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Investigating the Direction of Groundwater Flow, Drawdown and Over-Pumping in a Confined Aquifer

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ABSTRACT: The objective of this paper is to investigate the direction of groundwater flow, factors affecting drawdown and the effects of over-pumping in a confined aquifer using appropriate differential equations for steady and unsteady states. The results showed that groundwater flow from a region of higher elevation to a lower elevation, pumping time and distance from the well affect drawdown and when water is pumped faster than it is recharged, the water in the well dries up. The numerical methods used are efficient for all differential equations of real-life problems.

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Mathematical models are ways to describe the physical system using mathematical equations. They are based on solving an equation or a system of equations that describe the physical or real-life phenomenon. Such equations are called governing equations of the specified or real-life phenomenon (Muyinda *et. al.*, 2014). To develop models, it is helpful to first understand the general equation and its relationship to the underlying physical principles. The general equation varies in form depending on whether the flow is saturated or unsaturated, two-dimensional or three-dimensional, isotropic or anisotropic, and transient or steady state (Waghmare, 2016). For groundwater flow, the general groundwater flow equation, a partial differential equation is derived from combining Darcy's law with the principle of conservation of mass. (Atangana and Botha, 2013), (Waghmare, 2016), (Wang and Zheng, 2015). Groundwater flow models simulate either steady or unsteady states (transient flow). In steady-state systems, inputs (recharge) and outputs (discharge) are in equilibrium and solving steady state groundwater flow equation with or without sinks/sources is to calculate the hydraulic head (H) as a function of x and y. In unsteady state or transient simulations, the inputs (recharge) and outputs (discharge) are not in equilibrium so there is a net change in the systems with time that is the flow velocity and pressure are changing

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with time. To solve groundwater flow equations, hydrological inputs, hydraulic parameters, and initial and boundary conditions are required for calculations in groundwater modeling (Kakuba *et al.*, 2014). Some established techniques for solving the model's governing equations include the Finite Difference Method, Finite Element Method, Finite volume Method (Muyinda *et. al.*, 2014), Boundary element Method, and several others. The finite difference method converts the partial derivatives of the differential equation over a small interval by an algebraic expression that represents the properties and conditions of the aquifer. The problem domain is partitioned into a series of interconnected discrete points known as nodes. This approach replaces the continuous medium with a discrete set of points and assigns distinct hydrogeological parameters to each node (Thangarajan, 2007; Dhumal and Kiwne, 2014). Finite difference method (FDM) can be used to discretize both time and space. Partial derivatives are replaced using a difference operator that defines the spatio-temporal relationship between several parameters. The developed model is solved at each node by solving a set of algebraic equations at that node. There are various methods to solve these simplified equations. Some of the primary iterative numerical methods iclude the Jacobian method, Gauss-Seidel iterative method, iterative relaxation method and many more (Okiro *et. al.*, 2013; Kahlaf and Mhassin, 2021). The advantage of finite difference methods is that they are easy to understand and program, so complex systems with complex load paths and highly nonlinear behavior can be easily traced. It is, therefore, an economical way to solve large-scale nonlinear groundwater flow problems. Consequently, the objective of this paper is to investigate the direction of groundwater flow, factors affecting drawdown and the effects of over-pumping in a confined aquifer using appropriate differential equations for steady and unsteady states.

MATERIALS AND METHODS

The basic groundwater flow equation is:

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial h}{\partial z}\right)-Q = S_s\frac{\partial h}{\partial t} \quad (1)$$

h is the hydraulic head or piezometric head, K_x, K_y, K_z are the hydraulic conductivity along x, y, z axes, Q is the volumetric source or sink and S_s is the specific storage coefficient.

Derivation of Steady-state groundwater flow equation

The initial step in constructing developing a mathematical model involves formulating the general equations. In groundwater modeling, general equations are derived from two fundamental physical principles, Darcy's law and mass balance (continuity) equation (Kreysig, 2011).

Equation 2 is Darcy's law in three-dimensions along the x, y and z coordinates:

$$q_x = -K_x\frac{\partial \text{h}}{\partial x}, \qquad q_y = -K_y\frac{\partial \text{h}}{\partial y}, \qquad q_z = -K_z\frac{\partial \text{h}}{\partial z} \quad (2)$$

Where $K_x, K_y\ and\ K_x$ are the hydraulic conductivity in each of the coordinate direction respectively and $q_x, q_y\ and\ q_z$

Continuity equation in three dimensions is stated in equation 3

$$\frac{\partial}{\partial x}(q_x)+\frac{\partial}{\partial y}(q_{y)})+\frac{\partial}{\partial z}(q_z) = S_s\frac{\partial h}{\partial t} \quad (3)$$

In the steady-state, $\frac{\partial h}{\partial t} = 0$

substituting $\frac{\partial h}{\partial t} = 0$ and equation (2) in equation (3) gives equation (4)

$$\frac{\partial}{\partial x}\left(K_x\frac{\partial \text{h}}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial \text{h}}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial \text{h}}{\partial z}\right) = 0 \quad (4)$$

For an isotropic, homogeneous, confined aquifer, $K_x = K_y = K_z = K$

Equation (4) becomes

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}+\frac{\partial^2 h}{\partial z^2} = 0 \quad (5)$$

Therefore, equation (5) represents the governing equation for groundwater flow through an isotropic, homogeneous medium under steady-state condition in three-dimensions.

Allowing the possibility of a sink (for example, a pumping well) or a source of water (for example, an injection well or recharge) which is expressed as volume of per area of aquifer per time, R, so equation (5) becomes

$$\frac{\partial^2 h}{\partial x^2}+\frac{\partial^2 h}{\partial y^2}+\frac{\partial^2 h}{\partial z^2} = \frac{R}{T} \quad (6)$$

Equation (6) is called Poisson equation, which is the equation for steady-state flow equation with sinks/sources while equation (4) is a very famous equation called the Laplace equation.

Derivation of the unsteady-state groundwater flow equations

Equation (3) without sources/sinks that is when $\frac{\partial h}{\partial t} \neq 0$

$$\frac{\partial}{\partial x}(q_x) + \frac{\partial}{\partial y}(q_{y)} + \frac{\partial}{\partial z}(q_z) = S_s \frac{\partial h}{\partial t} \quad (7)$$

Substituting equation (2) in equation (7) gives

$$\frac{\partial}{\partial x}\left(K_x \frac{\partial \text{h}}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial \text{h}}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial \text{h}}{\partial z}\right) = S_s \frac{\partial h}{\partial t} \quad (8)$$

Re-writing equation (8)

$$K_x \frac{\partial^2 h}{\partial x^2} + K_y \frac{\partial^2 h}{\partial y^2} + K_z \frac{\partial^2 h}{\partial z^2} = S_s \frac{\partial h}{\partial t} \quad (9)$$

For confined, transient, isotropic and homogeneous groundwater flow, hydraulic conductivity is constant, that is $K_x = K_y = K_z = K$ and dividing both sides by K gives

equation (9) becomes

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = \frac{S_s}{K} \frac{\partial h}{\partial t} \quad (10)$$

The solution *h (x, y, z, t)* gives the value of *h* for any point in the flow field at time *t*. Specifying the boundary conditions and initial conditions. Saturated thickness, *b,* is not dependent on head, *h,* and assuming the aquifer thickness is constant, both sides of equation (10) can be multiplied by the aquifer thickness, *b* to give equation (11)

$$Kb\left[\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2}\right] = S_s b \frac{\partial h}{\partial t} \quad (11)$$

From the definition of Transmissivity, $T = Kb,$ and Storativity, $S_s b = S,$ equation (10) and dividing by T

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = \frac{S}{T} \frac{\partial h}{\partial t} \quad (12)$$

This is the unsteady-state equation in three-dimensions without sinks/sources.

In the availability of a sink (for example, a pumping well) or source of water (for example, an injection well or recharge) this is written as volume per area of aquifer per time, R, and dividing by T

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} + \frac{\partial^2 h}{\partial z^2} = \frac{S}{T} \frac{\partial h}{\partial t} - \frac{R}{T} \quad (13)$$

Where the flow is horizontal, unsteady-state flow in two-dimensions with and without sinks/sources in equation (14) and (15) respectively.

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{S}{T} \frac{\partial h}{\partial t} - \frac{R}{T} \quad (14)$$

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{S}{T} \frac{\partial h}{\partial t} \quad (15)$$

Steady-State Groundwater Flow Methods

Method 1: Alternating Direction Implicit Method (ADI)

The ADI Method was first introduced and used by Peaceman and Rachford for solving a two-dimensional the time-dependent heat equation. From the result, it was discovered that the method is unconditional stable which render it effective as a steady-state solver due to the possibility of employing large time steps for pseudo-time marching to a steady state (Abimbola and Bright, 2015), (Imam, 2015), (Esraa and Adel, 2020) (Shittu et. al., 2024).

Consider the two-dimensional steady state groundwater flow equation without sinks/sources below:

$\frac{\delta^2 H}{dx^2} + \frac{\delta^2 H}{dy^2} = 0$

Finite difference approximation, assuming $\Delta x = \Delta y$ is

$$H_{p+1,q} - 4H_{p,q} + H_{p-1,q} + H_{p,q+1} + H_{p,q-1} = 0 \quad (16)$$

For fix row formula, we have

$$H_{p+1,q} - 4H_{p,q} + H_{p-1,q} + H_{p,q+1} + H_{p,q-1} = 0 \quad (17)$$

Substituting n*th* approximation on the right as:

$$H_{p-1,q}^{(n+1)} - 4H_{p,q}^{(n+1)} + H_{p+1,q}^{(n+1)} = -H_{p,q+1}^{(n)} - H_{p,q-1}^{(n)} \quad (18)$$

In the next iteration, we alternate the direction by using the formula for fix column

$$H_{p,q+1} - 4H_{p,q} + H_{p,q-1} = -H_{p+1,q} - H_{p+1,q} \quad (19)$$

Substituting *n+1th* approximation:

$$H_{p,q-1}^{(n+2)} - 4H_{p,q}^{(n+2)} + H_{p,q+1}^{(n+2)} = -H_{p-1,q}^{(n+1)} - H_{p+1,q}^{(n+1)} \quad (20)$$

With sinks/sources, we have

$$H_{p-1,q}^{(n+1)} - 4H_{p,q}^{(n+1)} + H_{p+1,q}^{(n+1)} = -H_{p,q+1}^{(n)} - H_{p,q-1}^{(n)} - \frac{R(\Delta x)^2}{T} \quad (21)$$

$$H_{p,q-1}^{(n+2)} - 4H_{p,q}^{(n+2)} + H_{p,q+1}^{(n+2)} = -H_{p-1,q}^{(n+1)} - H_{p+1,q}^{(n+1)} - \frac{R(\Delta x)^2}{T} \quad (22)$$

Method 2: Improved Alternating Direction Implicit Method (IADI)

To improve the convergence of ADI, we introduce a parameter k (Imam, 2015; Shittu *et al.*, 2024, Esraa and Adel, 2020) equations (18) and (20), we have the improved formula for row and column

$$H_{p-1,q}^{(n+1)} - (2+k)H_{p,q}^{(n+1)} + H_{p+1,q}^{(n+1)} = -H_{p,q+1}^{(n)} - H_{p,q-1}^{(n)} + (2-k)H_{p,q}^{(n)} \quad (23)$$

$$H_{p,q-1}^{(n+2)} - (2+k)H_{p,q}^{(n+2)} + H_{p,q+1}^{(n+2)} = -H_{p-1,q}^{(n+1)} - H_{p+1,q}^{(n+1)} + (2-k)H_{p,q}^{(n+1)} \quad (24)$$

With sinks/sources, from equations (23) and (24), we add the source/sinks

$$H_{p-1,q}^{(n+1)} - (2+k)H_{p,q}^{(n+1)} + H_{p+1,q}^{(n+1)} = -H_{p,q+1}^{(n)} - H_{p,q-1}^{(n)} + (2-k)H_{p,q}^{(n)} - \frac{R(\Delta x)^2}{T} \quad (25)$$

$$H_{p,q-1}^{(n+2)} - (2+k)H_{p,q}^{(n+2)} + H_{p,q+1}^{(n+2)} = -H_{p-1,q}^{(n+1)} - H_{p+1,q}^{(n+1)} + (2-k)H_{p,q}^{(n+1)} - \frac{R(\Delta x)^2}{T} \quad (26)$$

Unsteady-State Groundwater Flow Methods

Method 1:Explicit Finite Difference Method (EFDM)

Explicit Finite Difference calculates solution at the next time step from the information of the previous time step. (Hoffmann and Chiang, 2000). This method is also called Forward Time Central Space Method (FTCS).

Consider the two-dimensional unsteady-state groundwater flow equation 27:

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = \frac{S}{T}\frac{\partial H}{\partial t} \quad (27)$$

Using the finite difference approximation, equation (27) becomes

$$\frac{H_{p+1,q}^{n} - 2H_{p,q}^{n} + H_{p-1,q}^{n}}{(\Delta x)^2} + \frac{H_{p,q+1}^{n} - 2H_{p,q}^{n} + H_{p,q-1}^{n}}{(\Delta y)^2} = \frac{S}{T}\frac{H_{p,q}^{n+1} - H_{p,q}^{n}}{\Delta t} \quad (28)$$

assume $\Delta x = \Delta y = a$

$$H_{p,q}^{n+1} = H_{p,q}^{n} + \frac{T\Delta t}{Sa^2}\left(H_{p+1,q}^{n} - 4H_{p,q}^{n} + H_{p-1,q}^{n} + H_{p,q+1}^{n} + H_{p,q-1}^{n}\right) \quad (29)$$

Let $\gamma = \frac{T\Delta t}{Sa^2}$ and collecting like terms, we have

$$H_{p,q}^{n+1} = H_{p,q}^{n}(1-4\gamma) + \gamma\left(H_{p+1,q}^{n} + H_{p-1,q}^{n} + H_{p,q+1}^{n} + H_{p,q-1}^{n}\right) \quad (30)$$

For the solution to be stable, the value of γ must be less than or equal to 0.25 while for one-dimensional case, γ must be less than or equal to 0.5 (Wang and Anderson, 1996).

Method 2: Crank Nicolson Method (CNM): Crank Nicolson Method was developed by John Crank and Phyllis Nicolson during the mid-20th century. This method is employed for solving partial differential equation. It exhibits second-order accuracy in space, operates implicitly in time ensures unconditional stability and offers enhanced accuracy level. (Fadugba *et al.*, 2013), (Ajeel and Gaftan, 2023), (Fernandes and Bhadkamkar, 2016), (Islam *et al.*, 2018), (Hoffmann and Chiang, 2000)

Consider the two-dimensional transient groundwater flow equation (27)

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = \frac{S}{T}\frac{\partial H}{\partial t}$$

Replace $\frac{\partial H}{\partial t}$ by forward difference approximation and central difference for the space derivative along at t and t+1 level, equation (27) becomes

$$\frac{H_{p+1,q}^{t} - 2H_{p,q}^{t} + H_{p-1,q}^{t}}{(\Delta x)^2} + \frac{H_{p,q+1}^{t} - 2H_{p,q}^{t} + H_{p,q-1}^{t}}{(\Delta y)^2} = \frac{S}{T}\frac{H_{p,q}^{t+1} - H_{p,q}^{t}}{\Delta t} \quad (31)$$

$$\frac{H_{p+1,q}^{t+1} - 2H_{p,q}^{t+1} + H_{p-1,q}^{t+1}}{(\Delta x)^2} + \frac{H_{p,q+1}^{t+1} - 2H_{p,q}^{t+1} + H_{p,q-1}^{t+1}}{(\Delta y)^2} = \frac{S}{T}\frac{H_{p,q}^{t+1} - H_{p,q}^{t}}{\Delta t} \quad (32)$$

Finding the average of equations (31) and (32) and assume $\Delta x = \Delta y$, we have

$$\frac{S}{T}\frac{H_{p,q}^{t+1} - H_{p,q}^{t}}{\Delta t} = \frac{1}{2}\left(\frac{H_{p+1,q}^{t+1} + H_{p-1,q}^{t+1} + H_{p,q+1}^{t+1} + H_{p,q-1}^{t+1} - 4H_{p,q}^{t+1}}{(\Delta x)^2} + \right.$$

$$+\left.\frac{H^{t}_{p+1,q} - 4H^{t}_{p,q} + H^{t}_{p-1,q} + H^{t}_{p,q+1} + H^{t}_{p,q-1}}{(\Delta x)^2}\right) \quad (33)$$

Let $\beta = \frac{T\Delta t}{S(\Delta x)^2}$ and re-arranging, it becomes

$$H^{t+1}_{p,q} = H^{t}_{p,q} + \frac{1}{2}\beta\Big(H^{t+1}_{p+1,q} + H^{t+1}_{p-1,q} + H^{t+1}_{p,q+1} + H^{t+1}_{p,q-1} - 4H^{t+1}_{p,q} + H^{t}_{p+1,q} - 4H^{t}_{p,q} + H^{t}_{p-1,q} + H^{t}_{p,q+1} + H^{t}_{p,q-1}\Big) \quad (34)$$

With sinks/sources

$$\frac{S}{T}\frac{\partial H}{\partial t} = \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{R}{T}$$

Using equation (33), we have

$$\frac{S}{T}\frac{H^{t+1}_{p,q} - H^{t}_{p,q}}{\Delta t} = \frac{1}{2}\left(\frac{H^{t+1}_{p+1,q} + H^{t+1}_{p-1,q} + H^{t+1}_{p,q+1} + H^{t+1}_{p,q-1} - 4H^{t+1}_{p,q}}{(\Delta x)^2} + \frac{H^{t}_{p+1,q} - 4H^{t}_{p,q} + H^{t}_{p-1,q} + H^{t}_{p,q+1} + H^{t}_{p,q-1}}{(\Delta x)^2}\right) + \frac{R}{T} \quad (35)$$

Let $\beta = \frac{T\Delta t}{S(\Delta x)^2}$

$$H^{t+1}_{p,q} = H^{t}_{p,q} + \frac{1}{2}\beta\Big(H^{t+1}_{p+1,q} + H^{t+1}_{p-1,q} + H^{t+1}_{p,q+1} + H^{t+1}_{p,q-1} - 4H^{t+1}_{p,q} + H^{t}_{p+1,q} -$$

Numerical Applications

Example 1:

1. In a confined isotropic, homogeneous aquifer, a single well is pumped to steady-state conditions. A square grid measuring 500m x 500m is imposed and the heads along the boundary of the grid are illustated below. Compute the hydraulic heads at the interior nodes (Karvonen, 2002).

Method 1: Alternating Direction Implicit Method (ADI)

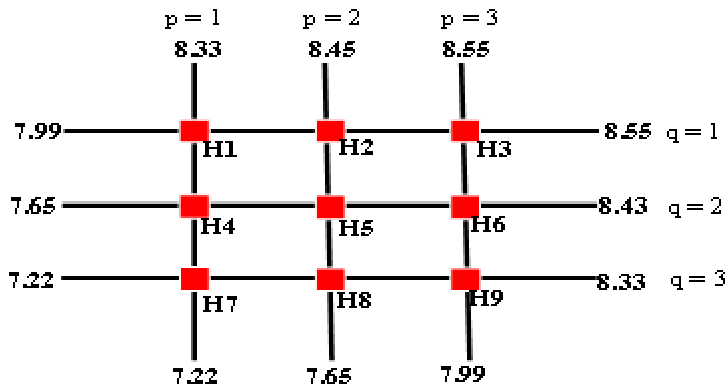

Fig 1. Schematic diagram for ADI of question 1

Using equation (18) and figure (1), we have the following equations for the row iterations

For Row q = 1

$$7.22 - 4H_7^{(n+1)} + H_8^{(n+1)} = -7.22 - H_4^{(n)}$$
for p = 1
$$H_7^{(n+1)} - 4H_8^{(n+1)} + H_9^{(n+1)} = -7.65 - H_5^{(n)}$$
for p = 2
$$H_8^{(n+1)} - 4H_9^{(n+1)} + 8.33 = -7.99 - H_6^{(n)}$$
for p = 3

For Row q = 2

$$7.65 - 4H_4^{(n+1)} + H_5^{(n+1)} = -H_7^{(n)} - H_1^{(n)}$$
for p = 1
$$H_4^{(n+1)} - 4H_5^{(n+1)} + H_6^{(n+1)} = -H_8^{(n)} - H_2^{(n)}$$
for p = 2
$$H_5^{(n+1)} - 4H_6^{(n+1)} + 8.43 = -H_9^{(n)} - H_3^{(n)}$$
for p = 3

For Row q = 3

$$7.99 - 4H_1^{(n+1)} + H_2^{(n+1)} = -H_4^{(n)} - 8.33$$
for p =1
$$H_1^{(n+1)} - 4H_2^{(n+1)} + H_3^{(n+1)} = -H_5^{(n)} - 8.43$$
for p = 2
$$H_2^{(n+1)} - 4H_3^{(n+1)} + 8.56 = -H_6^{(n)} - 8.55$$
for p = 3

Using equation (20) and figure (1), we have the following equations for the column iterations

For Column p = 1

$$7.22 - 4H_7^{(n+1)} + H_4^{(n+1)} = -H_8^{(n)} - 7.22$$
for q =1
$$H_7^{(n+1)} - 4H_4^{(n+1)} + H_1^{(n+1)} = -H_5^{(n)} - 7.65$$
for q = 2
$$H_4^{(n+1)} - 4H_1^{(n+1)} + 8.33 = -H_2^{(n)} - 7.99$$
for q = 3

For Column p = 2

$$7.65 - 4H_8^{(n+1)} + H_5^{(n+1)} = -H_7^{(n)} - H_9^{(n)}$$
for q = 1

$H_8{}^{(n+1)} - 4H_5^{(n+1)} + H_2^{(n+1)} = -H_4^{(n)} - H_6^{(n)}$ for q = 2

$H_5{}^{(n+1)} - 4H_2^{(n+1)} + 8.43 \quad = -H_1^{(n)} - H_3^{(n)}$ for q = 3

For Column p = 3

$7.99 - 4H_9^{(n+1)} + H_6^{(n+1)} \quad = -H_8^{(n)} - 8.33$ for q = 1

$H_9{}^{(n+1)} - 4H_6^{(n+1)} + H_3^{(n+1)} = -H_5^{(n)} - 8.43$ for q = 2

$H_6{}^{(n+1)} - 4H_3^{(n+1)} + 8.55 \quad = -H_2^{(n)} - 8.56$ for q = 3

The results converged after 20 iterations:

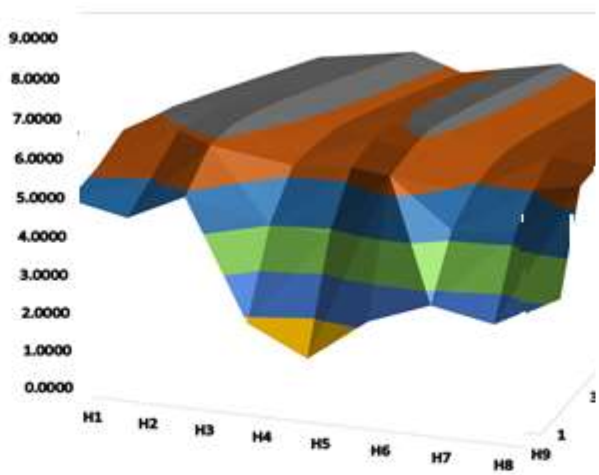

Fig 2: Surface Plot of Example 1 using ADI Method

Method 2: Improved Alternating Direction Implicit Method (IADI)

Using equations (23) and (24), we will use the following equations for the iterations

Assume k = 1.5, starting at p = 0

For Row n = 1

$7.22 - 3.5H_7^{(n+1)} + H_8^{(n+1)} = -7.22 - H_4^{(n)} + 0.5H_7^{(n)}$ for p =1

$H_7{}^{(n+1)} - 3.5H_8^{(n+1)} + H_9^{(n+1)} = -7.65 - H_5^{(n)} + 0.5H_8^{(n)}$ for p =2

$H_8{}^{(n+1)} - 3.5H_9^{(n+1)} + 8.33 = -7.99 - H_6^{(n)} + 0.5H_9^{(n)}$ for p =3

For Row n = 2

$7.65 - 3.5H_4^{(n+1)} + H_5^{(n+1)} = -H_7^{(n)} - H_1^{(n)} + 0.5H_4^{(n)}$ for p =1

$H_4{}^{(n+1)} - 3.5H_5^{(n+1)} + H_6^{(n+1)} = -H_8^{(n)} - H_2^{(n)} + 0.5H_5^{(n)}$ for p =3

$H_5{}^{(n+1)} - 3.5H_6^{(n+1)} + 8.43 \quad = -H_9^{(n)} - H_3^{(n)} + 0.5H_6^{(n)}$ for p =3

For Row n = 3

$7.99 - 3.5H_1^{(n+1)} + H_2^{(n+1)} \quad = -H_4^{(n)} - 8.33 + 0.5H_1^{(n)}$ for p =1

$H_1{}^{(n+1)} - 3.5H_2^{(n+1)} + H_3^{(n+1)} \; = -H_5^{(n)} - 8.43 + 0.5H_2^{(n)}$ for p =2

$H_2{}^{(n+1)} - 3.5H_3^{(n+1)} + 8.56 = -H_6^{(n)} - 8.55 + 0.5H_3^{(n)}$ for p =3

For Column m = 1

$7.22 - 3.5H_7^{(n+1)} + H_4^{(n+1)} \quad = -H_8^{(n)} - 7.22 + 0.5H_7^{(n)}$ for q =1

$H_7{}^{(n+1)} - 3.5H_4^{(n+1)} + H_1^{(n+1)} = -H_5^{(n)} - 7.65 + 0.5H_4^{(n)}$ for q =2

$H_4{}^{(n+1)} - 3.5H_1^{(n+1)} + 8.33 = -H_2^{(n)} - 7.99 + 0.5H_1^{(n)}$ for q =3

For Column m = 2

$7.65 - 3.5H_8^{(n+1)} + H_5^{(n+1)} \quad = -H_7^{(n)} - H_9^{(n)} + 0.5H_8^{(n)}$ for q =1

$H_8{}^{(n+1)} - 3.5H_5^{(n+1)} + H_2^{(n+1)} = -H_4^{(n)} - H_6^{(n)} + 0.5H_5^{(n)}$ for q =2

$H_5{}^{(n+1)} - 3.5H_2^{(n+1)} + 8.43 = -H_1^{(n)} - H_3^{(n)} + 0.5H_2^{(n)}$ for q =3

For Column m = 3

$7.99 - 3.5H_9^{(n+1)} + H_6^{(n+1)} \quad = -H_8^{(n)} - 8.33 + 0.5H_9^{(n)}$ for q =1

$-3.5H_6^{(n+1)} + H_3^{(n+1)} = -H_5^{(n)} - 8.43 + 0.5H_6^{(n)}$ for q =2

$H_6{}^{(n+1)} - 3.5H_3^{(n+1)} + 8.55 = -H_2^{(n)} - 8.56 + 0.5H_3^{(n)}$ for q =3

The results converged after six(11) iterations are

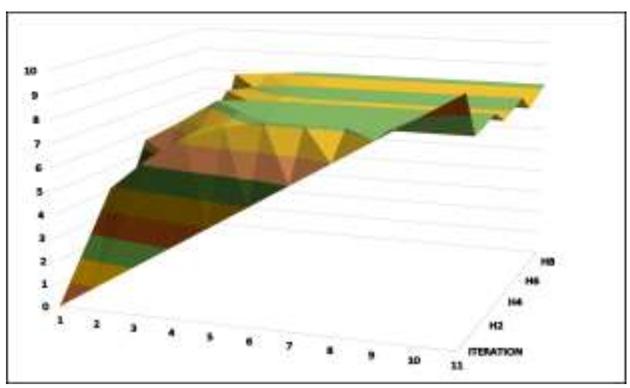

Fig 3: Surface Plot of Example 1 using IADI method

Example 2:

A domain is bounded at the top and bottom by Neumann boundaries with no flow. Water infiltration occurs at the rate of 1m and 2m per day on the left and right boundaries respectively based on the piezometric head. Express this mathematically and compute the hydraulic heads at the internal nodes.

These can be expressed mathematically as:

$\nabla^2 = 0 \qquad \{(x, y): \quad 0 < x < 1, 0 < y < 1$

$BC: \qquad H(0, y) = 1 \qquad 0 \le y \le 1$

$H(1, y) = 2 \qquad 0 \le y \le 1$

$H^1(x, 0) = 0 \qquad 0 \le x \le 1$

$H^1(x, 1) = 0 \qquad 0 \le x \le 1 \qquad N_y = 6, N_x = 6$

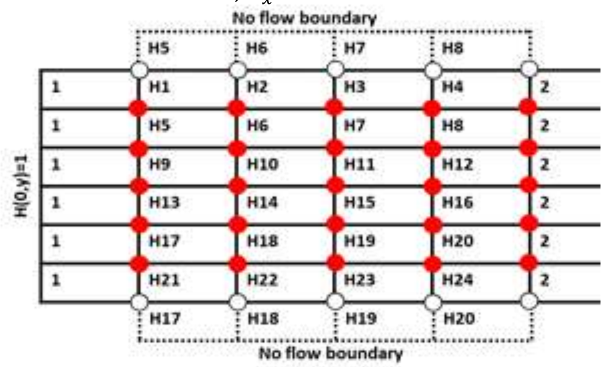

Fig 4: Schematic Diagram for *Example* 2

In figure 4, the central difference formula for the first derivative is used for the Neumann Boundary with no flow.

$\frac{dy}{dx} = \frac{H_{i+1,j} - H_{i-1,j}}{2\Delta x} = 0$ which implies that $H_{i+1,j} = H_{i-1,j}$ for example $H_{2,1} = H_{0,1}$

Method 1: Alternating Direction Implicit Method (ADI)

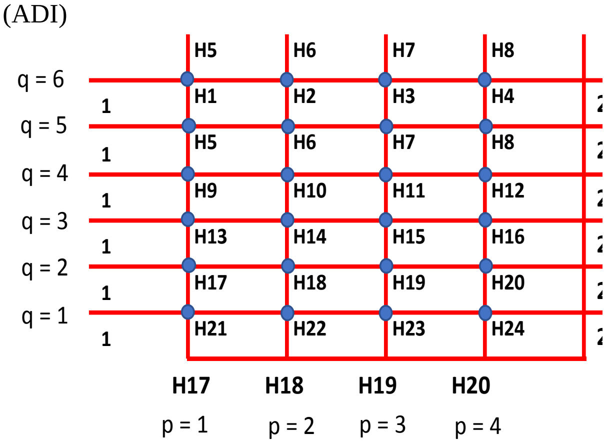

Fig 5: ADI Diagram for *Example* 2

From figure 5 and equations (18) and (20), we generate equations for the row iteration equations and the column iterations respectively. Starting at n = 0 and apply the boundary using the central difference for first derivative that is $H_{i+1,j} = H_{i-1,j}$ and $H_{i,j+1} = H_{i,j-1}$.

For Row q = 1

$1 - 4H_{21}^{(n+1)} + H_{22}^{(n+1)} = -2H_{17}^{(n)}$ for p = 1

$H_{21}{}^{(n+1)} - 4H_{22}^{(n+1)} + H_{23}^{(n+1)} = -2H_{18}^{(n)}$ for p = 2

$H_{22}{}^{(n+1)} - 4H_{23}^{(n+1)} + H_{24}^{(n+1)} = -2H_{19}^{(n)}$ for p = 3

$H_{23}{}^{(n+1)} - 4H_{24}^{(n+1)} + 2 = -2H_{20}^{(n)}$ for p = 4

For Row q = 2

$1 - 4H_{17}^{(n+1)} + H_{18}^{(n+1)} = -H_{21}^{(n)} - H_{13}^{(n)}$ for p = 1

$H_{17}{}^{(n+1)} - 4H_{18}^{(n+1)} + H_{19}^{(n+1)} = -H_{22}^{(n)} - H_{14}^{(n)}$ for p = 2

$H_{18}{}^{(n+1)} - 4H_{19}^{(n+1)} + H_{20}^{(n+1)} = -H_{23}^{(n)} - H_{15}^{(n)}$ for p = 3

$H_{19}{}^{(n+1)} - 4H_{20}^{(n+1)} + 2 = -H_{24}^{(n)} - H_{16}^{(n)}$ for p = 4

For Row q = 3

$1 - 4H_{13}^{(n+1)} + H_{14}^{(n+1)} = -H_{17}^{(n)} - H_{9}^{(n)}$ for p = 1

$H_{13}{}^{(n+1)} - 4H_{14}^{(n+1)} + H_{15}^{(n+1)} = -H_{18}^{(n)} - H_{10}^{(n)}$ for p = 2

$H_{14}{}^{(n+1)} - 4H_{15}^{(n+1)} + H_{16}^{(n+1)} = -H_{19}^{(n)} - H_{11}^{(n)}$ for p = 3

$H_{15}{}^{(n+1)} - 4H_{16}^{(n+1)} + 2 = -H_{20}^{(n)} - H_{12}^{(n)}$ for p = 4

This continues until it gets to Row q = 6

For the column iterations, we will use equation (20) and figure 5.

For Column p = 1

$2H_{17}^{(n+1)} - 4H_{7}^{(n+1)} = -1 - H_{22}^{(n)}$ for q =1

$H_{21}{}^{(n+1)} - 4H_{17}^{(n+1)} + H_{13}^{(n+1)} = -1 - H_{18}^{(n)}$ for q = 2

$H_{17}{}^{(n+1)} - 4H_{13}^{(n+1)} + H_{9}^{(n+1)} = -1 - H_{14}^{(n)}$ for q = 3

$H_{13}{}^{(n+1)} - 4H_{9}^{(n+1)} + H_{5}^{(n+1)} = -1 - H_{10}^{(n)}$ for q = 4

$H_{9}{}^{(n+1)} - 4H_{5}^{(n+1)} + H_{1}^{(n+1)} = -1 - H_{6}^{(n)}$ for q = 5

$2H_{5}{}^{(n+1)} - 4H_{1}^{(n+1)} = -1 - H_{2}^{(n)}$ for q = 6

For Column p = 2

$2H_{18}^{(n+1)} - 4H_{22}^{(n+1)} = -H_{21}^{(n)} - H_{23}^{(n)}$ for q = 1

$$H_{22}{}^{(n+1)} - 4H_{18}^{(n+1)} + H_{14}^{(n+1)} = -H_{17}^{(n)} - H_{19}^{(n)}$$
for q = 2
$$H_{18}{}^{(n+1)} - 4H_{14}^{(n+1)} + H_{10}^{(n+1)} = -H_{13}^{(n)} - H_{15}^{(n)}$$
for q = 3
$$H_{14}{}^{(n+1)} - 4H_{10}^{(n+1)} + H_{16}^{(n+1)} = -H_{9}^{(n)} - H_{11}^{(n)}$$
for q = 4
$$H_{10}{}^{(n+1)} - 4H_{6}^{(n+1)} + H_{2}^{(n+1)} = -H_{5}^{(n)} - H_{7}^{(n)}$$
for q = 5
$$2H_{6}^{(n+1)} - 4H_{2}^{(n+1)} = -H_{1}^{(n)} - H_{3}^{(n)}$$
for q = 6

This continues until it gets to Column p = 4
Now, starting the iterations of both row and column at n = 0
The iterations continue until the results the results converge
At iteration 18, the results converged and the results are:

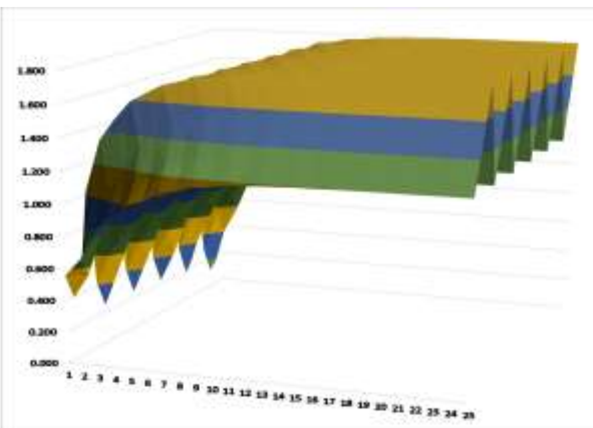

Fig 6: Surface Plot of ADI of example 2

Method 2: Improve Alternating Direction Implicit Method (IADI)

Using figure 5 and equation (23) for the row iterations and equation (24) for the column iterations.
Take k = 1.2, equation for row iteration at n = 0 is

$$H_{p-1,q}^{(1)} - (3.2)H_{p,q}^{(1)} + H_{p+1,q}^{(1)} = -H_{p,q-1}^{(0)} - H_{p,q+1}^{(0)} + (0.8)H_{p,q}^{(0)} \quad (37)$$

Equation for column iteration at n = 0 is

$$H_{p,q-1}^{(2)} - (3.2)H_{p,q}^{(2)} + H_{p,q+1}^{(2)} = -H_{p-1,q}^{(1)} - H_{p+1,q}^{(1)} + (0.8)H_{p,q}^{(1)} \quad (38)$$

The results of the iteration after 8 iterations are:

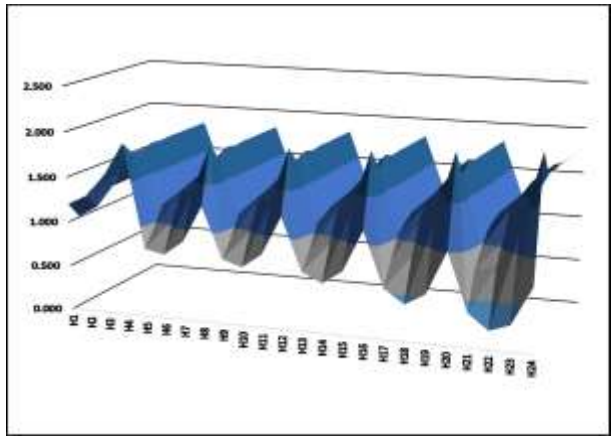

Fig 7: Surface Plot of IADI for example 2

Example 3 (Well Drawdown)
A fully penetrated well is situated in a horizontal isotropic aquifer of thickness 30m. The confined aquifer assumed to be of circular shape and the discharging well is located at the center of the aquifer, the radius of the homogeneous, isotropic aquifer is 1100 m. The transmissivity, T is 400 m^2/d and the pumping rate Q from the well is 2000 m^3/d. Before pumping, the static water level in the well is 30 m, the hydraulic head along the circular boundary of the aquifer is 30 m and it is assumed that drawdown extends out to a radial distance of 1100 m from the well. That is the static water level remains unaffected for distances that exceed 1100 m from the well. Take Δx = 100. (Karvonen, 2002).

Solution:
Δx = Δy = 100, T = 400 m^2/d, R = 0.2m/d, Q = 2000 m^3/d

$$R = -\frac{Q}{(\Delta x)^2} = \frac{2000}{10000} = -0.2$$
$$\frac{R(\Delta x)^2}{T} = 0.2 * 100 * \frac{100}{400} = 5$$

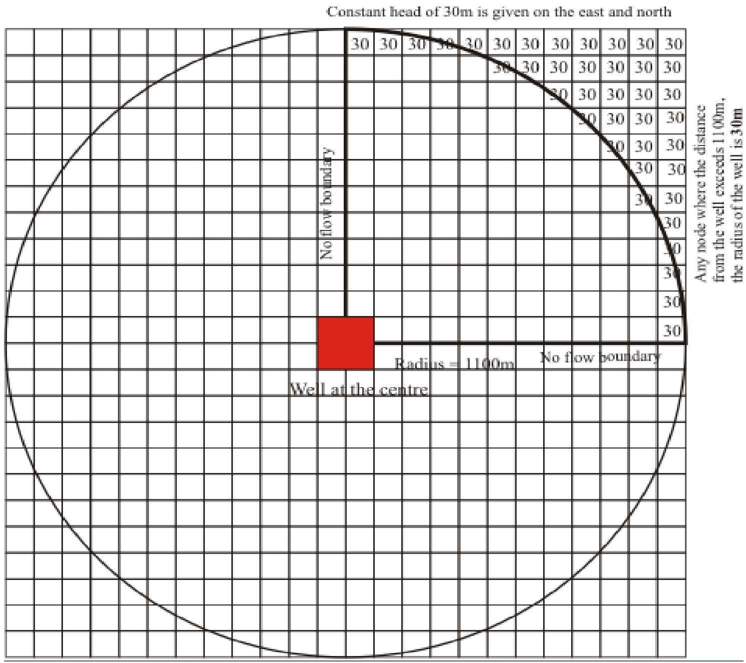

Fig 8: The circular aquifer with the well at the center

As shown in Figure 8, we will consider only one quadrant and use the solution of the one quadrant to get the solution of the other 3 quadrants. We will use rectangular coordinates for the solution. Because of symmetry, it is required that the line x and y axes be no-flow boundary. From Figure 8 and table 1, we can see that all the nodes that the distance from the well is greater than 1100m takes the value 30m. There is No flow boundary on the left and bottom of the quadrant and constant head on the right and top boundary = 30m.

Method 1: Alternating Direction Implicit Method (ADI)

Using figure 8, table 1 and equations (18) and (20), starting the iteration at p = 0 and applying the boundary conditions on the left and at the bottom since they are no flow boundaries

$$\frac{dy}{dx} = \frac{H_{p+1,q}-H_{p-1,q}}{2\Delta x} = 0$$ which implies that $H_{p+1,q} = H_{p-1,q}$ and $H_{p,q+1} = H_{p,q-1}$

$H_{2,1} = H_{0,1}$ and $H_{1,2} = H_{1,0}$

For Row Iteration at n = o,

$$H^{(1)}_{p-1,q} - 4H^{(1)}_{p,q} + H^{(1)}_{p+1,q} = -H^{(0)}_{p,q-1} - H^{(0)}_{p,q+1} \quad (39)$$

For Column Iteration at n = o,

$$H^{(1)}_{p,q-1} - 4H^{(1)}_{p,q} + H^{(1)}_{p,q+1} = -H^{(0)}_{p-1,q} - H^{(0)}_{p+1,q} \quad (40)$$

And for the pumping node, using equations (21) and (21)

For Row Iteration at n = o,

$$H^{(1)}_{p-1,q} - 4H^{(1)}_{p,q} + H^{(1)}_{p+1,q} = -H^{(0)}_{p,q-1} - H^{(0)}_{p,q+1} - 5 \quad (41)$$

For Column Iteration at n = o,

$$H^{(1)}_{p,q-1} - 4H^{(1)}_{p,q} + H^{(1)}_{p,q+1} = -H^{(0)}_{p-1,q} - H^{(0)}_{p+1,q} - 5 \quad (42)$$

Alternating the row and the column, the results after iteration 203 is presented in table 2

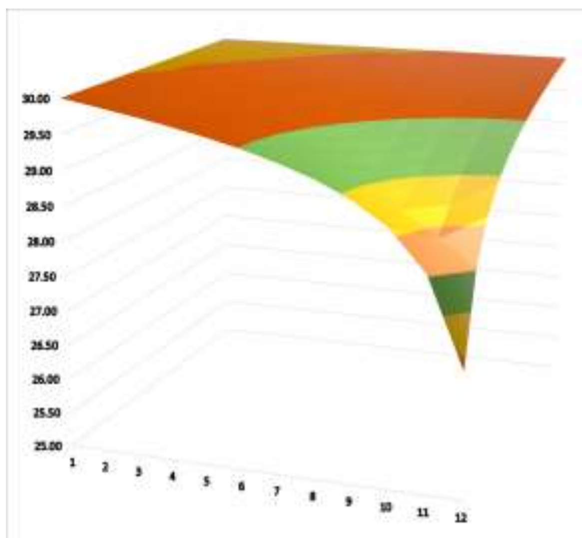

Fig 9: Surface Plot of ADI for Example 3

Method 2: Improved Alternating Direction Implicit Method (IADI)

Using Table 1 and equations (23) and (24), generate the equations for the Improved Alternating Direction implicit method (ADI), for n = 0 and k = 1.1, applying the boundary condition as in the second method

the row equation becomes

$$H_{p-1,q}^{(1)} - (3.1)H_{p,q}^{(1)} + H_{p+1,q}^{(1)} = -H_{p,q-1}^{(0)} - H_{p,q+1}^{(0)} + (0.9)H_{p,q}^{(0)} \quad (43)$$

the column equation becomes

$$H_{p,q-1}^{(2)} - (3.1)H_{p,q}^{(2)} + H_{p,q+1}^{(2)} = -H_{p-1,q}^{(1)} - H_{p+1,q}^{(1)} + (0.9)H_{p,q}^{(1)} \quad (44)$$

At the well node, for p = 0, k=1.1, using equations (25) and (26) for row and column iterations

$$H_{p-1,q}^{(1)} - (3.1)H_{p,q}^{(1)} + H_{p+1,q}^{(1)} = -H_{p,q-1}^{(0)} - H_{p,q+1}^{(0)} + (0.9)H_{p,q}^{(0)} - 5 \quad (45)$$

While the column iteration uses

$$H_{p,q-1}^{(2)} - (3.1)H_{p,q}^{(2)} + H_{p,q+1}^{(2)} = -H_{p-1,q}^{(1)} - H_{p+1,q}^{(1)} + (0.9)H_{p,q}^{(1)} - 5 \quad (46)$$

Alternating the row and the column, the result after iteration 48 is presented in Table 3

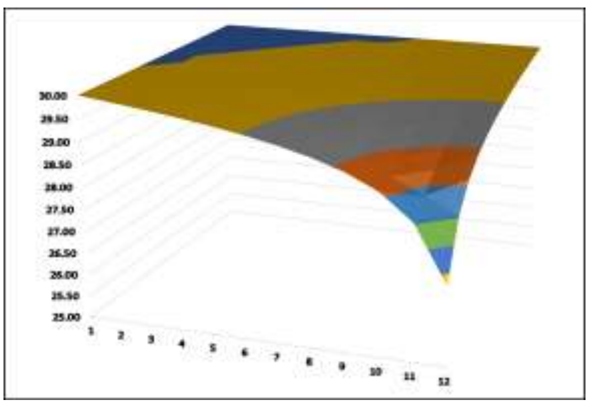

Fig 10: Surface Plot of IADI for Example 3

Example 4: Unsteady Flow

A rectangular aquifer is subdivided into a 7x7 grid with uniform grid spacing of 100m in both x and y directions. The Transmissivity is homogeneous at a value of $0.1m^2/s$. The Storage coefficient is 0.001. The North and South Boundaries are impervious with zero flux. The west and East boundaries are constant head boundaries with head values at 50m, which also serves as the initial head at all nodes at time t = 0. Recharge due to precipitation is zero (0). A pumping well starts operating at time t = 0 at a constant rate of $1m^3/s$, located at node (4, 4), take Δt = 10s, Compute heads for the interior nodes at time t = 50s and 100s. (Kinzelbach, 1986)

Given: $T = 0.1m^2/s$, S = 0.001, $R = 1m^3/s$, H (east and west) = 50m, H_y (north and south) = 0

Δt = 10s, Δx = Δy = 100m

$$\frac{R}{T} = \frac{1}{0.1} = 10, \quad r = \frac{T\Delta t}{S\Delta x^2} = \frac{0.1*10}{0.001*10000} = 0.1$$

where $\gamma = \frac{T\Delta t}{S(\Delta x)^2} = \frac{0.1x10}{0.001x10000} = 0.1$, $R = \frac{Q}{(\Delta x)^2} = \frac{1}{100^2} = 0.0001$

therefore, $\frac{R(\Delta x)^2}{T} = \frac{0.0001*100^2}{0.1} = 10$

Method 1: Explicit Finite Difference Method (EFDM)

Using table 1 and equation (47) for the nodes without a well at time t = 0, $\gamma = 0.1$

$$H_{p,q}^{1} = H_{p,q}^{0} + 0.1\left(H_{p+1,q}^{0} + H_{p-1,q}^{0} + H_{p,q+1}^{0} + H_{p,q-1}^{0} - 4H_{p,q}^{0}\right) \quad (47)$$

And equation (48) for the pumping well (pumping takes place at node **18)**

$$H_{p,q}^{1} = H_{p,q}^{0} + 0.1\left(H_{p+1,q}^{0} + H_{p-1,q}^{0} + H_{p,q+1}^{0} + H_{p,q-1}^{0} - 4H_{p,q}^{0} - 10\right) \quad (48)$$

All initial values = 50 and applying the boundary conditions

Use initial time t = 0 to compute the results at the next time step of time t = 10secs, and the results at time t = 100 seconds are presented in figure 11

Table 1: mathematical interpretation of Example 4 with the well at the centre

No flow boundary		**H1**	**H2**	**H3**	**H4**	**H5**	
grid	100	200	300	400	500	600	700
100	50	H1	H2	H3	H4	H5	50
200	50	H6	H7	H8	H9	H10	50
300	50	H11	H12	H13	H14	H15	50
400	50	H16	H17	**well**	H19	H20	50
500	50	H21	H22	H23	H24	H25	50
600	50	H26	H27	H28	H29	H30	50
700	50	H31	H32	H33	H34	H35	50
No flow boundary		**H31**	**H32**	**H33**	**H34**	**H35**	

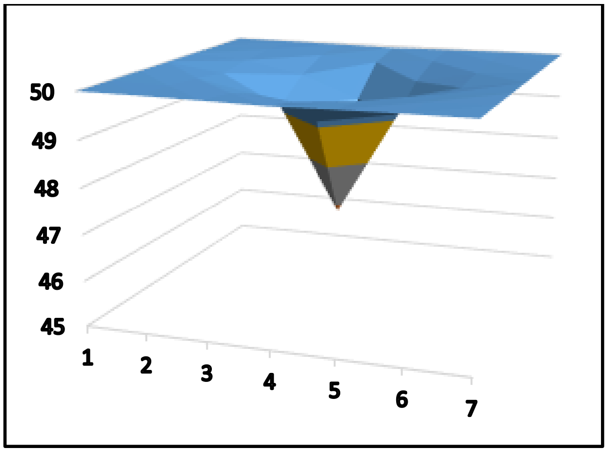

Fig 11: Surface Plot of EFDM for Example 4 at time t=100seconds

Method 2: Crank Nicolson Method

Using equation (49) for the nodes without the pumping well at time t = 0, $\varphi = 0.1$

$$H_{p,q}^{1} = H_{p,q}^{0} + \frac{1}{2}0.1\big(H_{p+1,q}^{1} + H_{p-1,q}^{1} + H_{p,q+1}^{1} + H_{p,q-1}^{1} - 4H_{p,q}^{1} + +H_{p+1,q}^{0} + +H_{p-1,q}^{0} + H_{p,q+1}^{0} + H_{p,q-1}^{0} - 4H_{p,q}^{0}\big) \quad (49)$$

And equation (50) for the nodes with the pumping well (Note that pumping takes place at node **18** that is H18**)**

$$H_{p,q}^{1} = H_{p,q}^{0} + \frac{1}{2}0.1\big(H_{p+1,q}^{1} + H_{p-1,q}^{1} + H_{p,q+1}^{1} + H_{p,q-1}^{1} - 4H_{p,q}^{1} + +H_{p+1,q}^{0} + + H_{p-1,q}^{0} + H_{p,q+1}^{0} + H_{p,q-1}^{0} - 4H_{p,q}^{0} - 2(10)\big) \quad (50)$$

Use the initial time t = 0 to compute the results at the next time step of time t = 10 seconds, and the results of distribution of heads at time t = 100s are presented in figure 12

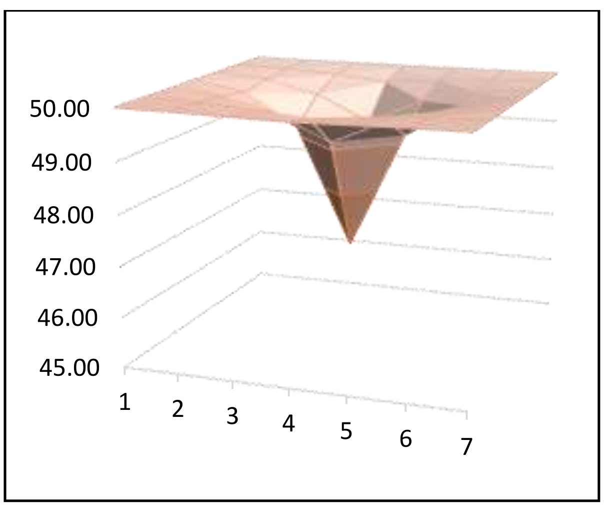

Fig 12: Surface Plot of EFDM for Example 4 at time t=100seconds

RESULTS AND DISCUSSION

Table 2 shows the results of ADI and IADI in example 1 and their comparison, the results of the comparison give zero error showing that the two methods can be used to solve steady-state groundwater flow equations. However, comparing the results using the number of iterations in Table 3 showed that IADI has the least number of iterations, this finding can be interpreted as support that IADI is the most effective method of solutions ((Karvonen, 2002; Shittu *et. al.*, 2024).

Table 2: Comparing the results of Example 1

HEADS	ADI	IADI	ERROR
H1	8.095937	8.095938	0.000001
H2	8.238661	8.238661	0.000000
H3	8.396830	8.396830	0.000000
H4	7.825089	7.825089	0.000000
H5	8.031875	8.031875	0.000000
H6	8.238661	8.238660	-0.000001
H7	7.522544	7.522544	0.000000
H8	7.825089	7.825089	0.000000
H9	8.095937	8.095937	0.000000

Table 3. Comparing the number of iterations of Example 1

Method	No. of Iterations
ADI	20
IADI	11

Table 4: Comparing the results of example 2

Heads	ADI	IADI	ERROR
H1	1.200000	1.200000	0.000000
H2	1.400000	1.400000	0.000000
H3	1.600000	1.600000	0.000000
H4	1.800000	1.800000	0.000000
H5	1.200000	1.200000	0.000000
H6	1.400000	1.400000	0.000000
H7	1.600000	1.600000	0.000000
H8	1.800000	1.800000	0.000000
H9	1.200000	1.200000	0.000000
H10	1.400000	1.400000	0.000000
H11	1.600000	1.600000	0.000000
H12	1.800000	1.800000	0.000000
H13	1.200000	1.200000	0.000000
H14	1.400000	1.400000	0.000000
H15	1.600000	1.600000	0.000000
H16	1.800000	1.800000	0.000000
H17	1.200000	1.200000	0.000000
H18	1.400000	1.400000	0.000000
H19	1.600000	1.600000	0.000000
H20	1.800000	1.800000	0.000000
H21	1.200000	1.200000	0.000000
H22	1.400000	1.400000	0.000000
H23	1.600000	1.600000	0.000000
H24	1.800000	1.800000	0.000000

Table 4 showed the results of the two methods of solution of example 2, ADI and IADI and they yield the same results with zero error, this illustrated that the two methods are reliable in solving the steady-state groundwater flow governing equation, but comparing using the number of iterations in Table 5 showed that IADI converged faster than ADI with the lowest number of iterations, this shows that IADI is reliable and the best method of solution (Shittu *et. al.*, 2024).

Table 5: Comparing the number of iterations of example 2

Method	No. of Iterations
ADI	45
IADI	19

Table 6 compares the results of the two methods used in example 3, it could be observed that the results give zero error, this shows that the two methods are applicable in finding the solution of the governing equation of the steady-state groundwater flow, however Table 7 illustrated the comparison of the two methods using the number of iterations and it was also observed that IADI has the least number of iterations which makes IADI more reliable than ADI (Karvonen, 2002).

Table 6: Comparing the results of example 3

HEADS	ADI	IADI	ERROR
H1	26.78456	26.78456	0.000000
H2	28.03456	28.03456	0.000000
H3	28.60147	28.60147	0.000000
H4	28.93602	28.93602	0.000000
H5	29.16972	29.16972	0.000000
H6	29.34955	29.34955	0.000000
H7	29.49618	29.49618	0.000000
H8	29.6204	29.6204	0.000000
H9	29.72873	29.72873	0.000000
H10	29.82566	29.82566	0.000000
H11	29.91487	29.91487	0.000000

Table 8: Comparing the result of example 4 at time t =100seconds

Heads	Explicit Method	Crank Nicolson Method
H1	49.90575	49.90061
H2	49.80435	49.80411
H3	49.90575	49.90061
H4	49.64371	49.65187
H5	49.14390	49.16925
H6	46.85207	49.65187
H7	49.14441	49.17025
H8	46.85207	46.90569
H9	49.14441	49.17025
H10	49.64371	49.65187
H11	49.14390	49.16925
H12	49.64371	49.65187

In table 8, the results of using Explicit finite difference method and Crank Nicolson method for example 4 is shown, and the results showed that the two methods can be effectively used in finding the solution of transient groundwater flow equation, therefore the two methods (EFDM and CNM) can be used for all real life engineering problems.

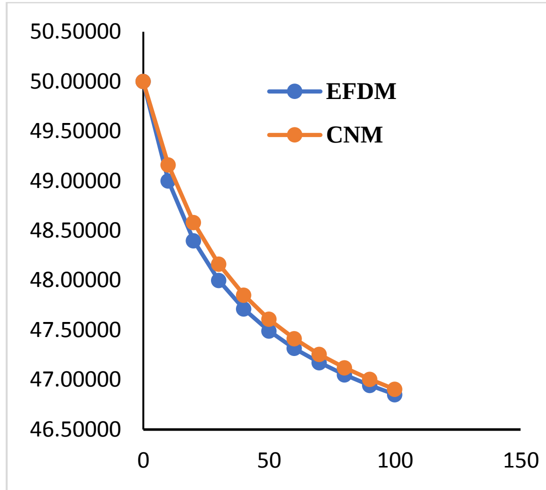

Fig 13: Pumping time against hydraulic head

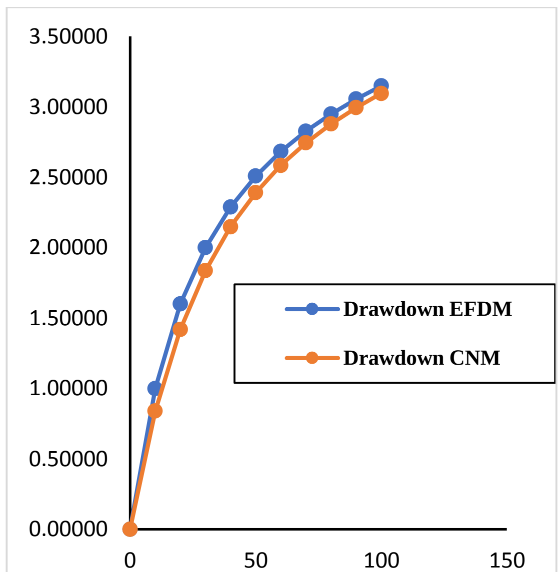

Fig 14: Pumping time against drawdown

Figure 13 shows that increase in time resulted in a decrease in hydraulic head which suggests that the direction of groundwater flow is from higher elevation to lower elevation However in Figure 14, it could be observed that drawdown increases with time which shows the effects of over-pumping on groundwater flow. When water is pumped faster than it is recharged, drawdown increases, this lowers the cone of depression to the bottom of the water and thereby the water in the well dries up (Kinzelbach, 1986).

Conclusion: In this study, the investigation of direction of groundwater flow and factors affecting drawdown and effects of over-pumping for steady and unsteady states has been carried out. The problems considered showed that the two methods used are efficient and reliable in solving steady and unsteady-state groundwater flow and can be used for all partial differential equations of real-life problems and that the direction of flow of groundwater is from higher hydraulic head to lower hydraulic head and that over pumping lowers the cone of depression to the bottom of the well and thereby dries up the water in the well.

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