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Hydrodynamics of Polymer in a Single Screw Extruder under Unsteady State Reaction

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ABSTRACT: Extrusion is the most important polymer processing operation. This paper focuses on obtaining an analytical solution for describing transient polymer movement and heat and mass transfer in the zone of polymer melting delay. The coupled nonlinear partial differential equations describing the phenomenon have been decoupled using the perturbation method and solved analytically using Eigenfunction expansion technique. The results obtained revealed that the Reynolds number and Eckert number have significant effects on the velocies, polymer temperature and mass flow rate.

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Most materials made from polymers must have passed through an extruder at least twice. (Vlachopoulos and Wayner, 2001). Screw extruders are polymer processing machines, they comprise of one or two rotating screws in a heated barrel. According to Vlachopoulos and Wayner (2001), the single screw extruder (SSE) is very essential to the plastics industry. The extruder channel comprises of three compartments or zones, namely the feed zone, compression zone, and the metering zone. In the feed zone, the polymer particles are compacted to form a solid bed by the rotation of the screw. In the compression zone, this solid bed is subjected to heat from the barrel heaters thus forming a thin film of molten polymer between the solid bed and the barrel wall. High rate of viscous dissipation results due to the high viscosity of the molten polymer. This viscous dissipation melts the solid bed within a short distance from when the melting started. At the metering zone, the melt flow is stabilized and the product leaves the extruder through the die (Vlachopoulos and Wagner, 2001). The hydrodynamics of polymeric flow under unsteady state reaction has been of research interest of late. Julian *et al.*, 2019 obtained a numerical solution of the equation for the synchronous unsteady motion of two spherical vesicles in incompressible viscous fluid in the presence of both Stokes drag and hydrodynamics memory. Hameed *et al.*, 2018 looked at the combined magneto hydrodynamic and electric field effect on an unsteady Maxwell Nano fluid flow over a stretching surface under the influence of variable heat and thermal radiation. Hassimi *et al.*, 2009 developed a mathematical model for describing the dynamic behaviour of the gas phase ethylene polymerization reactor. Pacelli *et al.*, 2001 reported on an investigation of the unsteady state flow of polymer solutions through granular porous media while Valdez and Tejero presented hydrodynamic interactions of dilute polymer solutions under shear flow in a narrow channel (Valdez and Tejero, 1994). This paper aimed at obtaining an analytical solution for describing transient polymer movement and heat and mass transfer in the zone of polymer melting delay.

Model Formulation: To construct a mathematical model of motion and heat and mass transfer in the zone of polymer melting delay of the extruder we assume the following:

1. The velocity of the solid bed is constant.
2. No flow along the channel depth. (z-axis) (i.e. w=0)
3. The screw channel is set on a plane and the rotation of the screw is restricted,

while the barrel and the screw rotates with the same speed but in the opposite direction.

4. Diffusion of heat along the channel (x-axis) is neglected.
5. Transient state is considered.

Based on the above assumptions, following Trufanova and Shcherbinin (2014) and introducing a new space variable as:

$$\varepsilon = x + y + z\sqrt{\frac{k_z}{k_y}} \qquad (1)$$

the system of constitutive equations for a liquid phase, which takes into account the variation of temperature in three directions, are:

$$\rho\frac{\partial u}{\partial t} + \frac{\partial p}{\partial \varepsilon} = b\frac{\partial}{\partial \varepsilon}\left(\mu\frac{\partial u}{\partial \varepsilon}\right) \qquad (2)$$

$$\rho\frac{\partial v}{\partial t} + \frac{\partial p}{\partial \varepsilon} = b\frac{\partial}{\partial \varepsilon}\left(\mu\frac{\partial v}{\partial \varepsilon}\right) \qquad (3)$$

$$G = \frac{\rho}{\sqrt{b}}\int_0^H u\,d\varepsilon \qquad (4)$$

$$\int_0^H v\,d\varepsilon = 0 \qquad (5)$$

$$\rho c_p\left(\frac{\partial T}{\partial t} + (u+v)\frac{\partial T}{\partial \varepsilon}\right) = k\frac{\partial^2 T}{\partial \varepsilon^2} + b\mu\left(\left(\frac{\partial u}{\partial \varepsilon}\right)^2 + \left(\frac{\partial v}{\partial \varepsilon}\right)^2\right) \qquad (6)$$

Where

$$b = \frac{k_z}{k_y} \qquad (7)$$

$$\mathrm{k} = \mathrm{k}_y\left(1 + b^2\right) \qquad (8)$$

The boundary conditions for the system are formulated as:

$$\left.\begin{array}{lll} u(\varepsilon,0)=0, & u(H,t)=V_0\cos\varphi, & u(H-h,t)=\mathrm{U} \\ v(\varepsilon,0)=0, & v(H,t)=V_0\sin\varphi, & v(H-h,t)=0 \\ T(\varepsilon,0)=T_0, & T(H,t)=T_b, & T(H-h,t)=T_m \end{array}\right\} \qquad (9)$$

Where

k is the coefficient of heat conductivity, c_p is the heat capacity, ρ is the density of the melted polymer, μ is the effective viscosity of the polymer melt, G is the mass flow rate, μ_0 is the consistency index, β is the temperature coefficient of viscosity (pre-exponential), I_2 is the square invariant of the strain rate tensor, n is the viscosity anomaly, T_b and T_m are the temperature of the barrel and the melting point of the polymer, h is the thickness of the melt film (the flight clearance), T_0 is the temperature of the solid polymer fed into the hopper, U is the velocity of the solid bed, T is the polymer temperature, P is the pressure, V_0 is the barrel velocity, u and v are the velocity components, x is the coordinate along channel, y is the coordinate along cross channel, z is the coordinate along channel height, t is time, L is the channel length, S is the channel width, H is the channel depth.

Method of Solution

Non-dimensionalization: Here, we non-dimensionalized equations (1) - (9), using the following dimensionless variables:

$$x' = \frac{x}{L},\quad \eta' = \frac{\eta}{h},\quad t' = \frac{U_0 t}{L},\quad U' = \frac{U}{U_0},\quad \mu' = \frac{\mu}{\mu_0},\quad \theta = \frac{T - T_0}{T_S - T_0} \qquad 10$$

And we obtain,

$$\frac{\partial u'}{\partial t'} + \frac{\partial p'}{\partial \varepsilon'} = \frac{1}{\mathrm{Re}}\frac{\partial}{\partial \varepsilon'}\left(\mu'\frac{\partial u'}{\partial \varepsilon'}\right) \qquad (11)$$

$$\frac{\partial v'}{\partial t'}+\frac{\partial p'}{\partial \varepsilon'}=\frac{1}{\mathrm{Re}}\frac{\partial}{\partial \varepsilon'}\left(\mu'\frac{\partial v'}{\partial \varepsilon'}\right) \quad (12)$$

$$G'=\frac{1}{\sqrt{b}}\int_0^1 u'd\varepsilon' \quad (13)$$

$$\int_0^1 v'd\varepsilon'=0 \quad (14)$$

$$\frac{\partial \theta}{\partial t'}+(u'+v')\frac{\partial \theta}{\partial \varepsilon'}=\frac{1}{Pe}\frac{\partial^2\theta}{\partial \varepsilon'^2}+\frac{Ec}{\mathrm{Re}}\mu'\left(\left(\frac{\partial u'}{\partial \varepsilon'}\right)^2+\left(\frac{\partial v'}{\partial \varepsilon'}\right)^2\right)$$
(15)

$$\mu=\left(\frac{I_2}{2}\right)^{\frac{n-1}{2}}\exp(\beta_1(1-\theta)) \quad (16)$$

Together with the following boundary conditions:

$$\left.\begin{array}{lll} u'(\varepsilon',0)=0, & u'(1,t')=\alpha\cos\varphi, & u'(1-a,t')=1 \\ v'(\varepsilon',0)=0, & v'(1,t')=\alpha\sin\varphi, & v'(1-a,t')=0 \\ \theta(\varepsilon',0)=0, & \theta(1,t')=\gamma, & \theta(1-a,t')=1 \end{array}\right\} \quad (17)$$

Where

$a=\dfrac{h}{H}$,

$\alpha=\dfrac{V_0}{U}$,

$E_c=\dfrac{U^2}{c_p(T_m-T_0)}=$ Eckert number,

$R_e=\dfrac{\rho UH^2}{L\mu_0}=$ Reynolds number,

$P_e=R_eP_r=\dfrac{\rho c_pUH^2}{kL}=$ Peclet number,

$\beta_1=\beta(T_m-T_0), \quad \sigma=\dfrac{(T_b-T_0)}{(T_m-T_0)}$

Analytical Solution: Here, we assume

$\dfrac{\partial p'}{\partial \varepsilon}=f=$ Constant and $\dfrac{\partial \theta}{\partial \varepsilon}=\dfrac{\partial \theta_m}{\partial \varepsilon}=g=$ constant, where θ_{m} is mean temperature defined as:

$$\theta_{\mathrm{m}}=\frac{2\int_0^L \pi xU\theta dx}{\pi x_0^2U_m} \quad (18)$$

The equations (11) – (17) was transformed using,

$$\bar{\varepsilon}=\varepsilon'+a-1 \quad (19)$$

By solving equations that resulted from the transformation we obtained the following solutions

$$u_0'(\bar{\varepsilon},t')=\sum_{n=1}^{\infty}B\left(1-e^{-st'}\right)\sin\left(\frac{n\pi}{a}\right)\bar{\varepsilon}+1+\left(\frac{\alpha\cos(\varphi)-1}{a}\right)\bar{\varepsilon} \quad (20)$$

$$v_0'(\bar{\varepsilon},t')=\sum_{n=1}^{\infty}B\left(1-e^{-st'}\right)\sin\left(\frac{n\pi}{a}\right)\bar{\varepsilon}+1+\left(\frac{\alpha\sin\varphi}{a}\right)\bar{\varepsilon} \quad (21)$$

$$\theta_0(\bar{\varepsilon},t')=1+\left(\frac{\gamma-1}{a}\right)\bar{\varepsilon}+\sum_{n=1}^{\infty}\phi_n(t')\sin\left(\frac{n\pi}{a}\right)\bar{\varepsilon} \quad (22)$$

$$\phi_n(t')=A_0\left(\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}A_1\left(\frac{1}{r}\left(1-e^{-rt'}\right)-\frac{2}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)+\frac{1}{(r-2s)}\left(e^{-2st'}-e^{-rt'}\right)\right)\right)$$
$$+\sum_{n=1}^{\infty}A_2\left(\frac{1}{r}\left(1-e^{-rt'}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)+\frac{A_3}{r}\left(1-e^{-rt'}\right) \quad (23)$$

$$u_1'(\bar{\varepsilon},t')=\sum_{n=1}^{\infty}u_{1n}(t')\sin\left(\frac{n\pi}{a}\bar{\varepsilon}\right) \quad (24)$$

$$u_{1n}(t')=\begin{pmatrix} B_4X_1+A_0(B_5-B_7)X_2+(B_5-B_7)X_3-X_4+A_3(B_5-B_7)X_5-B_5A_0X_6 \\ +B_5\sum_{n=1}^{\infty}A_2X_7+\frac{B_5A_3}{r}X_8-B_6X_9 \end{pmatrix} \qquad (25)$$

$$v_1'(\bar{\varepsilon},t')=\sum_{n=1}^{\infty}v_{1n}(t')\sin\left(\frac{n\pi}{a}\bar{\varepsilon}\right) \qquad (26)$$

$$v_{1n}(t')=\begin{pmatrix} B_4X_1+A_0(B_5-B_{10})X_2+(B_5-B_{10})X_3-X_4+A_3(B_5-B_{10})X_5-B_5A_0X_6 \\ +B_5\sum_{n=1}^{\infty}A_2X_7+\frac{B_5A_3}{r}X_8-B_9X_9 \end{pmatrix} \qquad (27)$$

$$\theta_1(\bar{\varepsilon},t')=\sum_{n=1}^{\infty}\theta_{1n}(t')\sin\left(\frac{n\pi}{a}\bar{\varepsilon}\right) \qquad (28)$$

$$\begin{aligned}\theta_{1n}(t')&=(B_{11}B_3+B_{12})X_{10}+A_3(B_5-B_7)X_{11}-B_5A_0X_{12}+B_5X_{13}+\frac{B_5A_3}{r}X_{14}+(B_{11}B_3+B_{12})X_{15}\\ &+A_0(B_5-B_{10})X_{16}+(B_5-B_{10})X_{17}+A_3(B_5-B_{10})X_{18}+B_5A_0X_{19}+B_5\sum_{n=1}^{\infty}A_2X_{20}+\frac{B_5A_3}{r}X_{21}\\ &-B_9X_{22}+B_{12}2B_4X_{23}+A_0(2B_s-B_7-B_{10})X_{24}+(2B_s-B_7-B_{10})X_{25}+A_3(2B_s-B_7-B_{10})X_{26} \qquad (29)\\ &-2A_0B_5X_{27}+2B_5X_{28}+\frac{2A_3B_5}{r}X_{29}+X_{30}-(B_6+B_9)X_{31}+\frac{B_{14}}{r}X_{32}+\frac{B_{15}}{(r-s)}X_{33}+\frac{B_{16}}{(r-2s)}X_{34}\\ &+B_{17}X_{35}+\left(X_{36}+\frac{A_3}{r}X_{37}+B_{19}X_{38}+\sum_{n=1}^{\infty}X_{39}+\frac{A_3}{r}X_{40}\right)\end{aligned}$$

Where

$$\left.\begin{aligned} &A_0=-\frac{4D\left(1-(-1)^{3n}\right)}{3n\pi},\; A_1=\frac{Bn\pi}{a},\; A_2=\begin{pmatrix}\frac{2D(\alpha(\cos\varphi+\sin\varphi)-1)Bn\pi\left(1-(-1)^{2n}\right)}{a^2n\pi}- \\ 2gB\end{pmatrix},\\ &A_3=\begin{pmatrix}\left(\frac{2D\left(\alpha^2\left(\cos^2\varphi+\sin^2\varphi\right)+1-2\alpha\cos\varphi\right)}{a^2}+2g\right)\frac{\left(1-(-1)^n\right)}{n\pi}- \\ \frac{2g(\alpha(\cos\varphi+\sin\varphi)-1)(-1)^n}{n\pi}\end{pmatrix}\end{aligned}\right\} \qquad (30)$$

$$\left.\begin{aligned} &B_1=\frac{n\pi}{a},\quad B_2=\frac{\gamma-1}{a},\quad B_3=\frac{\alpha\cos\varphi-1}{a},\\ &B_4=\sum_{n=1}^{\infty}BB_1^2+\sum_{n=1}^{\infty}\frac{BB_1^2B_2a\left(1+n^2\pi^2-(-1)^{2n}\right)}{2n^2\pi^2}-\sum_{n=1}^{\infty}\frac{BB_1B_2\left(1-(-1)^{2n}\right)}{n\pi},\\ &B_5=\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{2BB_1^2\left(2-3(-1)^n+(-1)^{3n}\right)}{3n\pi}-\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{BB_1^2\left(1-(-1)^{3n}\right)}{3n\pi},\\ &B_6=\frac{2B_2B_3\left(1-(-1)^n\right)}{n\pi},\; B_7=\sum_{n=1}^{\infty}\frac{B_1B_3\left(1-(-1)^{2n}\right)}{n\pi},\; B_8=(-1)^{2n}-1\\ &B_9=\frac{2B_2B_8\left(1-(-1)^n\right)}{n\pi},\end{aligned}\right\} \qquad (31)$$

$$\left.\begin{aligned} &B_{10}=\sum_{n=1}^{\infty}\frac{B_1B_8\left(1-(-1)^{2n}\right)}{n\pi},\qquad B_{11}=\sum_{n=1}^{\infty}2DB_1\left(\frac{1-(-1)^{2n}}{n\pi}\right),\\ &B_{12}=\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}4BB_1^2D\left(\frac{1-(-1)^{3n}}{3n\pi}\right)\end{aligned}\right\} \qquad (32)$$

$$\left.\begin{aligned}
&B_{13}=(-1)^n-1,\\
&B_{14}=\frac{B_2D\left(\alpha^2\left(\cos^2\varphi+\sin^2\varphi\right)-2\alpha\cos\varphi+1\right)2a(-1)^n}{n\pi a^2}\\
&-\sum_{n=1}^{\infty}BB_1\frac{2B_2Da\left(1-2(-1)^{2n}\right)\left(\alpha(\cos\varphi+\sin\varphi)-1\right)}{2n\pi a}+\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{4B^2B_1^2B_2Da(-1)^{3n}}{3n\pi}\\
&-\frac{2D\left(\alpha^2\left(\cos^2\varphi+\sin^2\varphi\right)-2\alpha\cos\varphi+1\right)\left(1-(-1)^n\right)}{n\pi a^2}\\
&-\sum_{n=1}^{\infty}2DBB_1\frac{2B_2Da\left(\alpha(\cos\varphi+\sin\varphi)-1\right)\left(1-(-1)^{2n}\right)}{n\pi a}-\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{4DB^2B_1^2\left(1-(-1)^{3n}\right)}{3n\pi},\\
&B_{15}=\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{8DB^2B_1^2\left(1-(-1)^{3n}\right)}{3n\pi}+\sum_{n=1}^{\infty}\frac{2DBB_1\left(\alpha(\cos\varphi+\sin\varphi)-1\right)\left(1-(-1)^{2n}\right)}{n\pi a}\\
&-\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{8DB^2B_1^2B_2a(-1)^{3n}}{3n\pi}+\sum_{n=1}^{\infty}\frac{2BB_1B_2Da\left(\alpha(\cos\varphi+\sin\varphi)-1\right)\left(1-2(-1)^{2n}\right)}{2n\pi a},\\
&B_{16}=\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{4B^2B_1^2B_2Da(-1)^{3n}}{3n\pi}-\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{4DB^2B_1^2\left(1-(-1)^{3n}\right)}{3n\pi},\\
&B_{17}=-\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{DB^2B_1^2}{2}-\frac{D\left(\alpha^2\left(\cos^2\varphi+\sin^2\varphi\right)-2\alpha\cos\varphi+1\right)}{a^2},\\
&B_{18}=\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}DB^2B_1^2,\quad B_{19}=-\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{DB^2B_1^2}{2},\ B_{20}=\frac{4\gamma(-1)^h}{n^2\pi^2},\ B_{21}=\frac{4}{n^2\pi^2}
\end{aligned}\right\}\qquad(33)$$

$$X_1=\left(\frac{1}{s}\left(1-e^{-st'}\right)-t'e^{-st'}\right)\qquad(34)$$

$$X_2=\left(\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}A_1\left(\begin{aligned}&\frac{1}{r}\left(\frac{1}{s}\left(1-e^{-st'}\right)-\frac{1}{(s-r)}\left(e^{-rt'}-e^{-st'}\right)\right)-\frac{2}{(r-s)}\left(t'e^{-st'}-\frac{1}{(s-r)}\left(e^{-rt'}-e^{-st'}\right)\right)\\&+\frac{1}{(r-2s)}\left(\frac{1}{s}\left(e^{-2st'}-e^{-st'}\right)-\frac{1}{s-r}\left(e^{-rt'}-e^{-st'}\right)\right)\end{aligned}\right)\right.$$
$$\left.-\frac{1}{(r-s)}\left(t'e^{-st'}-\frac{1}{(s-r)}\left(e^{-rt'}-e^{-st'}\right)\right)\right)\qquad(35)$$

$$X_3=\sum_{n=1}^{\infty}A_2\left(\frac{1}{r}\left(\frac{1}{s}\left(1-e^{-st'}\right)-\frac{1}{(r-2s)}\left(e^{-rt'}-e^{-st'}\right)\right)\right)\qquad(36)$$

$$X_4=\frac{1}{(r-s)}\left(t'e^{-st'}-\frac{1}{(s-r)}\left(e^{-rt'}-e^{-st'}\right)\right)\qquad(37)$$

$$X_5=\frac{1}{r}\left(\frac{1}{s}\left(1-e^{-st'}\right)-\frac{1}{(s-r)}\left(e^{-rt'}-e^{-st'}\right)\right)\qquad(38)$$

$$X_6=\left(\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}A_1\left(\begin{aligned}&\frac{1}{r}\left(t'e^{-st'}+\frac{1}{r}\left(e^{-(s+r)t'}-e^{-st'}\right)\right)\\&-\frac{2}{(r-s)}\left(-\frac{1}{s}\left(1-e^{-st'}\right)-\frac{1}{r}\left(e^{-(s+r)t'}-e^{-st'}\right)\right)\\&+\frac{1}{(r-2s)}\left(-\frac{1}{2s}\left(e^{-3st'}-e^{-st'}\right)+\frac{1}{r}\left(e^{-(s+r)t'}-e^{-st'}\right)\right)\end{aligned}\right)\right)\qquad(39)$$

$$X_7=\sum_{n=1}^{\infty}A_2\left(\begin{aligned}&\frac{1}{r}\left(t'e^{-st'}+\frac{1}{r}\left(e^{-(s+r)t'}-e^{-st'}\right)\right)\\&-\frac{1}{(r-s)}\left(-\frac{1}{s}\left(e^{-2st'}-e^{-st'}\right)-\frac{1}{r}\left(e^{-(s+r)t'}-e^{-st'}\right)\right)\end{aligned}\right)\qquad(40)$$

$$X_8=\left(t'e^{-st'}+\frac{1}{r}\left(e^{-(s+r)t'}-e^{-st'}\right)\right)\qquad(41)$$

$$X_9 = \frac{1}{s}\left(1 - e^{-st'}\right) \tag{42}$$

$$X_{10} = \begin{pmatrix} B_4\left(\frac{1}{s}\left(\frac{1}{r}\left(1-e^{-rt'}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)-\left(\frac{t'e^{-st'}}{(r-s)}-\frac{\left(e^{-st'}-e^{-rt'}\right)}{(r-s)^2}\right)\right)+ \\ A_0\left(B_5-B_7\right)\left(\sum_{n=1}^{\infty}\sum_{n-1}^{\infty}A_1\frac{1}{r}\begin{pmatrix}\frac{1}{s}\left(\frac{1}{r}\left(1-e^{-rt'}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)-\frac{1}{(s-r)} \\ \left(t'e^{-rt'}-\frac{1}{(r-s)}\left(e^{-st}-e^{-rt}\right)\right)\end{pmatrix}\right) \\ -\frac{2}{(r-s)}\begin{pmatrix}\left(\frac{t'e^{-st'}}{(r-s)}-\frac{\left(e^{-st'}-e^{-rt'}\right)}{(r-s)^2}\right)-\frac{1}{(s-r)} \\ \left(t'e^{-rt'}-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)\end{pmatrix}+\frac{1}{(r-2s)}\begin{pmatrix}-\frac{1}{s}\left(\frac{1}{(r-2s)}\left(e^{-2st'}-e^{-rt'}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right) \\ -\frac{1}{(s-r)}\left(t'e^{-rt'}-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)\end{pmatrix} \\ +\left(B_5-B_7\right)\left(\sum_{n-1}^{\infty}A_2\begin{pmatrix}\frac{1}{r}\left(\frac{1}{s}\left(\frac{1}{r}\left(1-e^{-rt'}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)-\frac{1}{(s-r)}\left(t'e^{-rt'}-\frac{1}{(r-s)}\left(e^{-st}-e^{-rt}\right)\right)\right) \\ -\frac{1}{(r-s)}\left(\left(\frac{t'e^{-st'}}{r-2}-\frac{\left(e^{-st'}-e^{-rt'}\right)}{(r-s)^2}\right)-\frac{1}{(s-r)}\left(te^{-rt'}-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)\right)\end{pmatrix}\right)\end{pmatrix} \tag{43}$$

$$X_{11} = \frac{1}{r}\left(\frac{1}{s}\left(\frac{1}{r}\left(1-e^{-rt'}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)-\frac{1}{(s-r)}\left(t'e^{-rt'}-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)\right) \tag{44}$$

$$X_{12} = \left(\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}A_1\begin{pmatrix}\frac{1}{r}\left(\left(\frac{te^{-st'}}{(r-s)}-\frac{\left(e^{-st'}-e^{-rt'}\right)}{(r-s)^2}\right)+\frac{1}{r}\left(-\frac{1}{s}\left(e^{-(r-s)t'}-e^{-rt}\right)-\frac{1}{(r-s)}\begin{pmatrix}e^{-st'}- \\ e^{-rt'}\end{pmatrix}\right)\right) \\ -\frac{2}{(r-s)}\begin{pmatrix}-\frac{1}{s}\left(\frac{1}{r}\left(1-e^{-rt'}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right) \\ -\frac{1}{r}\left(-\frac{1}{s}\left(e^{-(r+s)t'}-e^{-rt'}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)\end{pmatrix} \\ +\frac{1}{(r-2s)}\begin{pmatrix}-\frac{1}{2s}\left(\frac{1}{(r-3s)}\left(e^{-3st'}-e^{-rt'}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right) \\ +\frac{1}{r}\left(\frac{1}{s}\left(e^{-(r+s)t'}-e^{-rt'}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)\end{pmatrix}\end{pmatrix}\right) \tag{45}$$

$$X_{13} = \sum_{n=1}^{\infty}A_2\begin{pmatrix}\frac{1}{r}\left(\left(\frac{te^{-st'}}{(r-s)}-\frac{\left(e^{-st'}-e^{-rt'}\right)}{(r-s)^2}\right)+\frac{1}{r}\left(-\frac{1}{s}\left(e^{-(r+s)t'}-e^{-rt}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)\right) \\ -\frac{1}{(r-s)}\begin{pmatrix}-\frac{1}{s}\left(\frac{1}{(r-2s)}\left(e^{-2st'}-e^{-rt'}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right) \\ +\frac{1}{r}\left(-\frac{1}{s}\left(e^{-(r+s)t'}-e^{-rt'}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)\end{pmatrix}\end{pmatrix}$$

$$X_{14} = \begin{pmatrix}\left(\frac{te^{-st'}}{(r-s)}-\frac{\left(e^{-st'}-e^{-rt'}\right)}{(r-s)^2}\right)+\frac{1}{r}\left(-\frac{1}{s}\left(e^{-(r+s)t'}-e^{-rt}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right) \\ B_6\frac{1}{s}\left(\frac{1}{r}\left(1-e^{-rt'}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)\end{pmatrix} \tag{46}$$

$$X_{15}=\left(B_4\left(\frac{1}{s}\left(\frac{1}{r}\left(1-e^{-rt'}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)-\left(\frac{t'e^{-st'}}{(r-s)}-\frac{\left(e^{-st'}-e^{-rt'}\right)}{(r-s)^2}\right)\right)\right) \quad (47)$$

$$X_{16}=\left(\sum_{n=1}^{\infty}\sum_{n-1}^{\infty}A_1\left(\frac{1}{r}\left(\begin{array}{l}\frac{1}{s}\left(\frac{1}{r}\left(1-e^{-rt'}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)-\frac{1}{(s-r)}\\ \left(t'e^{-rt'}-\frac{1}{(r-s)}\left(e^{-st}-e^{-rt}\right)\right)\\ -\frac{2}{(r-s)}\left(\begin{array}{l}\left(\frac{t'e^{-st'}}{(r-s)}-\frac{\left(e^{-st'}-e^{-rt'}\right)}{(r-s)^2}\right)-\frac{1}{(s-r)}\\ \left(t'e^{-rt'}-\frac{1}{(s-r)}\left(e^{-st'}-e^{-rt'}\right)\right)\end{array}\right)\\ +\frac{1}{(r-2s)}\left(\begin{array}{l}-\frac{1}{s}\left(\frac{1}{(r-2s)}\left(e^{-2st'}-e^{-rt'}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)\\ -\frac{1}{(s-r)}\left(t'e^{-rt'}-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)\end{array}\right)\end{array}\right)\right)\right) \quad (48)$$

$$X_{18}=\left(\frac{1}{r}\left(\begin{array}{l}\frac{1}{s}\left(\frac{1}{r}\left(1-e^{-rt'}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)-\frac{1}{(s-r)}\\ \left(t'e^{-rt'}-\frac{1}{(r-s)}\left(e^{-st}-e^{-rt}\right)\right)\end{array}\right)\right) \quad (49)$$

$$X_{17}=\left(\sum_{n-1}^{\infty}A_2\left(\frac{1}{r}\left(\begin{array}{l}\frac{1}{s}\left(\frac{1}{r}\left(1-e^{-rt'}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)-\frac{1}{(s-r)}\left(\begin{array}{l}t'e^{-rt'}-\frac{1}{(r-s)}\\ \left(e^{-st}-e^{-rt}\right)\end{array}\right)\\ -\frac{1}{(r-s)}\left(\begin{array}{l}\left(\frac{t'e^{-st'}}{(r-s)}-\frac{\left(e^{-st'}-e^{-rt'}\right)}{(r-s)^2}\right)-\frac{1}{(s-r)}\\ \left(t'e^{-rt'}-\frac{1}{(s-r)}\left(e^{-st'}-e^{-rt'}\right)\right)\end{array}\right)\end{array}\right)\right)\right) \quad (50)$$

$$X_{19}=\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}A_1\left(\begin{array}{l}\frac{1}{r}\left(\left(\frac{te^{-st'}}{(r-s)}-\frac{\left(e^{-st'}-e^{-rt'}\right)}{(r-s)^2}\right)+\frac{1}{r}\left(-\frac{1}{s}\left(e^{-(r+s)t'}-e^{-rt}\right)-\frac{1}{(r-s)}\left(\begin{array}{l}e^{-st'}-\\ e^{-rt'}\end{array}\right)\right)\right)\\ -\frac{2}{(r-s)}\left(\begin{array}{l}-\frac{1}{s}\left(\frac{1}{r}\left(1-e^{-rt'}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)\\ -\frac{1}{r}\left(-\frac{1}{s}\left(e^{-(r+s)t'}-e^{-rt'}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)\end{array}\right)\\ +\frac{1}{(r-2s)}\left(\begin{array}{l}-\frac{1}{2s}\left(\frac{1}{(r-3s)}\left(e^{-3st'}-e^{-rt'}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)\\ +\frac{1}{r}\left(\frac{1}{s}\left(e^{-(r+s)t'}-e^{-rt'}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)\end{array}\right)\end{array}\right) \quad (51)$$

$$X_{20}=\sum_{n=1}^{\infty}A_2\left(\begin{array}{l}\frac{1}{r}\left(\left(\frac{te^{-st'}}{(r-s)}-\frac{\left(e^{-st'}-e^{-rt'}\right)}{(r-s)^2}\right)+\frac{1}{r}\left(-\frac{1}{s}\left(e^{-(r+s)t'}-e^{-rt}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)\right)\\ -\frac{1}{(r-s)}\left(\begin{array}{l}-\frac{1}{s}\left(\frac{1}{(r-2s)}\left(e^{-2st'}-e^{-rt'}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)\\ +\frac{1}{r}\left(-\frac{1}{s}\left(e^{-(r+s)t'}-e^{-rt'}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)\end{array}\right)\end{array}\right) \tag{52}$$

$$\left.\begin{array}{l}X_{21}=\left(\left(\frac{te^{-st'}}{(r-s)}-\frac{\left(e^{-st'}-e^{-rt'}\right)}{(r-s)^2}\right)+\frac{1}{r}\left(-\frac{1}{s}\left(e^{-(r+s)t'}-e^{-rt}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)\right)\\ X_{22}=\frac{1}{s}\left(\frac{1}{r}\left(1-e^{-rt'}\right)-\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)\right)\\ X_{23}=\left(\frac{1}{s}\left(\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)-\frac{1}{(r-2s)}\left(e^{-2st'}-e^{-rt'}\right)\right)-\left(\frac{t'e^{-2st'}}{(r-2s)}-\frac{\left(e^{-2st'}-e^{-rt'}\right)}{(r-2s)^2}\right)\right)\end{array}\right\} \tag{53}$$

$$X_{24}=\left(\sum_{n=1}^{\infty}\sum_{n-1}^{\infty}A_1\left(\frac{1}{r}\left(\begin{array}{l}\frac{1}{s}\left(\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)-\frac{1}{(r-2s)}\left(e^{-2st'}-e^{-rt'}\right)\right)-\frac{1}{(s-r)}\\ \left(-\frac{1}{s}\left(e^{-(r+s)t'}-e^{-rt'}\right)-\frac{1}{(r-2s)}\left(e^{-2st'}-e^{-rt'}\right)\right)\\ -\frac{2}{(r-s)}\left(\begin{array}{l}\left(\frac{te^{-2st'}}{(r-2s)}-\frac{\left(e^{-2st'}-e^{-rt'}\right)}{(r-2s)^2}\right)-\frac{1}{(s-r)}\\ \left(-\frac{1}{s}\left(e^{-(r+s)t'}-e^{-rt'}\right)-\frac{1}{(r-2s)}\left(e^{-2st'}-e^{-rt'}\right)\right)\end{array}\right)\\ +\frac{1}{(r-2s)}\left(\begin{array}{l}-\frac{1}{s}\left(\frac{1}{(r-3s)}\left(e^{-3st'}-e^{-rt'}\right)-\frac{1}{(r-2s)}\left(e^{-2st'}-e^{-rt'}\right)\right)\\ -\frac{1}{(s-r)}\left(-\frac{1}{s}\left(e^{-(r+s)t'}-e^{-rt'}\right)-\frac{1}{(r-2s)}\left(e^{-2st'}-e^{-rt'}\right)\right)\end{array}\right)\end{array}\right)\right)\right) \tag{54}$$

$$\left.\begin{array}{l}X_{25}=\sum_{n-1}^{\infty}A_2\left(\frac{1}{r}\left(\begin{array}{l}\frac{1}{s}\left(\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)-\frac{1}{(r-2s)}\left(e^{-2st'}-e^{-rt'}\right)\right)-\frac{1}{(s-r)}\left(-\frac{1}{s}\left(e^{-(r+s)t'}-e^{-rt'}\right)-\frac{1}{(r-2s)}\left(e^{-2st'}-e^{-rt'}\right)\right)\\ -\frac{1}{(r-s)}\left(\left(\frac{te^{-2st'}}{(r-2s)}-\frac{\left(e^{-2st'}-e^{-rt'}\right)}{(r-2s)^2}\right)-\frac{1}{(s-r)}\left(-\frac{1}{s}\left(e^{-(r+s)t'}-e^{-rt'}\right)-\frac{1}{(r-2s)}\left(e^{-2st'}-e^{-rt'}\right)\right)\right)\end{array}\right)\right)\\ )\\ X_{26}=\left(\frac{1}{r}\left(\begin{array}{l}\frac{1}{s}\left(\frac{1}{(r-s)}\left(e^{-st'}-e^{-rt'}\right)-\frac{1}{(r-2s)}\left(e^{-2st'}-e^{-rt'}\right)\right)-\frac{1}{(s-r)}\\ \left(-\frac{1}{s}\left(e^{-(r+s)t'}-e^{-rt'}\right)-\frac{1}{(r-2s)}\left(e^{-2st'}-e^{-rt'}\right)\right)\end{array}\right)\right)\end{array}\right| \tag{55}$$

$$X_{27} = \sum_{n=1}^{\infty} A_1 \left(\begin{array}{l} \frac{1}{r}\left(\left(\frac{t e^{-st'}}{(r-2s)} - \frac{\left(e^{-2st'} - e^{-rt'}\right)}{(r-2s)^2} \right) + \frac{1}{r}\left(\begin{array}{l} -\frac{1}{2s}\left(e^{-(r+2s)t'} - e^{-rt}\right) - \\ \frac{1}{(r-2s)}\left(e^{-2st'} - e^{-rt'}\right) \end{array} \right) \right) \\ -\frac{2}{(r-s)}\left(\begin{array}{l} -\frac{1}{s}\left(\frac{1}{(r-s)}\left(e^{-st'} - e^{-rt'}\right) - \frac{1}{(r-2s)}\left(e^{-2st'} - e^{-rt'}\right) \right) \\ -\frac{1}{r}\left(-\frac{1}{s}\left(e^{-(r+2s)t'} - e^{-rt'}\right) - \frac{1}{(r-2s)}\left(e^{-2st'} - e^{-rt'}\right) \right) \end{array} \right) \\ +\frac{1}{(r-2s)}\left(\begin{array}{l} -\frac{1}{2s}\left(\frac{1}{(r-4s)}\left(e^{-4st'} - e^{-rt'}\right) - \frac{1}{(r-2s)}\left(e^{-2st'} - e^{-rt'}\right) \right) \\ +\frac{1}{r}\left(-\frac{1}{2s}\left(e^{-(r+2s)t'} - e^{-rt'}\right) - \frac{1}{(r-2s)}\left(e^{-2st'} - e^{-rt'}\right) \right) \end{array} \right) \end{array} \right) \quad (56)$$

$$\left. \begin{array}{l} X_{28} = \sum_{n=1}^{\infty} A_2 \left(\begin{array}{l} \frac{1}{r}\left(\left(\frac{te^{-st'}}{(r-2s)} - \frac{\left(e^{-2st'} - e^{-rt'}\right)}{(r-2s)^2} \right) + \frac{1}{r}\left(-\frac{1}{2s}\left(e^{-(r+2s)t'} - e^{-rt}\right) - \frac{1}{(r-2s)}\left(e^{-2st'} - e^{-rt'}\right) \right) \right) \\ -\frac{1}{(r-s)}\left(\begin{array}{l} -\frac{1}{s}\left(\frac{1}{(r-3s)}\left(e^{-3st'} - e^{-rt'}\right) - \frac{1}{(r-2s)}\left(e^{-2st'} - e^{-rt'}\right) \right) + \\ \frac{1}{r}\left(-\frac{1}{2s}\left(e^{-(r+2s)t'} - e^{-rt'}\right) - \frac{1}{(r-2s)}\left(e^{-2st'} - e^{-rt'}\right) \right) \end{array} \right) \end{array} \right) \\ X_{29} = \left(\frac{te^{-st'}}{(r-2s)} - \frac{\left(e^{-2st'} - e^{-rt'}\right)}{(r-2s)^2} \right) \\ X_{30} = \frac{1}{r}\left(-\frac{1}{2s}\left(e^{-(r+2s)t'} - e^{-rt}\right) - \frac{1}{(r-2s)}\left(e^{-2st'} - e^{-rt'}\right) \right) \\ X_{31} = \frac{1}{s}\left(\frac{1}{(r-s)}\left(e^{-st'} - e^{-rt'}\right) - \frac{1}{(r-2s)}\left(e^{-2st'} - e^{-rt'}\right) \right) \\ X_{32} = \left(1 - e^{-rt'}\right) \\ X_{33} = \left(e^{-st'} - e^{-rt'}\right) \\ X_{34} = \left(e^{-2st'} - e^{-rt'}\right) \end{array} \right\} \quad 57$$

$$X_{35}=\left(A_0\left(\begin{array}{l}\displaystyle\sum_{n=1}^{\infty}\sum_{n-1}^{\infty}A_1\left(\frac{1}{r}\left(\begin{array}{l}\displaystyle\frac{1}{r}\left(\left(1-e^{-rt'}\right)-t'e^{-rt'}\right)-\frac{2}{(r-s)}\left(\frac{1}{(r-s)}\left(e^{-st}-e^{-rt}\right)-t'e^{-rt'}\right)\\ \displaystyle+\frac{1}{(r-2s)}\left(\frac{1}{(r-2s)}\left(e^{-2st'}-e^{-rt'}\right)-t'e^{-rt'}\right)\end{array}\right)\right)\\ \displaystyle+\sum_{n-1}^{\infty}A_2\left(\begin{array}{l}\displaystyle\frac{1}{r}\left(\frac{1}{r}\left(\left(1-e^{-rt'}\right)-t'e^{-rt'}\right)-\frac{1}{(r-s)}\left(\frac{1}{(r-s)}\left(e^{-st}-e^{-rt}\right)-t'e^{-rt'}\right)\right)\\ \displaystyle+\frac{A_3}{r}\left(\frac{1}{r}\left(\left(1-e^{-rt'}\right)-t'e^{-rt'}\right)\right)\\ \displaystyle+B_{18}\left(\begin{array}{l}\displaystyle A_1\frac{1}{r}\left(\frac{1}{(r-s)}\left(e^{-st}-e^{-rt}\right)+\frac{1}{s}\left(e^{-rt'-rs}-e^{-rt'}\right)\right)\\ \displaystyle A_0\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}-\frac{2}{(r-s)}\left(\frac{1}{(r-2s)}\left(e^{-2st}-e^{-rt}\right)+\frac{1}{s}\left(e^{-rt'-rs}-e^{-rt'}\right)\right)\\ \displaystyle+\frac{1}{(r-s)}\left(\frac{1}{(r-3s)}\left(e^{-3st'}-e^{-rt'}\right)+\frac{1}{s}\left(e^{-rt'-rs}-e^{-rt'}\right)\right)\end{array}\right)\end{array}\right)\end{array}\right)\right) \quad 58)$$

$$\left(\begin{array}{l}\displaystyle X_{36}=\sum_{n=1}^{\infty}\left(\begin{array}{l}\displaystyle A_1\frac{1}{r}\left(\frac{1}{(r-s)}\left(e^{-st}-e^{-rt}\right)+\frac{1}{s}\left(e^{-rt'-rs}-e^{-rt'}\right)\right)\\ \displaystyle-\frac{1}{(r-s)}\left(\frac{1}{(r-2s)}\left(e^{-2st}-e^{-rt}\right)+\frac{1}{s}\left(e^{-rt'-rs}-e^{-rt'}\right)\right)\end{array}\right)\\ \displaystyle X_{37}=\left(\frac{1}{(r-s)}\left(e^{-st}-e^{-rt}\right)+\frac{1}{s}\left(e^{-rt'-rs}-e^{-rt'}\right)\right)\\ \displaystyle X_{38}=\left(A_0\sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left(\begin{array}{l}\displaystyle A_1\frac{1}{r}\left(\frac{1}{(r-s)}\left(e^{-st}-e^{-rt}\right)+\frac{1}{2s}\left(e^{-rt'-rs}-e^{-rt'}\right)\right)\\ \displaystyle-\frac{2}{(r-s)}\left(\frac{1}{(r-3s)}\left(e^{-3st}-e^{-rt}\right)+\frac{1}{2s}\left(e^{-rt'-rs}-e^{-rt'}\right)\right)\\ \displaystyle+\frac{1}{(r-s)}\left(\frac{1}{(r-4s)}\left(e^{-4st'}-e^{-rt'}\right)+\frac{1}{2s}\left(e^{-rt'-rs}-e^{-rt'}\right)\right)\end{array}\right)\right)\\ \displaystyle X_{39}=\sum_{n=1}^{\infty}\left(\begin{array}{l}\displaystyle A_2\frac{1}{r}\left(\frac{1}{(r-2s)}\left(e^{-2st}-e^{-rt}\right)+\frac{1}{2s}\left(e^{-rt'-rs}-e^{-rt'}\right)\right)\\ \displaystyle-\frac{1}{(r-s)}\left(\frac{1}{(r-3s)}\left(e^{-3st}-e^{-rt}\right)+\frac{1}{2s}\left(e^{-rt'-rs}-e^{-rt'}\right)\right)\end{array}\right)\\ \displaystyle X_{40}=\left(\frac{1}{(r-2s)}\left(e^{-2st}-e^{-rt}\right)+\frac{1}{2s}\left(e^{-rt'-rs}-e^{-rt'}\right)\right)\end{array}\right) \quad (59)$$

The computations were done using computer symbolic algebraic package MAPLE.

RESULTS AND DISCUSSION

Analytical solutions of equations (9) - (15) are computed using MAPLE 16. Figures 1, 2, 3, 4 and 5 explained the velocities, temperature and mass flow rate distribution against different dimensionless parameters. For the purpose of the graphs, $\bar{\varepsilon} = x$. Figure 1 depicts the effect of Re on the velocity along the channel in a 3D plot. It is observed that velocity along the channel increases and later decreases along distance x while it increases and later becomes steady with time but maximum velocity along the channel increases as Re increases.

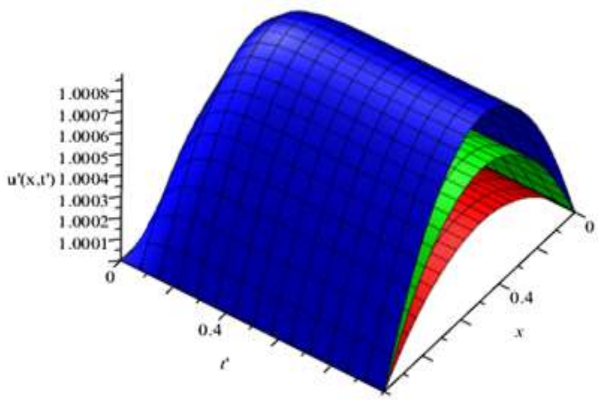

Fig 1: Relationship between Velocity along the Channel, Time and Distance for different values of Re. $\mathrm{Re} = 0.3$ (Red), $\mathrm{Re} = 0.5$ (Green) and $\mathrm{Re} = 0.7$ (Blue)

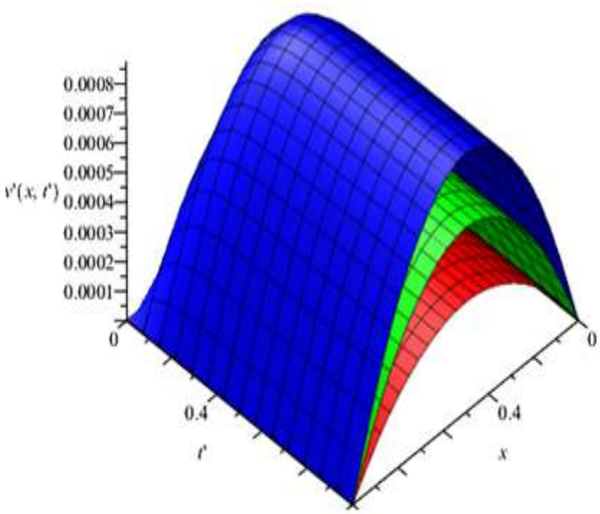

Fig 2: Relationship between Velocity across the Channel, Time and Distance for different values of Re. $\mathrm{Re} = 0.3$ (Red), $\mathrm{Re} = 0.5$ (Green) and $\mathrm{Re} = 0.7$ (Blue)

Figure 2 presents the effect of Re on the velocity along the cross channel in a 3D plot. It is observed that velocity along the cross channel increases and later decreases along distance x while it increases and later becomes steady with time but maximum velocity along the channel increases as Re increases. Figure 3 displays the effect of Re on the polymer temperature in a 3D plot.

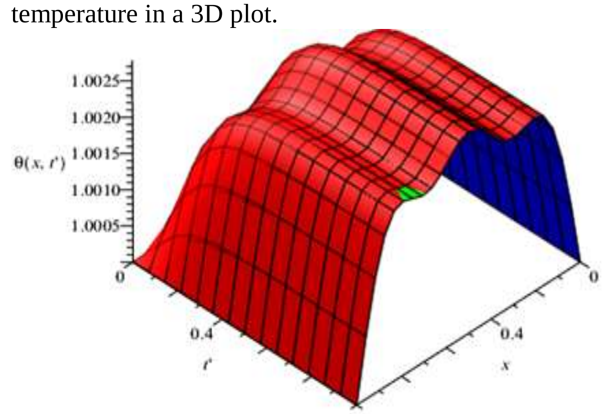

Fig 3: Relationship between Polymer Temperature, Time and Distance for different values of Re. $\mathrm{Re} = 0.3$ (Red), $\mathrm{Re} = 0.5$ (Green) and $\mathrm{Re} = 0.7$ (Blue)

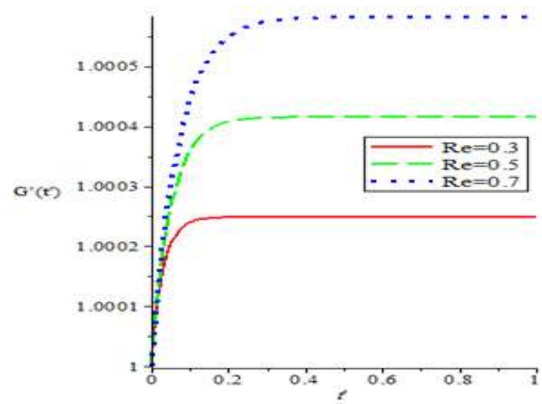

Fig 4: Relationship between Mass Flow Rate and Time for different values of Re

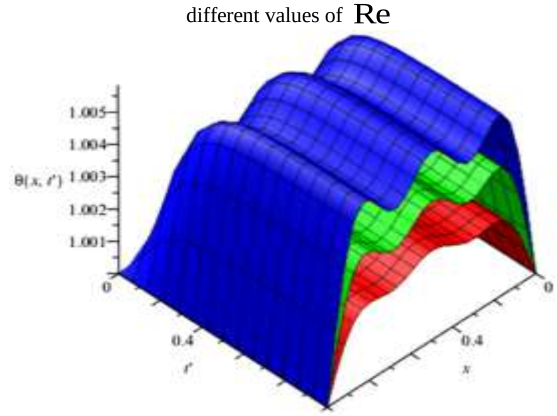

Fig 5: Relationship between Polymer Temperature, Time and Distance for different values of Ec. $Ec = 1$ (Red), $Ec = 2$ (Green) and $Ec = 3$ (Blue)

It is observed that polymer temperature oscillates along distance x while it increases and later becomes steady with time but Re did not show any noticeable effect on maximum temperature. Figure 4 shows the effect of Re on the mass flow rate. It is observed that mass flow rate increases and later becomes steady with time but mass flow rate increase as Re increases. Figure 5 presents the effect of Ec on the polymer temperature in a 3D plot. It is observed that temperature increases and later becomes steady with time while it increases and later decreases along distance x but maximum polymer temperature increases as Ec increases.

Conclusion: We have formulated and solved analytically a mathematical model of polymer movement and heat and mass transfer in the zone of polymer melting delay to determine the velocity and temperature distributions and mass flow rate. From the results obtained, we conclude that Reynolds' number enhanced mass flow rate and both velocities along the channel and across the channel Eckert number enhanced the polymer temperature.

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