



Elucidation of the Fourth-Order Generalized Schrodinger Problem for Hydrogen Atom using Frobenius Series Application

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ABSTRACT: The hydrogen atom is a fundamental system in quantum mechanics that has been extensively studied due to its simplicity and relevance to atomic physics. The accurate determination of the energy levels of the hydrogen atom is of paramount importance for various applications in physics and chemistry. Hence, the objective of this work is to solve the fourth-order generalized Schrodinger problem for the hydrogen atom using the Frobenius series application. Findings obtained demonstrate the effectiveness and efficiency in determining the energies of the hydrogen atom, thereby contributing to a deeper understanding of quantum mechanical systems.

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In quantum mechanics, point interactions refer to idealized interactions that occur at a single point in space. These interactions are used in theoretical models to simplify the description of certain physical systems (Arnbak *et al.*, 2011; Calcada *et al.*, 2019; Coutinho *et al.*, 1997; Gadella *et al.*, 2009; Schmidt *et al.*, 2002; Zolotaryuk, 2010)

One of the most common examples of a point interaction is the delta function potential, often denoted as $\alpha\delta(x)$, where α is a constant and $\delta(x)$ is the Dirac delta function. The Dirac delta function is a mathematical construct that is zero everywhere except at $x = 0$, where it is infinite, yet integrates to 1 over the entire real line.

In the context of quantum mechanics, a delta function potential can model, for instance, an electron

interacting with a localized impurity or a particle interacting with an infinitely thin, infinitely high potential barrier (Lieb, 1963; Lieb and Liniger, 1963). Other applications are seen in (Comtet *et al.*, 2010; Exner, 1995; Kundu, 1999). Solving the Schrödinger equation with point interactions can be quite challenging because the delta function potential introduces singularities in the equations. One-dimensional non-relativistic quantum mechanics' simplest point interaction is provided by the well-defined and well-known Dirac's δ -function potential. However, attempts to consider more generic interactions, like those connected to a potential, have been known to run into problems with the description of the interaction (Albeverio *et al.*, 1993; Griffiths,

1993; Roman and Tarrach, 1996; Seba, 1986a, 1986b; Zhao, 1992)

It's important to note that point interactions are idealized models and do not represent real-world physical interactions which are typically smoothed out over some finite region. They serve as useful tools for gaining insight and understanding in specific situations. The one-dimensional hydrogen atom, often referred to as a quantum mechanical model of a hydrogen atom in one spatial dimension, is a simplified theoretical framework used to gain insights into the behavior of electrons in hydrogen-like systems. In this model, the electron's motion is constrained to a single dimension, typically represented along the x-axis. One key aspect of the one-dimensional hydrogen atom is the singularity at $x = 0$ in the potential energy term. This singularity is indicative of a point interaction, which means that the potential energy goes to infinity as the distance between the electron and the nucleus approaches zero. This singularity is a characteristic feature of the Coulomb potential (Palma and Raff, 2006). Due to this singularity, the one-dimensional hydrogen atom has some unique properties and poses mathematical challenges. The solutions to the Schrödinger equation in this scenario involve specialized functions known as irregular and regular Coulomb wave functions, which are complex and not as well-behaved as the solutions in three dimensions. Due to the existence of singularities at the origin, one-dimensional quantum mechanical issues can be more difficult and complex from a topological standpoint than their three-dimensional counterparts. The one-dimensional hydrogen atom, which has been solved using a wide range of techniques, serves as an excellent example of this (Abramovici and Avishai, 2009; Calçada *et al.*, 2014; Calçada *et al.*, 2019; Carrillo-Bernal *et al.*, 2015; Coutinho and Amaku, 2009; Gebremedhin and Weatherford, 2015; Ivetic, 2018; Jaramillo *et al.*, 2009; Kurasov, 1996; Loudon, 2016; Palma and Raff, 2006)

Despite these methods and clear conclusions, there is still controversy in the literature regarding the parity of the solutions and whether or not the ground state is limited. It is also evident that the various approaches limit their solutions to only second order in treating the one-dimensional hydrogen atom. The fourth order equation has not yet yielded the approximate bound and scattering state solutions that perturbative methods did for the harmonic oscillator problem (Koffa *et al.*, 2013; Walter, 1971). The exact solution of the fourth-order differential equation has not been looked into, despite the fact that these works are fascinating and greatly progress novel physics. A full

understanding of the characteristics of the hydrogen atom in one dimension is anticipated from such a consideration. Hence, the objective of this work is to solve the fourth-order generalized Schrodinger problem for the hydrogen atom using the Frobenius series application.

Theory

To the fourth order of \hbar , our complete quantum mechanical wave equation for the hydrogen atom reduces to

$$i\hbar \frac{\partial}{\partial t} \Psi(r, t) = \left(m_0 c^2 - \frac{\hbar^2}{2m_0} \nabla^2 - \frac{\hbar^4}{8m_0^3} \nabla^4 - \frac{ze^2}{r} \right) \Psi(r, t) \quad (1)$$

Where Ψ is the fourth order quantum mechanical wavefunction of the hydrogen atom. The fourth order quantum mechanical wave function of the hydrogen atom is precisely the second order quantum mechanical wave function for the hydrogen atom but it is augmented by the most natural and unique (with no analogy in any previous theory of quantum mechanics) and hence hitherto unknown quantum mechanical energy wave operator for more physical completeness, based upon the experimental physical facts available.

$$- \frac{\hbar^4}{8m_0^3} \nabla^4 \quad (2)$$

Now the variable may be separated in the fourth order quantum mechanical wavefunction for the hydrogen atom as;

$$\Psi(r, t) = U(r) \exp\left(\frac{-iEt}{\hbar}\right) \quad (3)$$

Where E is the fourth order quantum mechanical energy and U is the corresponding fourth order quantum mechanical energy wavefunction which is given by;

$$0 = \frac{\hbar^4}{8m_0^3} \nabla^4 U(r) + \frac{\hbar^2}{2m_0} \nabla^2 U(r) + \left[(E - m_0 c^2) + \frac{ze^2}{r} \right] U(r) \quad (4)$$

Subject to the conditions of uniqueness and regularity everywhere and continuity across all boundaries and normalization. The fourth order quantum mechanical energy wave equation for the hydrogen atom (4) may be written equivalently and more precisely as;

$$0 = \alpha \nabla^4 U(r) + \beta \nabla^2 U(r) + \left[(E - m_0 c^2) + \frac{ze^2}{r} \right] U(r) \quad (5)$$

Where

$$\gamma = \frac{\hbar^4}{8m_0^3} \text{ and } \beta = \frac{\hbar^2}{2m_0} \quad (5)$$

Precisely as in the solution of the second-order quantum mechanical wave equation above, we seek the solution of (5) as

$$U(r) = R(r)\gamma_{lm}(\theta, \Phi) \quad (6)$$

Where γ_{lm} is the spherical harmonic function of order lm and $R(r)$ is the fourth order quantum mechanical wave energy radial wavefunction.

The choice of equation (6) transforms equation (5) into the form;

$$0 = \gamma \left[R''''(r) + \frac{4}{r}R'''(r) - \frac{2l(l+1)}{r^2}R''(r) - \frac{4}{r^3}R'(r) + \frac{l(l+1)(l^2+l-2)}{r^4}R(r) \right] + \beta \left[R''(r) + \frac{2}{r}R'(r) - \frac{l(l+1)}{r^2}R(r) \right] + \left[(E - m_0c^2) + \frac{ze^2}{r} \right] R(r) \quad (7)$$

Next as in the solution of the second order quantum mechanical wave equation above, we seek the solution of equation (7) as

$$R(r) = \frac{1}{r}Q(r) \quad (8)$$

Which upon substitution into eqn (7) yields

$$0 = \gamma \left[Q''''(r) - \frac{2l(l+1)}{r^2}Q''(r) + \frac{4l(l+1)}{r^3}Q'(r) + \frac{l(l+1)(l^2+l-6)}{r^4}Q(r) \right] + \beta \left[Q''(r) - \frac{l(l+1)}{r^2}Q(r) \right] + \left[(E - m_0c^2) + \frac{ze^2}{r} \right] Q(r) \quad (9)$$

By introducing the dimensionless variable ξ defined by;

$$\xi = \frac{r}{a} \quad (10)$$

Where “a” is a constant parameter. Then, the fourth order quantum mechanical energy radial wave equation for the hydrogen atom (7) transforms as;

$$0 = \frac{\gamma}{a^4} [Q''''(\xi) - 2l(l+1)\xi^{-2}Q''(\xi) + 4l(l+1)\xi^{-3}Q'(\xi) + l(l+1)(l^2+l-6)\xi^{-4}Q(\xi)] + \frac{\beta}{a^2} [Q''(\xi) - l(l+1)\xi^{-2}Q(\xi)] + \left[(E - m_0c^2) + \frac{ze^2}{a}\xi^{-1} \right] Q(\xi) \quad (11)$$

We proceed to seek the solution of eqn (11) in the form

$$Q(\xi) = \xi^{l+1}W(\xi) \quad (12)$$

This choice further transforms eqn (12) as follows;

$$0 = \frac{\gamma}{a^4} [W'''' + 4(l+1)\xi^{-1}W''' + 4l(l+1)\xi^{-2}W'' - 4l(l+1)\xi^{-3}W'] + \frac{\beta}{a^2} [W'' + 2(l+1)\xi^{-1}W'] + \left[(E - m_0c^2) + \frac{ze^2}{a}\xi^{-1} \right] W \quad (13)$$

We seek the solution of eqn (13) in the form

$$W = \exp(-\lambda\xi)F \quad (14)$$

It follows that eqn (14) can be expressed as;

$$0 = F'''' + [-4\lambda + 4(l+1)\xi^{-1}]F''' + \left[(6\lambda^2 + \frac{\beta a^2}{\gamma}) - 12\lambda(l+1)\xi^{-1} + 4l(l+1)\xi^{-2} \right] F'' + \left\{ \left[-4\lambda^3 - \frac{2\lambda\beta a^2}{\gamma} \right] + \left[12\lambda^2(l+1) + \frac{2(l+1)\beta a^2}{\gamma} \right] \xi^{-1} - 8\lambda l(l+1)\xi^{-2} - 4l(l+1)\xi^{-3} \right\} F' + \left\{ \left[\lambda^4 + \frac{\beta a^2}{\gamma} \lambda^2 + \frac{a^4(E - m_0c^2)}{\gamma} \right] + \left[-4\lambda^3(l+1) - \frac{2(l+1)\lambda\beta a^2}{\gamma} + \frac{a^3ze^2}{\gamma} \right] \xi^{-1} + 4\lambda^2 l(l+1)\xi^{-2} + 4\lambda l(l+1)\xi^{-3} \right\} F \quad (15)$$

For the purpose of the equivalence of the independent variable ξ in both the second and fourth order quantum mechanical energy wave equations for the hydrogen atom, we choose the parameter;

$$a = \frac{\hbar^2}{m_o e^2} \tag{16}$$

We also choose the parameter λ such that the constant term in the coefficient of F in eqn (15) vanishes.

$$0 = \lambda^4 + \frac{\beta a^2}{\gamma} \lambda^2 + \frac{a^4(E - m_o c^2)}{\gamma} \tag{17}$$

Explicitly, using (16) and (5a)

$$E = m_o c^2 - \left(\frac{m_o e^4}{2\hbar^2}\right) \lambda^2 - \left(\frac{m_o e^8}{8c^2 \hbar^4}\right) \lambda^4 \tag{18}$$

Now, with the above choices, the fourth order quantum mechanical energy radial wave equation (15) becomes

$$0 = F'''' + [-4\lambda + 4(l + 1)\xi^{-1}]F''' + \left[\left(6\lambda^2 + \frac{\beta a^2}{\gamma}\right) - 12(l + 1)\lambda\xi^{-1} + 4l(l + 1)\xi^{-2}\right]F'' + \left\{\left[-4\lambda^2 - \frac{2\beta a^2 \lambda}{\gamma}\right] + \left[12(l + 1)\lambda^2 + \frac{2(l+1)\beta a^2}{\gamma}\right]\xi^{-1} - 8l(l + 1)\lambda\xi^{-2} - 4l(l + 1)\xi^{-3}\right\}F' + \left\{\left[-4(l + 1)\lambda^3 - \frac{2(l+1)\beta a^2 \lambda}{\gamma} + \frac{z a^3 e^2}{\gamma}\right]\xi^{-1} + 4l(l + 1)\lambda^2\xi^{-2} + 4l(l + 1)\lambda\xi^{-3}\right\}F \tag{19}$$

The quantum mechanical energy radial wave equation (19) has thus been reduced to the form that admits series solution. Thus, using the Frobenius series solution of the form

$$F = \sum_{n=0}^{\infty} A_n \xi^{s+n} \tag{20}$$

Where ‘s’ is a constant index and A_n are constants. Substituting (20) into (19) with necessary transformations yields

$$0 = \sum_{n=0}^{\infty} (s + n)(s + n - 1)(s + n - 2)(s + n - 3)A_n \xi^{s+n-4} - 4\lambda \sum_{n=1}^{\infty} (s + n - 1)(s + n - 2)(s + n - 3)A_{n-1} \xi^{s+n-4} + 4(l + 1) \sum_{n=0}^{\infty} (s + n)(s + n - 1)(s + n - 2)A_n \xi^{s+n-4} + \left[6\lambda^2 + \frac{\beta a^2}{\gamma}\right] \sum_{n=2}^{\infty} (s + n - 2)(s + n - 3)A_{n-2} \xi^{s+n-4} - 12(l + 1)\lambda \sum_{n=1}^{\infty} (s + n - 1)(s + n - 2)A_{n-1} \xi^{s+n-4} + 4l(l + 1) \sum_{n=0}^{\infty} (s + n)(s + n - 1)A_{n-2} \xi^{s+n-4} + \left[-4\lambda^2 - \frac{2\beta a^2 \lambda}{\gamma}\right] \sum_{n=3}^{\infty} (s + n - 3)A_{n-3} \xi^{s+n-4} + \left[12(l + 1)\lambda^2 + \frac{2(l+1)\beta a^2}{\gamma}\right] \sum_{n=2}^{\infty} (s + n - 2)A_{n-2} \xi^{s+n-4} - 8l(l + 1)\lambda \sum_{n=1}^{\infty} (s + n - 1)A_{n-1} \xi^{s+n-4} - 4l(l + 1) \sum_{n=0}^{\infty} (s + n)A_n \xi^{s+n-4} + \left[-4(l + 1)\lambda^3 - \frac{2(l+1)\beta a^2 \lambda}{\gamma} + \frac{z a^3 e^2}{\gamma}\right] \sum_{n=3}^{\infty} (s + n)A_{n-3} \xi^{s+n-4} + 4l(l + 1)\lambda^2 \sum_{n=2}^{\infty} (s + n)A_{n-2} \xi^{s+n-4} + 4l(l + 1)\lambda \sum_{n=1}^{\infty} A_{n-1} \xi^{s+n-4} \tag{21}$$

Now setting $n = 0$, the indicial equation can be expressed as

$$s^2 + 4ls + 4l^2 - 1 = 0 \tag{22}$$

Therefore, the solutions of the indicial equation are

$$s = 0; \quad s = 2; \quad s = -2l + 1; \quad s = -2l - 1 \tag{23}$$

The general recurrence relation is given by

$$0 = [(s + n)(s + n - 1)(s + n - 2)(s + n - 3) + 4(l + 1)(s + n)(s + n - 1)(s + n - 2) + 4l(l + 1)(s + n)(s + n - 1) - 4l(l + 1)(s + n)]A_n + [-4(s + n - 1)(s + n - 2)(s + n - 3) - 12(l + 1)(s + n - 1)(s + n - 2) - 8l(l + 1)(s + n - 1) + 4l(l + 1)]\lambda A_{n-1} + \left\{[6(s + n - 2)(s + n - 3) + 12l(l + 1)(s + n - 2) + 4l(l + 1)]\lambda^2 + (s + n - 2)(s + n + 2l - 1)\frac{\beta a^2}{\gamma}\right\}A_{n-2} + \left[-4(s + n + l - 2)\lambda^3 - 2(s + n + l - 2)\frac{\beta a^2}{\gamma}\lambda + \frac{z e^2 a^3}{\gamma}\right]A_{n-3}; \quad n = 3,4,5 \dots \tag{24}$$

RESULTS AND DISCUSSION

For ground level of the hydrogen atom, we choose the coefficient of A_0 in the recurrence relation for A_3 in (24) to vanish.

$$0 = -4(s + n + l - 2)\lambda^3 - 2(s + l + 1)\frac{\beta a^2}{\gamma}\lambda + \frac{z e^2 a^3}{\gamma} \tag{25}$$

The above equation determines the quantum mechanical parameter λ which we shall demote by λ_{nlm} and the corresponding quantum mechanical Eigen energy for the ground level of the hydrogen atom denoted by E_{0lm} . From the case of the ground level, it is obvious that the general level of the hydrogen atom is obtained by choosing the coefficient of A_{n-3} in the recurrence relation for A_n in (24) to vanish. This can be written conveniently expressed as

$$0 = -4(s + n + l + 1)\gamma\lambda^3 - 2(s + n + l + 1)\beta a^2 \lambda + ze^2 a^3 \tag{26}$$

Neglecting the first term containing γ as small compared to others, it follows that;

$$\lambda = \frac{z}{s + n + l + 1} \tag{27}$$

Consequently, the fourth order quantum mechanical eigenvalue reduces to the second order quantum mechanical Eigenvalue as the first approximation for all levels of the hydrogen atom.

To obtain the general solution of the quantum mechanical Eigenvalue equation for the hydrogen atom, we write (26) more conveniently as;

$$0 = \lambda^3 + B\lambda + D \tag{28}$$

Where

$$B = \frac{\beta a^2}{2\gamma} = \frac{2\hbar^2 c^2}{e^4} \tag{29}$$

$$D = \frac{ze^2 a^3}{4(s + n + l + 1)\gamma} = -\frac{2z\hbar^2 c^2}{(s + n + l + 1)e^4} \tag{30}$$

Equation (28) is therefore a cubic equation in standard form with positive determinant. Since $B > 0$, it follows from the cubic formula that equation (28) possesses exactly one real root given by;

$$\lambda = \left(\frac{4B}{3}\right)^{1/2} \sinh \left\{ \frac{1}{3} \sinh^{-1} \left[\frac{-2D}{B \left(\frac{4B}{3}\right)} \right] \right\} \tag{31}$$

now follows that for each of the quantum number , n, the value of the quantum mechanical parameter λ for the hydrogen atom is given by (31) and which may be denoted as λ_{nlm} and the quantum mechanical energy relation (18) defines the exact quantum mechanical Eigen energies for the general level of the hydrogen atom which may be denoted as E_{nlm} .

It can be shown that by series expansion and a lot of manipulations, from equation (31) that

$$\lambda_{nlm} = \frac{z}{(s + n + l + 1)} \left[1 - \frac{z^2 e^4}{2(s + n + l + 1)^2 \hbar^2 c^2} \dots \right] \tag{32}$$

Consequently, the quantum mechanical Eigen energies of the hydrogen atom are given approximately but more explicitly from equations (18) and (33) as;

$$E_{nlm} = m_0 c^2 - \frac{m_0 z^2 e^4}{2(s + n + l + 1)^2 \hbar^2} \left[1 - \frac{z^2 e^4}{2(s + n + l + 1)^2 \hbar^2 c^2} \dots \right] - \frac{m_0 z^4 e^8}{8(s + n + l + 1)^4 \hbar^4 c^2} \left[1 - \frac{z^2 e^4}{2(s + n + l + 1)^2 \hbar^2 c^2} \dots \right] \tag{33}$$

This is the expression for the quantum mechanical energies of the hydrogen atom. For each level of the hydrogenic atom, $n = 0,1,2,3, \dots$, the quantum mechanical Eigen value λ_{nlm} , given by equation (32) together with the recurrence relations determine four linearly independent solutions of the quantum mechanical wave equation (20) denoted by $F_{nlm,i,i=0,1,2,3}$, at least one of which gives the corresponding quantum mechanical wave function. The first profound physical results of the work in this paper is the discovery of the indefinitely fine

corrections of the famous Schrodinger's quantum mechanical Eigen energies of the hydrogenic atom which are more significant as the atomic number becomes lager (or equivalently as the electron moves faster compared with the speed of light in vacuo. The second profound physical result of our work is the discovery of the indefinitely fine extension of the renowned Schrodinger's quantum mechanical Eigen functions of the hydrogenic atom which are also very significant with larger atomic number (equivalently, as the electron moves faster compared with the speed of light in vacuo). This is where lies a profound

experimental challenge of our new discovery to the law of quantum mechanical theory. The mathematical result of this paper is also the discovery of infinitely many complete sequences of orthogonal functions which are bases of the mathematical space $C(0, \infty)$ or $L(0, \infty)$. Each of our bases is unprecedented and mathematically the most elegant and interesting generalization or extension of the established basis of confluent hypergeometric functions. Each of our bases is generated from a Sturm-Liouville differential equation of order greater than two, compared to the second order confluent hypergeometric differential equation. Consequently, our discovery to the law of quantum mechanics in paper has opened up the profound mathematical revolution into fourth and higher order Sturm-Liouville differential equations in the mathematical space $C(0, \infty)$ Herein lies an eternal mathematical interest of our discovery to the law of quantum mechanics in this paper.

Hydrogenic atom is the basis of all atomic physics, consequently, the fourth and higher order of \hbar to the Eigen solution of the quantum mechanical problem of the hydrogenic atom in this paper imply corresponding unprecedented revolutions in atomic physics and all areas of it applications.

It is well-known that the Laguerre polynomial function L_n of mathematical physics are related to the confluent hypergeometric function F by;

$$L_n(x) = \Gamma(n + 1)F(-n, l, x) \quad (34)$$

for all-natural number n . Consequently, by replacing the confluent hypergeometric function by each of the formerly un-known generalization of the confluent hypergeometric function in this paper, we obtain corresponding hitherto unknown generalization of the classic Laguerre polynomial of mathematical physics. The generalized Laguerre function L_n^p are related to the confluent hypergeometric functions F by

$$L_n^p(x) = \frac{\Gamma(p + n + 1)}{\Gamma(p + 1)}F(-n, p + 1, x) \quad (35)$$

For arbitrary p and all natural numbers n . consequently by replacing the confluent hypergeometric functions in (35) by each of the previously unknown generalization of hypergeometric functions of this paper, we obtain corresponding generalization of the classic generalized hitherto unknown Laguerre functions of mathematical physics.

It is well known that the Hermite polynomials H_n are related to the generalized Laguerre function by;

$$H_n(x) = L_n^{\pm \frac{1}{2}}(x) \quad (36)$$

For all natural numbers n . Consequently, by replacing the confluent hypergeometric function in (36) by each of the once unfamiliar generalizations of the confluent hypergeometric functions in this paper, we obtain corresponding hitherto unknown generalizations of the classic generalized Hermit's polynomial functions of mathematical physics.

It has been established that although Dirac's relativistic law of quantum mechanics predicts corrections to the pure Schrodinger's quantum mechanical energy eigen values for the hydrogenic atom, Dirac's theory does not lead to any new sequence of orthogonal functions similar to the wave functions of the hydrogenic atom in this paper. It may be noted that for each level of the quantum mechanical energy eigen function for the hydrogenic atom, $F_{4,ntm}$ is a unique (hitherto unknown in any previous theory of quantum mechanics) and mathematically most elegant and natural generalization or extension of the second order quantum mechanical energy Eigen function of the hydrogenic atom $F_{2,ntm}$ (which are the well-established hypergeometric functions of mathematical physics). Consequently, our new addition to the law of quantum mechanics for all entities of non-zero rest masses at all speeds less than the speed of light in vacuo in all inertial reference frames applied to the electron in the coulomb's interaction field of a nucleus yielded the most natural and unique generalization of the confluent hypergeometric differential equation of mathematical physics. It may also be shown that, precisely as in the case of the second order quantum mechanical Eigen functions, the sequence of fourth order quantum mechanical Eigen functions for the hydrogen atom constitute an unknown complete orthonormal basis for the mathematical space $C(0, \infty)$ of all piecewise smooth and continuous functions on the interval $(0, \infty)$.

Conclusion: Conclusively, the profound discovery and contribution in this paper is the quantum mechanical wave equation for the hydrogen atom and its energies and eigen functions correct to the order of \hbar^2 and \hbar^4 . Therefore, the door is henceforth opened for the derivations of the Eigen energies and Eigen functions of the hydrogen atom correct to all orders of \hbar^{2n} ; $n = 1, 2, 3, \dots$ and hence their physical and mathematical applications.

$$i\hbar \frac{\partial}{\partial t} \Psi(r, t) = \left\{ m_0 c^2 \left[1 + \frac{\hbar^2}{m_0^2 c^2} \nabla^2 \left(1 - \frac{\hbar^2}{m_0^2 c^2} \nabla^2 \right)^{-1} \right]^{\frac{1}{2}} - \frac{ze^2}{r} \right\} \Psi(r, t) \quad (37)$$

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