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Application of Two-Parameter Laplace-Type R-transform to Solve Linear Ordinary Differential Equations

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ABSTRACT: There are thirty one-parameter Laplace-type integral transforms, however, two-parameter Laplace-type integral transforms are few. Hence, the objective in the paper is to introduce and apply a new two-parameter Laplace-type integral transform, called the R-transform, to solve linear ordinary differential equations. Linear ordinary differential equations with constant coefficients were solved by taking the inverse R-transform. Also, two important theorems about the R-transform were proved. The theorems furnish us with the essential results for applications. Computations were manually done and confirmed with *Mathematica* software.

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Integral transforms are powerful operational methods for solving linear differential and integral equations (Debnath and Bhatta, 2007). The Laplace transform, introduced by the French mathematician Pierre-Simon Laplace (1749 – 1827), is a one-parameter integral transform. It was systematically developed by the British physicist Oliver Heaviside (1850 – 1925) to simplify the solution of many differential equations that describe physical processes. Laplace-type integral transforms are those that have the exponential kernel. There are, generally, two classes of Laplace-type integral transforms, namely: one-parameter Laplace-type integral transforms and two-parameter Laplace-type integral transforms. The first one-parameter Laplace-type integral transform, called the Elzaki transform, was introduced by (Elzaki, 2011). Since then, more than thirty-one-parameter Laplace-type integral transforms have been proposed by different scholars. These include the Aboodh transform (Aboodh, 2013), the Mahgoub transform (Mahgoub, 2013), the Kamal transform (Abdelilah & Hassa, 2016), the Mohand transform (Mahgoub, 2017), the Sawi transform (Mahgoub, 2019), the Iman transform (Iman, 2023), just to mention few. The first two-parameter Laplace-type integral transform, called the Sumudu transform, was introduced by G.K. Watugala (Watugala, 1993). Since then, only very few two-parameter Laplace-type integral transforms have been proposed, such as the Natural transform (Khan and Khan, 2008), the Shehu transform (Maitama and Zhao, 2019), the ZZ transform (Zafar, 2016), the NE transform (Musta, 2023), etc. Why the two-parameter Laplace-type integral transforms when one-parameter Laplace-type integral transforms can solve many differential and integral equations that describe physical phenomena? Well, by introducing an additional parameter, one can discover new interesting properties of the transforms (two-parameter Laplace-type integral transforms) and extend their range of application. This is mainly because the operational calculus involved in proposing and applying a new two-parameter Laplace-type integral transform is more

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complicated than the case of one-parameter Laplace-type integral transforms. The two-parameter Laplace-type integral transform offers certain advantages over the one-parameter Laplace-type integral transform, such as allowing for more flexibility in representing functions and systems with multiple variables and parameters, and is useful for solving higher-dimensional partial differential equations and problems involving more than one independent variable. Hence, the objective in this paper is to evaluate the application of two-parameter Laplace-type R-transform to solve linear ordinary differential equations.

Definition

The R-transform of a function $f(t)$ is defined as:

$$R\{f(t)\} = A(s,u) = s\int_0^\infty f(ut)e^{-\frac{t}{s}}\,dt \quad (1)$$

R-Transform of Some Functions: In this section, we find the R-transform of some simple functions.

(i) Let $f(t) = a$, then

$$R\{a\} = s\int_0^\infty ae^{-\frac{t}{s}}\,dt$$

$$R\{a\} = -as^2\left[e^{-\frac{t}{s}}\right]_0^\infty = -as^2[0-1]$$
$$= as^2 \quad (2)$$

(ii) Let $f(t) = t$, then

$$R\{t\} = s\int_0^\infty ut\,e^{-\frac{t}{s}}\,dt$$

$$= su\int_0^\infty t\,e^{-\frac{t}{s}}\,dt$$

Integrating by parts, we have:

$$R\{t\} = s^3u \quad (3)$$

(iii) Let $f(t) = t^2$, then

$$R\{t^2\} = s\int_0^\infty (ut)^2\,e^{-\frac{t}{s}}\,dt$$

$$= su^2\int_0^\infty t^2\,e^{-\frac{t}{s}}\,dt$$

Integrating by parts, we have:

$$R\{t^2\} = 2s^4u^2 \quad (4)$$

(iv) Let $f(t) = t^3$, then

$$R\{t^3\} = s\int_0^\infty (ut)^3\,e^{-\frac{t}{s}}\,dt$$
$$= su^3\int_0^\infty t^3\,e^{-\frac{t}{s}}\,dt$$

Integrating by parts, we have:

$$R\{t^3\} = 6s^5u^3 \quad (5)$$

(v) In general, if $n > 0$ is an integer,

Then $R\{t^n\} = n!\,s^{n+2}u^n$

(vi) Let $f(t) = e^{at}$, then

$$R\{e^{at}\} = s\int_0^\infty e^{aut}\,e^{-\frac{t}{s}}\,dt$$

$$= s\int_0^\infty e^{-(\frac{1}{s}-au)t}\,dt$$

$$R\{e^{at}\} = \frac{s^2}{1-asu} \quad (6)$$

Similarly, $R\{e^{-at}\} = \frac{s^2}{1+asu}$

(vii) Let $f(t) = sinat$, then

$$R\{sinat\} = s\int_0^\infty sinaut\,e^{-\frac{t}{s}}\,dt$$

$$= s\int_0^\infty \left(\frac{e^{iaut}-e^{iaut}}{2i}\right)e^{-\frac{t}{s}}dt$$

$$R\{sinat\} = \frac{as^3u}{1+(asu)^2} \quad (7)$$

(viii) Let $f(t) = cosat$, then

$$R\{cosat\} = s\int_0^\infty cosaut\,e^{-\frac{t}{s}}\,dt$$

$$= s\int_0^\infty \left(\frac{e^{iaut}-e^{iaut}}{2}\right)e^{-\frac{t}{s}}\,dt$$

$$R\{cosat\} = \frac{s^2}{1+(asu)^2} \quad (8)$$

(ix) Let $f(t) = sinhat$, then

$$R\{sinhat\} = s\int_0^\infty sinhaut\, e^{-\frac{t}{s}}\, dt$$

$$= s\int_0^\infty \left(\frac{e^{aut} - e^{-aut}}{2}\right) e^{-\frac{t}{s}} dt$$

$$R\{sinhat\} = \frac{as^3u}{1-(asu)^2} \qquad (9)$$

(x) Let $f(t) = coshat$, then

$$R\{coshat\} = s\int_0^\infty coshaut\, e^{-\frac{t}{s}}\, dt$$

$$= s\int_0^\infty \left(\frac{e^{aut} + e^{aut}}{2}\right) e^{-\frac{t}{s}} dt$$

$$R\{coshat\} = \frac{s^2}{1-(asu)^2} \qquad (10)$$

Theorem 3.1
Let $B(s,u)$ be the Y-transform of $\left[R\big(f(t)\big) = A(s,u)\right]$, then:

(i) $R\{f'(t)\} = \frac{1}{su}A(s,u) - \frac{s}{u}f(0) \qquad (11)$

(ii) $R\{f''(t)\} = \frac{1}{(su)^2}A(s,u) - \frac{1}{u^2}f(0) - \frac{s}{u}f'(0) \qquad (12)$

(iii)
$$R\{f^{(n)}(t)\} = \frac{1}{(su)^n}A(s,u) - \sum_{k=0}^{n-1}\frac{s^{2-n+k}}{u^{n-k}}f^{(k)}(0) \qquad (13)$$

Proof

(i) $\mathrm{R}\{f'(t)\} = s\int_0^\infty f'(ut)e^{-\frac{t}{s}}\, dt$

Integrating by parts, we have:

$$\mathrm{R}\{f'(t)\} = s\left\{\left[e^{-\frac{t}{s}}\left(\frac{1}{u}f(ut)\right)\right]_0^\infty - \int_0^\infty \frac{1}{u}f(ut)\left(-\frac{e^{-\frac{t}{s}}}{s}\right)dt\right\}$$

$$= s\left\{\left[\frac{1}{u}f(ut)e^{-\frac{t}{s}}\right]_0^\infty + \frac{1}{su}\int_0^\infty f(ut)e^{-\frac{t}{s}}dt\right\}$$

$$= s\left[-\frac{f(0)}{u} + \frac{1}{s^2u}A(s,u)\right]$$

$$= \frac{1}{su}A(s,u) - \frac{s}{u}f(0)$$

(*ii*) $\mathrm{R}\{f''(t)\} = s\int_0^\infty f''(ut)e^{-\frac{t}{s}}\, dt$

Let $g(t) = f'(t), \Rightarrow g'(t) = f''(t)$

So that,

$$\mathrm{R}\{f''(t)\} = s\int_0^\infty g'(ut)e^{-\frac{t}{s}}\, dt$$

Using integrating by parts:

$$\mathrm{R}\{f''(t)\} = s\left\{\left[\frac{1}{u}g(ut)e^{-\frac{t}{s}}\right]_0^\infty + \frac{1}{su}\int_0^\infty g(ut)e^{-\frac{t}{s}}dt\right\}$$

$$= s\left\{-\frac{f'(0)}{u} + \frac{1}{su}\left[-\frac{f(0)}{u} + \frac{1}{s^2u}A(s,u)\right]\right\}$$

$$= \frac{1}{(su)^2}A(s,u) - \frac{1}{u^2}f(0) - \frac{s}{u}f'(0)$$

(iii) This can be proved by mathematical induction.

Theorem 3.2 (Linearity Property of the R-transform)
Let $f_1(t)$ and $f_2(t)$ be two functions of t and c_1 and c_2 be any two constants, then

$$R\{c_1f_1(t) \pm c_2f_2(t)\} = c_1R\{f_1(t)\} \pm c_2R\{f_2(t)\} \quad (14)$$

Proof
Using the definition of the R-transform, we have that:

$$R\{c_1 f_1(t) \pm c_2 f_2(t)\} = s\int_0^\infty [c_1 f_1(ut) \pm c_2 f_2(ut)]e^{-\frac{t}{s}}\, dt$$

$$= s\left\{\int_0^\infty [c_1 f_1(ut)]e^{-\frac{t}{s}}\, dt \pm \int_0^\infty [c_2 f_2(ut)]e^{-\frac{t}{s}}\, dt\right\}$$

$$= s\left\{c_1\int_0^\infty f_1(ut)e^{-\frac{t}{s}}\, dt \pm c_2\int_0^\infty f_2(ut)e^{-\frac{t}{s}}\, dt\right\}$$

$$= c_1 s\int_0^\infty f_1(ut)e^{-\frac{t}{s}}\, dt \pm c_2 s\int_0^\infty f_2(ut)e^{-\frac{t}{s}}dt$$

$$= c_1 R\{f_1(t)\} \pm c_2 R\{f_2(t)\}$$

Alternative Notation: We make our working neater by adopting the following notation.

Let $y(0) = y_0, y'(0) = y_1,\ y''(0) = y_2, \dots, y^{(n)}(0) = y_n$

Also, we denote the R-transform of y by $\bar{y}$,

i.e. $\bar{y} = R\{y\} = R\{f(t)\} = A(s,u)$

So, $R\{y\} = R\{f(t)\} = \bar{y}$

$R\{y'\} = \mathrm{R}\{f'(t)\} == \frac{1}{su}A(s,u) - \frac{s}{u}f(0)$

$R\{y''\} = \mathrm{R}\{f''(t)\} = \frac{1}{(su)^2}A(s,u) - \frac{1}{u^2}f(0) - \frac{s}{u}f'(0)$

Applications: In this section, we will apply the R-transform to solve some linear ordinary differential equations with constant coefficients.

Example 1
Consider first order differential equation

$$y' + y = 0, \quad y(0) = 1 \qquad (15)$$

Take the R-transform of both sides of (15):

$$\frac{1}{su}\bar{y} - \frac{s}{u}y_0 + \bar{y} = 0$$

$$\bar{y} = \frac{s^2}{1+su}$$

$$y(x) = R^{-1}\left\{\frac{s^2}{1+su}\right\} = e^{-x} \qquad (16)$$

Example 2
Consider first order differential equation

$$y' + 2y = x, \quad y(0) = 1 \qquad (17)$$

Take the R-transform of both sides of (17):

$$\frac{1}{su}\bar{y} - \frac{s}{u}y_0 + 2\bar{y} = u$$

$$\bar{y} = \frac{s^4u^3 + s^2u}{u + 2su}$$

$$\bar{y} = \frac{1}{2}s^3u + \frac{5}{4}\left[\frac{s^2}{1+2su}\right] - \frac{1}{4}s^2$$

$$y(x) = R^{-1}\left\{\frac{1}{2}s^3u + \frac{5}{4}\left[\frac{s^2}{1+2su}\right] - \frac{1}{4}s^2\right\}$$

$$y(x) = \frac{1}{2}x + \frac{5}{4}\, e^{-2x} - \frac{1}{4} \qquad (18)$$

Example 3
Consider second order differential equation

$$y'' + y = 0, \quad y(0) = y'(0) = 1 \qquad (19)$$

Take the R-transform of both sides of (19):

$$\frac{1}{(su)^2}\bar{y} - \frac{1}{u^2}y_0 - \frac{s}{u}y_1 + \bar{y} = 0$$

$$\bar{y} = \frac{s^3u}{1+(su)^2} + \frac{s^2}{1+(su)^2}$$

$$y(x) = R^{-1}\left\{\frac{s^3u}{1+(su)^2} + \frac{s^2}{1+(su)^2}\right\}$$

$$y(x) = sinx + cosx \qquad (20)$$

Example 4
Consider second order differential equation

$$y'' - 3y' + 2y = 0, \quad y(0) = 1, \quad y'(0) = 4 \quad (21)$$

Take the R-transform of both sides of (21):

$$\frac{1}{(su)^2}\bar{y} - \frac{1}{u^2}y_0 - \frac{s}{u}y_1 - 3\left(\frac{1}{su}\bar{y} - \frac{s}{u}y_0\right) + 2\bar{y} = 0$$

$$\bar{y} = \frac{s^2(1+su)}{2(su)^2 - 3su + 1}$$

Expressing in partial fractions, we have:

$$\bar{y} = \frac{3s^2}{1-2su} - \frac{2s^2}{1-su}$$

$$y(x) = R^{-1}\left\{\frac{3s^2}{1-2su} - \frac{2s^2}{1-su}\right\}$$

$$y(x) = 3e^{2x} - 2e^{x} \qquad (22)$$

Conclusion: In this paper, we introduce a new integral transform called the R-transform. We have shown that the R-transform has very interesting properties. We have applied the new integral transform to solve some linear ordinary differential equations with constant coefficients.

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Data Availability Statement: Data are available upon request from the corresponding author.

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