of the Gromov-Hausdorff stability, for $P \geq 1$, let $f: X \to X$ and $g: W_p(X) \to W_p(X)$ be homeomorphism of compact metric space X and its Wasserstein hyperspaces $W_p(X)$ respectively. Suppose f and g are isometric while f is topologically Gromov-Haursdorff stable, fix another isometry $h: W_p(X) \to X$ such that $f = hogoh^{-1}$. Let $\varepsilon > 0$ and $\delta > 0$ be given by the topological Gromov-Hausdorff stability of f. Let $h^1: Y^1 \to Y^1$ be a homeomorphism of a compact metric space Y^1 such that

$$d_{\mathcal{GH}}^{W_p(X)}(g, h^1) < \frac{\delta}{2}\ldots\ldots\ldots\ldots\ldots\ldots(34)$$

Then,

$d_{\mathcal{GH}}^{W_p(X)}(f,h) \leq 2\left(d_{\mathcal{GH}}^{W_p(X)}(f,h) + d_{\mathcal{GH}}^{W_p(X)}(g,h^1)\right) < 2.\frac{\delta}{2} = \delta$……………………………………...(35)

Thus, by the choice of δ in equation (34) and (35), there is a continuous ε – isometry $v: Y^1 \rightarrow X$ such that $fov = voh^1$ and $hogoh^1 = voh^1$. Then, by defining $v^1 = hov$, we get a continuous ε – isometry $v^1: Y^1 \rightarrow W_p(X)$ which satisfying $gov^1 = v^1oh^1$. Therefore, g is topologically Gromov-Hausdorff stable. This ends the proof. In the next result, we are going to follow the technique of *Lemma 2* below to achieve our aim in the Gromov sense.

Lemma 2 (Victor MP and Yoav, Z, 2020)**:** The Wasserstein space $W_p(X)$ is complete and separable if and only if X does.

Theorem 5: Let X_1 and X_2 be two separable metric spaces. Let $\varepsilon > 0$ and $f: X_1 \rightarrow X_2$ be an $\varepsilon - \mathcal{GH}$ approximation. If $g: X_2 \rightarrow X_1$ be the inverse $\varepsilon - \mathcal{GH}$ approximation of f. Then, (i) for each $d_{X_1}\left(f(x_1), f_1(x_1)\right) \leq 2\varepsilon$, and $\sup_{x_2 \in X_2} d_{X_2}\left(g(x_2), g_1(x_2)\right) \leq 2\varepsilon$, there exist a $5\varepsilon - \mathcal{GH}$ approximation $f_1: X_1 \rightarrow X_2$ such that f_1 is measurable for every $x_1 \in X_1$ (ii) for each $\sup_{x_1 \in X_1} d_{X_1}\left(x_1, (gof)(x_1)\right) \leq 4\epsilon$ and $\sup_{x_2 \in X_2} d_{X_2}\left(x_2, (fog_1)(x_2)\right) \leq 4\epsilon$, there exist a $9\varepsilon - \mathcal{GH}$ approximation $g_1: X_2 \rightarrow X_1$ such that g_1 is measurable.

Proof: Since X_1 is separable metric space, by *Lemma 2***,** there exist a countable everywhere dense subset $\{x_n\}_{n \in N}$ of X_1. We put $B_n = B_\varepsilon^1(x_n)$ and $B_{n+1} = B_\varepsilon^1(x_{n+1}) \setminus \bigcup_{j=1}^n B_\varepsilon^1(x_j)$, for $n \geq 1$. Then, the sequence $\{B_n\}_{n \in N}$ which is a disjoint covering of X_1 and B_n is measurable for every $n \in N$ so that for every $x_1 \in X_1$, $\exists$ a unique $n \in N$ such that $x_1 \in B_n$. We can now define the measurable map $f_1: X_1 \rightarrow X_2$ by $f(x_1) = f(x_n)$ such that for every $x_1 \in B_n$, we have

$d_{X_1}\left(f(x_1), f_1(x_1)\right) = d_{X_1}\left(f(x_1), f(x_n)\right) \leq d_{X_1}(x_1, x_n) + \varepsilon \leq 2\varepsilon$…………………… (36)

Therefore, for every $x_1, x^1 \in X_1$, from equation (36), we get $\left|d_{X_1}\left(f_1(x_1), f_1(x^1)\right) - d_{X_1}(x_1, x^1)\right| \leq \left|d_{X_1}\left(f_1(x_1), f_1(x^1)\right) - d_{X_1}\left(f(x_1), f(x^1)\right)\right| + \left|d_{X_1}\left(f_1(x_1), f(x^1)\right) - d_{X_1}\left(f(x_1), f(x^1)\right)\right| + \left|d_{X_1}\left(f(x_1), f(x^1)\right) - d_{X_1}(x_1, x^1)\right|$

$\leq d_{X_1}\left(f_1(x^1), f(x^1)\right) + d_{X_1}\left(f_1(x_1), f(x_1)\right) + \varepsilon \leq 5\varepsilon$………………………………………… (37)

Therefore, f_1 is a measurable $5\varepsilon - \mathcal{GH}$ approximation from X_1 to X_2.
(ii) provided that X_2 is separable, we can also find $9\varepsilon - \mathcal{GH}$ approximation $g_1: X_2 \rightarrow X_1$ such that g_1 is measurable and $\sup_{x_2 \in X_2} d_{X_2}\left(g(x_2), g_1(x_2)\right) \leq 2\varepsilon$.

For every $x^1 \in X_1$,

$$d_{X_1}\left(x^1, g_1of(x)\right) \leq d_{X_1}\left(x^1, gof(x^1)\right) + d_{X_1}\left(gof(x^1), g_1of(x^1)\right)$$

$$\leq 2\varepsilon + 2\varepsilon = 4\varepsilon \quad \text{…………………………..….(37)}$$

and for every $x_2 \in X_2$.

$$d_{X_2}\left(x_2, fog_1(2)\right) \leq d_{X_2}\left(x_2, fog(x_2)\right) + d_{X_2}\left(fog(x_2), fog_1(x_2)\right) \leq \varepsilon + d_{X_2}\left(g(x_2), g_1(x_2)\right) + \varepsilon \leq 4\varepsilon \quad \text{……………………………………. (38)}$$

Then, by adding equation (37) to (38), we have

$d_{X_1}\left(x^1, g_1of(x)\right) + d_{X_2}\left(x_2, fog_1(2)\right) \leq 4\varepsilon + 4\varepsilon + \epsilon \leq 9\epsilon$.

Therefore, $g_1$is a measurable 9ϵ- $\mathcal{GH}$ approximation from X_2 to X_1.

Conclusion: We concluded that Wasserstein hyperspace is a mapping which sends one probability distribution to another with the help of distance preserving map. So, for this, we conclude that the Wasserstein hyperspace is an isometric space.

REFERENCES

Arbieto, A; Morales. (2017). Topological Stability from Gromov – Hausdorff view point. *Institute de Matematica, University federal do Rio de Janeiro. Discrete and continuous Dynamical systems.* 7 (37): 3531-3543

Abebe, R; Kasy, M (2020). Fairness, Equality and Power in Algorithmic Decision Making. *NeurlPS Workshop on dataset curation and* security, p.1-5

Antonyan, SA (2020). The Gromov – Hausdorff hyperspace of a Eauclidean space. *Advances in Mathematics*, p.1881-1890

Black, E., Yeom; S; Fredrikson. M (2020). Fairness testing via optimal transport. *In proceeding of the 2020 Conference on Fairness, Accountability and Transparency*, p.111-121

Cheeger, J; Tobias, H (1996). Lower bounds on Ricci curvature and the almost rigidity of warped product. *Ann of Math.* 144(2): 189-237.

Dong, M (2018). Group Actions from measure theoretical viewpoint. *Ph.D. thesis, Chungnan National University in Daejeon*

Demetci, P; Rebecca, S; William. S; Ritambhara, S (2020). Gromov – Wasserstein Optimal transport to align single cell multi – Omics data. BiORxiv. p. 321

Facundo, Memoli; Zhengchao (2021). Charaterization of Gromov – type geodesics. *Department of Mathematics and Computer Science and Engineering, The Ohio State University.*

Gromov, M (1998). Groups of polynomial growth and expanding maps. *Inst. Hautes Etudes. Sci. publ, Math.* 53: 53-73

Hongteng, X; Dixin L; Lawrence, C (2019). Scalable Gromov – Wasserstein Learning for Graph Pertitioning and Matching. *In advances in Neural Transformation Processing System,* 3052-3062

Inavov, A; Tuzhilin, A (2019). Isometric group of Gromov – Hausdorff space. *Matematicki Vesnik,* 71(1-2): 123-154

Jiwon, C (2019). Gromov – Hausdorff distance between metric Graph. St Mary's School.2: 120-152

Jeff, C; Kenji, F; Mikhael, G (1992). Nilpotent structures and invariant metrics on collapsed manifolds *J. Amer. Math.* 2(5): 327-372

Kweku, K; Aggrey; Sarah, MB (2020). Measuring Bias with Wasserstein Distances. *NeurlPS Workshop on dataset curation and security.* 1-5

Mikhailov, IA (2018). Hausdorff mapping: 1 Lipschitz and isometry properties. *Moscow University Math. Bull.* 73(6):211-216

Meihua, D (2021). Gromov-Hausdorff stability for Group Actions. *Department of Mathematics, College of Science. Discret and continuous Dynamical systems.* 3(41):1347-1357

Metzger, R; Morale, CA; Thieullen, P (2020). Topological Stability in Set-valued Dynamics. *Discrete contin. Dyn. Syst.* Ser. B22: 1965-1975

Nhan-phu, C; Keonhee (2018). Topological stability and pseudo – orbit tracing property of group actions. *Proc. Amer. Math. Soc.* 146, number 3: 1047-1057

Nhan-phu, C. (2019). Gromov – Hausdorff Distances for dynamical systems. *2010 Mathematics subject classification.* 37C85, 37C40, 37C75. No. 2: 1-20.

Sturm, KT (2020). Adendum – the space of spaces: curvature bounds and gradient flows on the space of metric measure space. *Personal communication*

Victor, D; Natalia, J; Ananda, L (2021). Some Notes On Induced Functions and Group Actions On Hyperspace*2010 Mathematics Subjects Classification. Department De Mathematicas, Facultad de Ciencias, Mexico.* Volume 4, p. 1-23.

Villani, C (2008). Optimal transport: old and new. *Volume 338. Springer Science and Business Media.*

Xiaochum, R; Shicheng, X (2012). Stability of e^{ε} – Lipschitz maps in Gromov Hausdorff topology. *Adv. Math.* 2(231): 774-797