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Exponentiated Complementary Kumaraswamy-G Power Series Family of Distributions with Application to Lifetime Data

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ABSTRACT. This paper proposed the Exponentiated Complementary Kumaraswamy - G Power (ECK-GP) Series family of distributions which is an extension of the Complementary Kumaraswamy-G Power Series family of distributions. The expressions for its densities, basic statistical properties and parameters estimation using the method of Maximum Likelihood were derived and established. An application of the Exponentiated Complementary Kumaraswamy Exponential Poisson (ECKEP) distribution to a real lifetime dataset clearly reveals its suitability and flexibility in fitting real life dataset.

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In many applied areas like lifetime analysis, there is a clear need for new distributions which are more flexible to model real data that present a high degree of skewness and kurtosis. Recent developments focus on new techniques by compounding of distributions and adding parameters to existing distributions thereby building classes of more flexible distributions. Complementary risk motivation has led to the development of several compound models which have been applied in several areas, Louzada-Neto *et al* (2011). Problems in reliability have been solved by the wide application of the exponential distribution to statistical distribution, Nadarajah and Kotz (2006). But, the memory-less property with a constant failure rate exhibited by the distribution makes it unsuitable for real life problems. As a result, a vital problem is created in statistical modeling and applications. To address this problem, several researchers have proposed different extensions of the exponential distribution and some of these recent studies focusedon the generalization of exponential distribution. Adamidis and Loukas (1998) proposed a variation of the Exponential distribution, with decreasing hazard function called the Exponential Geometric (EG) distribution. Lemonte *et al* (2013) proposed the exponentiated Kumaraswamy distribution and its log-transform. The Exponentiated Kumaraswamy Dagum distribution was proposed by Shujiao and Broderick (2014). Rodrigues and Silva (2015) introduced the exponentiated Kumaraswamy exponential distribution using Kumaraswamy distribution as a baseline distribution. Aryal and Yousof (2017) proposed the exponentiated Generalized-G Poisson family of distributions. Eissa (2017) introduced the exponentiated Kumaraswamy-Weibull distribution with application to real data. The exponential inverse exponential distribution with applications to lifetime data was introduced by

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Oguntunde *et al* (2017). Ibrahim and Khaleel (2018) proposed the exponentiated Kumaraswamy exponentiated Weibull distribution that extends the exponentiated Weibull distribution. Ieren and Kuhe (2018) proposed the Lomax-exponential distribution. Abdullahi and Ieren (2018) proposed the transmuted Exponential Lomax distribution and Eraikhuemen *et al* *(*2020a) proposed the complementary Kumaraswamy Weibull Poisson distribution. This paper proposed a new lifetime family of distributions called the Exponentiated Complementary Kumaraswamy-G Power (ECK-GP) Series obtained by compounding the Kumaraswamy-G family and power series family of distributions.

MATERIALS AND METHODS

The cumulative distribution function (cdf) and probability density function (pdf) of the exponential distribution are given by

$$G(x) = 1 - \exp(-\alpha x) \quad (1)$$

and

$$g(x) = \alpha \exp(-\alpha x), \qquad x, \alpha > 0 \quad (2)$$

respectively, where α is the scale parameter.

Cordeiro and de Castro (2011) proposed a generalized class of distribution called the Kumaraswamy generated (K-G) distribution whose cumulative distribution function (cdf) is given by

$$Q_{a,b}(x) = [1 - G(x)^a]^b, \qquad x, a, b > 0 \quad (3)$$

and the corresponding probability density function (pdf) given by

$$q_{a,b}(x) = abg(x)G(x)^{a-1}[1 - G(x)^a]^{b-1}, \quad x, a, b > 0 \quad (4)$$

Where a and b are additional shape parameters and g(x) is the first derivative of G(x).
Substituting equations1 and 2 into equations 3 and 4gives respectively the cdf and pdf of the Kumaraswamy Exponential distribution.

$$F(x) = \left(1 - \left[1 - \exp\{(-\alpha x)\}\right]^a\right)^b \quad (5)$$

and

$$f(x) = ab\alpha \exp(-\alpha x)\left[1 - \exp(-\alpha x)\right]^{a-1}\left(1 - \left[1 - \exp\{(-\alpha x)\}\right]^a\right)^{b-1} \quad (6)$$

By the compounding procedure pioneered by Marshall and Olkin (1997), let N be a discrete random variable having a power series distribution with probability mass function (pmf) given as

$$P(M = m) = \frac{c_m \theta^m}{D(\theta)}, \qquad m = 1, 2, \ldots \quad (7)$$

Where $C_m \geq 0$ depends only onm, $D(\theta) = \sum_{m=1}^{\infty} C_m \theta^m$ and $\theta > 0$ is such that D(θ) is finite and its first, second and third derivatives exist and are defined by $D'(\theta)$, $D''(\theta)$ and $D'''(\theta)$,(Noack,1950). The various properties of the power series family of distributions were explored by Patil, (1961; 1962).

In Table 1, the Power Series family of distributions defined by equation 7 is shown with their respective c_m, $D(\theta)$, $D'(\theta)$,$D''(\theta)$ and $D'''(\theta)$, where m is the number of trials and n is the number of successes in a sequence of m trials, Kosambi (1949) and Noack (1950).

Table 1: Useful Quantities for Some Power Series Distribution

Distribution	c_n	$D(\theta)$	$D'(\theta)$	$D''(\theta)$	$D'''(\theta)$	$D(\theta)^{-1}$	parameter space
Possion	$(n!)^{-1}$	$e^{\theta}-1$	e^{θ}	e^{θ}	e^{θ}	$\log(\theta+1)$	$\theta \in (0,\infty)$
Geometric	1	$\theta(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$2(1-\theta)^{-3}$	$6(1-\theta)^{-4}$	$\theta(\theta+1)^{-1}$	$\theta \in (0,1)$
Logarithmic	$(n)^{-1}$	$-\log(1-\theta)$	$(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$2(1-\theta)^{-3}$	$1-e^{-\theta}$	$\theta \in (0,1)$
Binomial	[illegible]	$(\theta+1)^m-1$	$m(1+\theta)^{m-1}$	$\frac{m(m-1)}{(\theta+1)^{2-m}}$	$\frac{m(m-1)(m-2)}{(\theta+1)^{3-m}}$	$(\theta-1)^{1/m}-1$	$\theta \in (0,\infty)$

The Exponentiated Complementary Kumaraswamy-G Power Series Distribution: Now, let M be a discrete random variable with pmf defined by equation 7. Let $X_1, X_2, \ldots, X_M$ be M independent and identically distributed (iid) random variables with cdf given by equation 3 and let $X_{(m)} = \max\{x_i\}_{i=1}^{M}$. The conditional cumulative distribution of $X_{(M)} / M = m$ is given as

$$T_{X_{(M)}/M=m}(x) = \left\{1 - Q_{a,b}(x)\right\}^m \quad (8)$$

Equation 8 has the exponentiated form of the general class as given in equation 3.Thus, we obtain

$$P(X_{(m)} \le x, M = m) = \frac{c_m\theta^m}{D(\theta)}\{1-Q_{a,b}(x)\}^m, \qquad x>0, m\ge 0 \quad (9)$$

The complementary class of distributions is defined by the marginal cdf of $X_{(M)}$ as

$$F_{\theta,a,b}(x) = \frac{D\left[\theta\left(1-Q_{a,b}(x)\right)\right]}{D(\theta)}, \quad x,a,b,\theta>0 \quad (10)$$

and the pdf is given by

$$f_{\theta,a,b}(x) = q_{a,b}(x)\theta\frac{D'\left(\theta\{1-Q_{a,b}(x)\}\right)}{D(\theta)}, \quad x,a,b,\theta>0 \quad (11)$$

where $q_{a,b}(x) = ab\theta g(x)G(x)^{a-1}[1-G(x)^a]^{b-1}$

The cdf and pdf of the ECK-GPS class of distribution are

$$F_{\theta,a,b}(x) = \left\{\frac{D\left[\theta\left(1-Q_{a,b}(x)\right)\right]}{D(\theta)}\right\}^{\beta}, \quad x,a,b,\theta,\beta>0 \quad (12)$$

and

$$f_{\theta,a,b}(x) = q_{a,b}(x)\theta\beta\left\{\frac{D'\left(\theta\{1-Q_{a,b}(x)\}\right)}{D(\theta)}\right\}^{\beta-1},, \quad x,a,b,\theta,\beta>0 \quad (13)$$

respectively, where a,b, β are shape parameters.

Sub - Models of the ECKGPS Distribution: Here, we present four sub-models of the ECKGPS family of distributions. These are as follows:

(1) With $D(\theta) = e^{\theta}-1$ the ECK-GPS distributions reduces to the Exponentiated Complementary Kumaraswamy-G Poisson (ECK-GP) distribution with cdf

$$F_{(x;a,b,\alpha,\beta)}(x) = \frac{\left\{\left[e^{\theta}\left(1-\left(1-[G(x)]^a\right)^b\right)\right]-1\right\}^{\beta}}{\{e^{\theta}-1\}^{\beta}}, \quad x,a,b,\theta,\beta>0 \quad (14)$$

and

(2) With $D(\theta) = \theta[1-\theta]^{-1}$ the ECK-GPS distributions reduces to the Exponentiated Complementary Kumaraswamy-G Geometry (ECK-GG) distribution with cdf

$$F_{(x;a,b,\alpha,\theta)}(x) = \frac{\left\{\theta\left[1-\theta\left(1-\left(1-[G(x)]^a\right)^b\right)\right]^{-1}\right\}^{\beta}}{\left\{\theta[1-\theta]^{-1}\right\}^{\beta}}, \quad x,a,b,\theta,\beta>0 \quad (15)$$

(3) With $D(\theta) = (1+\theta)^m-1$ the ECK-GPS distributions reduces to the Exponentiated Complementary Kumaraswamy-G Binomial (ECK-GB) distribution with cdf

$$F_{(x;a,b,\alpha,\theta)}(x) = \frac{\left\{\left[1+\theta\left(1-\left(1-[G(x)]^a\right)^b\right)\right]^m-1\right\}^{\beta}}{\{(1+\theta)^m-1\}^{\beta}}, \quad x,a,b,\theta,\beta>0 \quad (16)$$

(4) With $D(\theta) = -In(1-\theta)$ the ECK-GPS distributions reduces to the Exponentiated Complementary Kumaraswamy-G Logarithmic (ECK-GL) distribution with cdf

$$F_{(x;a,b,\alpha,\theta)}(x) = \frac{\left\{-In\left\{1-\theta\left[1-\left(1-[G(x)]^a\right)^b\right]\right\}\right\}^{\beta}}{\{-In(1-\theta)\}^{\beta}}, \quad x,a,b,\beta>0, \quad 0<\theta<1 \quad (17)$$

Other sub-models considered are as follows:

1. If $G(x) = [1-\exp(-\alpha x)]$ is Exponential distribution, equation 14 reduces to the Exponentiated Complementary Kumaraswamy Exponential Poisson (ECKEP) distribution with cdf and pdf given respectively as

$$F_{ECKEP}(x) = \frac{\left\{\left[e^{\theta}\left(1-\left(1-[1-\exp(-\alpha x)]^a\right)^b\right)\right]-1\right\}^{\beta}}{\{e^{\theta}-1\}^{\beta}}, \quad x,a,b,\theta,\beta>0 \quad (18)$$

$$f_{ECKEP}(x) = \frac{ab\alpha\theta\beta\exp(-\alpha x)(1-\exp(-\alpha x))^{a-1}\left(1-[1-\exp(-\alpha x)]^a\right)^{b-1}\left\{\left(e^{\theta}\left[1-\left(1-[1-\exp(-\alpha x)]^a\right)^b\right]\right)\right\}^{\beta-1}}{(e^{\theta}-1)^{\beta}} \quad (19)$$

$$x,\alpha,a,\beta,b,\theta>0$$

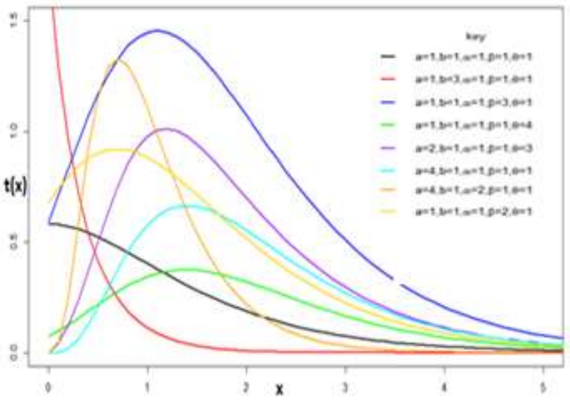

Fig 1: The plot of the ECKEP density function for some parameter values

Figure 1 illustrates the PDF of the ECKEP distribution for different values of the parameters

2. If $G(x) = [1-\exp(-\alpha x)]$ is Exponential distribution, equation 15 reduces to the Exponentiated Complementary Kumaraswamy Exponential Geometric (ECKEG) distribution with cdf

$$F_{(x;a,b,\alpha,\theta)}(x) = \left\{\frac{\theta\left[1-\theta\left(1-\left(1-[1-\exp(-\alpha x)]^a\right)^b\right)\right]^{-1}}{\theta[1-\theta]^{-1}}\right\}^{\beta}, \quad x,a,b,\theta,\beta>0 \quad (20)$$

3. If $G(x)=\left[1-\exp(-\alpha x)\right]$ is Exponential distribution, equation 16 reduces to the Exponentiated Complementary Kumaraswamy Exponential Binomial (ECKEB) distribution with cdf

$$F_{(x;a,b,\alpha,\theta)}(x)=\left\{\frac{\left[1+\theta\left(1-\left(1-\left[1-\exp(-\alpha x)\right]^{a}\right)^{b}\right)\right]^{m}-1}{(1+\theta)^{m}-1}\right\}^{\beta}, \quad x,a,b,\theta,\beta>0 \tag{21}$$

4. If $G(x)=\left[1-\exp(-\alpha x)\right]$ is Exponential distribution, equation 17 reduces to the Exponentiated Complementary Kumaraswamy Exponential Logarithmic (ECKEL) distribution with cdf

$$F_{(x;a,b,\alpha,\theta)}(x)=\left\{\frac{-In\left\{1-\theta\left[1-\left(1-\left[1-\exp(-\alpha x)\right]^{a}\right)^{b}\right]\right\}}{-In(1-\theta)}\right\}^{\beta}, \quad x,a,b,\beta>0, \quad 0<\theta<1 \tag{22}$$

Also, given that $G(x)=\left[1-\exp(-\alpha x)\right]$ is Exponential distribution and $a=b=1$, then:

5. If $D(\theta)=e^{\theta}-1$ the ECK-GPS family of distributions reduces to the Exponentiated Complementary Exponential Poisson (ECEP) distribution.

6. If $D(\theta)=\theta(1-\theta)^{-1}$ the ECK-GPS family of distributions reduces to the Exponentiated Complementary ExponentialGeometric (ECEG) distribution.

7. If $D(\theta)=(\theta+1)^{m}-1$ the ECK-GPS family of distributions reduces to the Exponentiated Complementary Exponential Binomial (ECEB) distribution.

8. If $D(\theta)=-In(1-\theta)$ the ECK-GPS family of distributions reduces to the Exponentiated Complementary Exponential Logarithmic (ECEL) distribution.

Again, given that $G(x)=\left[1-\exp(-\alpha x)\right]$ is Exponential distribution and $a=b=1, \beta=1$,

9. For $D(\theta)=e^{\theta}-1$, the ECK-GPS family of distributions reduces to the Complementary Exponential Poisson (CEP) distribution.

10. For $D(\theta)=\theta(1-\theta)^{-1}$, the ECK-GPS family of distributions reduces to the Complementary Exponential Geometric (CEG) distribution.

11. For $D(\theta)=(\theta+1)^{m}-1$ the ECK-GPS family of distributions reduces to the Complementary Exponential Binomial (CEB) distribution.

12. For $D(\theta)=-In(1-\theta)$ the ECK-GPS family of distributions reduces to the Complementary Exponential Logarithmic (CEL) distribution.

Statistical Properties of ECKEPD: In what follows, we give a detail discussion of the statistical properties of the ECKEP distribution.

Expansion of the Density Function of the ECKEP distribution: Here, the expansion of the pdf of the ECKEP distribution is presented. For $\beta>0$ any real non-integer, we use the power series representation

$$(1-z)^{\beta-1}=\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(\beta)z^{j}}{\Gamma(\beta-j)j!}, \quad \beta>0, \mathrm{I}z\mathrm{I}<1 \tag{23}$$

Using the power series in equation 23 on equation 9, we can express the pdf for real non-integer $(a,b,\alpha,\theta,\beta)$ as

$$f(x;a,b,\alpha,\theta,\beta)=ab\alpha\beta\theta\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{j+k+l}\Gamma(\beta)\Gamma[b(j+1)]\Gamma[a(k+1)]\exp[-\alpha(l+1)x][e^{\theta}-1]^{-\beta}}{\Gamma(\beta-j)\Gamma[b(j+1)-k]\Gamma[a(k+1)-l]j!k!l!} \tag{24}$$

Reliability Function: For a continuous distribution function with pdf f(x) and cdf F(x), the survival function of the ECKEP distribution is given by

$$S_{ECKEP}(x)=\frac{\left\{\left(\mathrm{e}^{\theta}-1\right)-\left\{e^{\theta}\left[1-\left(1-\left[\left[\theta\exp(-\alpha x)\right]\right]^{a}\right)^{b}\right]\right\}-1\right\}^{\beta}}{\left(\mathrm{e}^{\theta}-1\right)^{\beta}}, \qquad x,\alpha,a,b,\theta,\beta>0 \tag{25}$$

Hazard Rate Function: For a continuous distribution function with pdf f(x) and cdf F(x), the hazard rate function of the ECKEP distribution is given by

$$h_{ECKEP}(x)=\frac{ab\alpha k\theta\exp(-\alpha x)(1-\exp(-\alpha x))^{a-1}\left(1-\left[1-\exp(-\alpha x)\right]^{a}\right)^{b-1}\left\{\left(e^{\theta}\left[1-\left(1-\left[1-\exp(-\alpha x)\right]^{a}\right)^{b}\right]\right)-1\right\}^{k}}{\left\{\left(\mathrm{e}^{\theta}-1\right)-\left\{e^{\theta}\left[1-\left(1-\left[\left[\theta\exp(-\alpha x)\right]\right]^{a}\right)^{b}\right]\right\}-1\right\}^{k}} \tag{26}$$

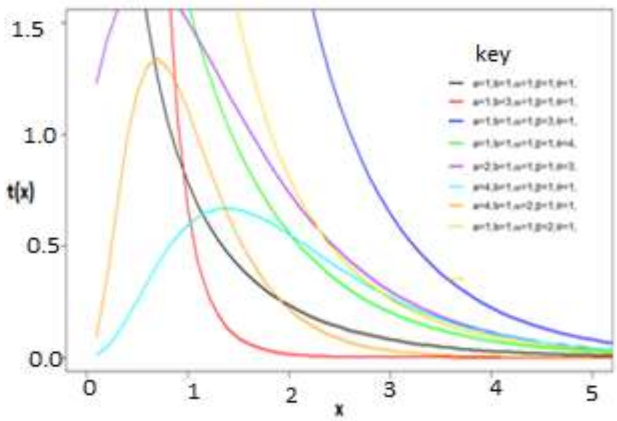

Fig 2: Plot of the ECKEP Distribution Hazard Rate Functions for Some Parameter Values

Figure 2 illustrates the behavior of the hazard function of ECKEP distribution for selected values of the parameters α, β, θ, a and b.

Moments: Now, we discuss the r^{th} moment of a random variable X following the ECKEP distribution. Moments are necessary and important in any statistical analysis, especially in applications.
Using equation24, we write

$$E(X^r) = \int_0^\infty x^r f(x;a,b,\alpha,\theta,\beta)dx \quad (27)$$

$$= ab\alpha\beta\theta \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{j+k+l}\Gamma(\beta)\Gamma[b(j+1)]\Gamma[a(k+1)][e^{\theta}-1]^{-\beta}}{\Gamma(\beta-j)\Gamma[b(j+1)-k]\Gamma[a(k+1)-i]j!k!l!} \times \int_0^\infty x^r \exp[-\alpha(l+1)x]dx$$

Applying the transformation:
$u = \alpha(l+1)x, \quad x = u[\alpha(l+1)]^{-1}, \quad du = \alpha(l+1)dx, \quad dx = [\alpha(l+1)]^{-1}du$ and the gamma function

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt, \quad (28)$$

we obtain the r^{th} moment of ECKEP distribution given as

$$E(X^r) = ab\alpha\beta\theta \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{j+k+l}\Gamma(\beta)\Gamma[b(j+1)]\Gamma[a(k+1)]r!}{\Gamma(\beta-j)\Gamma[b(j+1)-k]\Gamma[a(k+1)-i][\alpha(l+1)]^{r+1}j!k!l![e^{\theta}-1]^{-\beta}} \quad (29)$$

The first moment about the origin (the mean) of the ECKEP distribution is obtained by setting r = 1 in equation 29 to get

$$E(X) = \mu = ab\alpha\beta\theta \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{j+k+l}\Gamma(\beta)\Gamma[b(j+1)]\Gamma[a(k+1)][e^{\theta}-1]^{-\beta}}{\Gamma(\beta-j)\Gamma[b(j+1)-k]\Gamma[a(k+1)-i][\alpha(l+1)]^2 j!k!l!} \quad (30)$$

The moment generating function of the ECKEP distribution is given as

$$M(t) = ab\alpha\beta\theta \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{j+k+l}\Gamma(\beta)\Gamma[b(j+1)]\Gamma[a(k+1)][e^{\theta}-1]^{-\beta}}{\Gamma(\beta-j)\Gamma[b(j+1)-k]\Gamma[a(k+1)-i][\alpha(l+1)-u]j!k!l!} \quad (31)$$

If $u < \alpha(l+1)$, the corresponding characteristic function is given as

$$\phi(t) = ab\alpha\beta\theta \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{j+k+l}\Gamma(\beta)\Gamma[b(j+1)]\Gamma[a(k+1)][e^{\theta}-1]^{-\beta}}{\Gamma(\beta-j)\Gamma[b(j+1)-k]\Gamma[a(k+1)-i][\alpha(l+1)-iu]j!k!l!} \quad (32)$$

where $i = \sqrt{-1}$

The variance of the ECKEP distribution is given as

$$Var(X) = \mu'_2 - (\mu'_1)^2 = \sigma^2 =$$

$$= \left\{ ab\alpha\beta\theta \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{j+k+l}\Gamma(\beta)\Gamma[b(j+1)]\Gamma[a(k+1)]2[e^{\theta}-1]^{-\beta}}{\Gamma(\beta-j)\Gamma[b(j+1)-k]\Gamma[a(k+1)-i][\alpha(l+1)]^3 j!k!l!} \right\} \quad (33)$$

$$- \left\{ ab\alpha\beta\theta \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{j+k+l}\Gamma(\beta)\Gamma[b(j+1)]\Gamma[a(k+1)][e^{\theta}-1]^{-\beta}}{\Gamma(\beta-j)\Gamma[b(j+1)-k]\Gamma[a(k+1)-i][\alpha(l+1)]^2 j!k!l!} \right\}^2$$

The coefficient of skewness and kurtosis of the ECKEP distribution are given as

$$Sk = \frac{E[(X-\mu)^3]}{[E(X-\mu)^2]^{3/2}} = \frac{\mu'_3 - 3\mu\mu'_2 + 2\mu^3}{(\mu'_2 - \mu^2)^{3/2}} \qquad (34)$$

and

$$Ku = \frac{E[(X-\mu)^4]}{[E(X-\mu)^2]^2} = \frac{\mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4}{(\mu'_2 - \mu^2)^2} \qquad (35)$$

The Quantile of ECKEPD: The quantile of the ECKEP distribution is the real solution of the equation F(x)=U, where F(x) is the cdf of ECKEP distribution.

U=F(x)
and is given as

$$U = \frac{\left\{\left[e^{\theta}\left(1-\left(1-[1-\exp(-\alpha x)]^{a}\right)^{b}\right)\right]-1\right\}^{\beta}}{\left\{e^{\theta}-1\right\}^{\beta}}$$

$$1-\exp(-\alpha x) = \left[1-\left\{1-\left(\frac{1+\left[U\left\{e^{\theta}-1\right\}^{\beta}\right]^{\frac{1}{\beta}}}{e^{\theta}}\right)\right\}^{\frac{1}{b}}\right]^{\frac{1}{a}}$$

$$x = \frac{1}{\alpha}\left\{-\log\left(1-\left[1-\left\{1-\left(\frac{1+\left[U\left\{e^{\theta}-1\right\}^{\beta}\right]^{\frac{1}{\beta}}}{e^{\theta}}\right)\right\}^{\frac{1}{b}}\right]^{\frac{1}{a}}\right)\right\} \qquad (36)$$

where U is a uniform random variable on unit interval (0, 1).

The median of the ECKEP distribution is obtained by setting U=0.5 in equation36 to get

$$x = \frac{1}{\alpha}\left\{-\log\left(1-\left[1-\left\{1-\left(\frac{1+\left[\left\{e^{\theta}-1\right\}^{\beta}\right]^{\frac{1}{\beta}}}{2e^{\theta}}\right)\right\}^{\frac{1}{b}}\right]^{\frac{1}{a}}\right)\right\} \qquad (37)$$

Order Statistics

Suppose $X_{1,}\ X_{2,},\ X_{3,},\ldots,X_{n,}$ is a random sample from ECKEP distribution. Let $X_{1,}$ denote the minimum time to failure and $X_{n,}$ denote the maximum time to failure. The trials are independent and identically distributed. The pdf of the k^{th} order statistics from the ECKEP distribution is given as

$$g_{k:n}(x) = \frac{n!\,f(x)}{(k-1)!(n-k)!}\left[F(x)\right]^{k-1}\left[1-F(x)\right]^{n-1} \quad (38)$$

and using the identity

$$(1-z)^{n-1} = \sum_{p=0}^{\infty}(-1)^p \binom{n-1}{p} z^p$$

gives

$$g_{k:n}(x) = \frac{n!\,f(x)}{(k-1)!(n-k)!}\sum_{p=0}^{\infty}(-1)^p\binom{n-k}{p}\left[F(x)\right]^{p+k-1}$$

where F(x) and f(x) are the cdf and pdf of the ECKEP distribution, respectively. It follows from equations18 and 19 that

$$g_{i;n}(x;p) = \frac{q(x)\left[e^{\theta}\left\{1-\left(1-\exp(-\alpha x)\right)\right\}\right]}{B(i,n-i+1)\left[(e^{\theta}-1)^{\beta}\right]^{i+j}}.\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j\left\{\left[e^{\theta}\left\{1-\left(1-\exp(-\alpha x)\right)\right\}\right]-1\right\}^{i+j-1} \quad (39)$$

where B (.,.) denote the beta function.
Hence,

$$g_{i:n}(x;p)=\frac{q(x)\sum_{k=1}^{\infty}ka_k\theta^{k-1}\left\{1-(1-\exp(-\alpha x))\right\}^{k-1}}{B(i,n-i+1)\left[(e^{\theta}-1)^{\beta}\right]^{i+j}}\cdot\sum_{j=0}^{n-1}\binom{n-1}{j}(-1)^j\sum_{s=0}^{\infty}d_{i+j-1;s}a^{i+j-1}\theta^{i+j+s-1}\left\{1-(1-\exp(-\alpha x))\right\}^{l+j+s-1}$$

$$=\frac{q(x)}{B(i,n-i+1)((e^{\theta}-1)^{\beta})^{i+j}}\cdot\sum_{j=0}^{n-1}\sum_{s=0}^{\infty}\sum_{k=0}^{\infty}\binom{n-1}{j}(-1)^j b_k(k+1)\,d_{i+j-1;s}a^{i+j}\theta^{i+j+s+k}\left\{1-(1-\exp(-\alpha x))\right\}^{(l+j+s+k)}$$

$$=\sum_{j=0}^{n-1}\sum_{s=0}^{\infty}\sum_{k=0}^{\infty} w_{i,j,s,k}\left\{1-(1-\exp(-\alpha x))\right\}^{(l+j+s+k)} \qquad (40)$$

where

$$w_{i,j,s,k}=\frac{q(x)\binom{n-1}{j}(-1)^j b_k(k+1)d_{i+j-1;s}a_1^{\,i+j}\theta^{i+j+s+k}}{B(i,n-i+1)((e^{\theta}-1)^{\beta})^{i+j}}$$ and $q(x)=abg(x)G(x)^{a-1}[1-G(x)^a]^{b-1}$

Equation 40 is the order statistics of the ECKEP distribution.
To obtain the smallest order statistic pdf, we substitute i=1 into equation40 to obtain

$$g_{1;n}(x;p)=\sum_{j=0}^{n-1}\sum_{s=0}^{\infty}\sum_{k=0}^{\infty} w_{1,j,s,k}\left\{1-(1-\exp(-\alpha x))\right\}^{(j+s+k+1)} \qquad 0<x<\infty \qquad (41)$$

The pdf of the largest order statistics is obtained by substituting $i=n$ into equation40 to give

$$g_{1;n}(x;p)=\sum_{s+0}^{\infty}\sum_{k+0}^{\infty} w_{n,i,s,k}\left\{1-(1-\exp(-\alpha x))\right\}^{(j+s+k+n)}, \qquad 0<x<\infty \qquad (42)$$

where $w_{n,j,s,k}=\dfrac{nq(x)(-1)^j b_k(k+1)d_{n+j-1;m}a_1^{\,j+n}\theta^{j+s+k+n}}{((e^{\theta}-1)^{\beta})^{j+n}}$

INFERENCE with ECKEPD
Maximum likelihood estimation

Let $x_1, x_2, x_3, ..., x_n$ denote a random sample drawn from the ECKEP distribution given by equation19 with parameters $\theta, \alpha, \beta, a\ and\ b$ .The likelihood function of the ECKEP distribution is given by

$$L(x;\alpha,\beta,a,b,\theta)=\prod_{i=1}^{n}ab\alpha\theta\beta\exp(-\alpha x)\left(1-\exp(-\alpha x)\right)^{a-1}\left(1-\left[1-\exp(-\alpha x)\right]^{a}\right)^{b-1}$$
$$\times\left\{\left(e^{\theta}\left[1-\left(1-\left[1-\exp(-\alpha x)\right]^{a}\right)^{b}\right]\right)-1\right\}^{\beta}(e^{\theta}-1)^{-\beta} \qquad (43)$$

and

$$L=n\log(a)+n\log(b)+n\log(\alpha)+n\log(\beta)+n\log(\theta)-\alpha\sum_{i=1}^{n}x_i \qquad (44)$$
$$+(a-1)\sum\log(1-\exp(-\alpha x_i))+(b-1)\sum\log\left[1-(1-\exp(-\alpha x_i))^{a}\right]$$
$$+(\beta-1)\sum\log\left(e^{\theta}\left[1-\left(1-\left[1-\exp(-\alpha x_i)\right]^{a}\right)^{b}\right]\right)-\beta\sum\log(e^{\theta}-1)$$

Let $\Theta=(\alpha,\beta,a,b,\theta)^T$ be the unknown parameter vector. The score vector which is the gradient of the log-likelihood function with respect to the parameters being estimated is given as

$$U(\Theta)=\left(\frac{\partial L}{\partial a},\frac{\partial L}{\partial b},\frac{\partial L}{\partial \alpha},\frac{\partial L}{\partial \beta},\frac{\partial L}{\partial \theta}\right)^T \qquad (45)$$

The components of the score function are the partial derivatives with respect to each of the parameters. The maximum likelihood estimate of Θ can be obtained by solving the non-linear system of equation $U_n(\Theta)=0$. That is:

$$\frac{\partial L}{\partial \alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}\frac{\alpha(a-1)}{\exp(\alpha x_i)-1}+\sum_{i=1}^{n}\frac{\alpha a(1-b)(1-\exp(-\alpha x_i))^{a-1}}{\left[1-(1-\exp(-\alpha x_i))^{a}\right]\exp(\alpha x_i)}-\sum_{i=1}^{n}x_i \tag{46}$$

$$+\sum_{i=1}^{n}\frac{\alpha ab(\beta-1)(1-\exp(-\alpha x_i))^{a-1}\left(1-[1-\exp(-\alpha x_i)]^{a}\right)^{b-1}}{\left(e^{\theta}\left[1-\left(1-[1-\exp(-\alpha x_i)]^{a}\right)^{b}\right]\right)\exp(\alpha x_i)}=0$$

$$\frac{\partial L}{\partial a}=\frac{n}{a}+(1-b)\sum_{i=1}^{n}\frac{(1-\exp(-\alpha x_i))^{a}\log(1-\exp(-\alpha x_i))}{1-(1-\exp(-\alpha x_i))^{a}}+\sum_{i=1}^{n}\log(1-\exp(-\alpha x_i)) \tag{47}$$

$$+b(\beta-1)\sum_{i=1}^{n}\frac{\left[1-(1-\exp(-\alpha x_i))^{a}\right]^{b-1}(1-\exp(-\alpha x_i))^{a}}{\left(e^{\theta}\left[1-\left(1-[1-\exp(-\alpha x_i)]^{a}\right)^{b}\right]\right)}\times\log(1-\exp(-\alpha x_i))=0$$

$$\frac{\partial L}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}\log\left[1-(1-\exp(-\alpha x_i))^{a}\right]+(\beta-1)\sum_{i=1}^{n}\frac{\left[1-(1-\exp(-\alpha x_i))^{a}\right]^{b}}{\left(e^{\theta}\left[1-\left(1-[1-\exp(-\alpha x_i)]^{a}\right)^{b}\right]\right)}\times\log\left[1-(1-\exp(-\alpha x_i))^{a}\right]=0 \tag{48}$$

$$\frac{\partial L}{\partial \beta}=\frac{n}{\beta}++\sum_{i=1}^{n}\log\left(e^{\theta}\left[1-\left(1-[1-\exp(-\alpha x_i)]^{a}\right)^{b}\right]\right)-\sum_{i=1}^{n}\log(e^{\theta}-1)=0 \tag{49}$$

$$\frac{\partial L}{\partial \theta}=\frac{n}{\theta}+\sum_{i=1}^{n}\left(e^{\theta}\left[1-\left(1-[1-\exp(-\alpha x_i)]^{a}\right)^{b}\right]\right)-\sum_{i=0}^{n}(e^{\theta})=0 \tag{50}$$

Solving the non-linear system of equation given in equations 46 through 50numerically produce the maximum likelihood estimates of parameters $\widehat{a}, b, \alpha, \beta, and \widehat{\theta}$.

RESULTS AND DISCUSSION

This section presents the application of the ECKEP distribution to a real life dataset. For a comparative study, the Kumaraswamy Exponential Weibull Distribution (KEWD), Kumaraswamy Weibull Distribution (KWD), Kumaraswamy Weibull Poisson Distribution (KWPD) and Kumaraswamy Modified Weibull Distribution (KMWD) are also applied to the dataset. In order to evaluate the efficiency of the ECKEP distribution, a package named "maxLik" was used in R Statistical software environment, with method "SANN". The performance of these distributions are evaluated and compared using some Model Selection Information Criteria (MSIC) which include *AIC* (Akaike Information Criterion), *CAIC* (Consistent Akaike Information Criterion), *BIC* (Bayesian Information Criterion) and *HQIC* (Hannan Quin Information Criterion). The MSIC are given as follows:

$AIC=-2ll+2k$, $BIC=-2ll+k\log(n)$,

$CAIC=-2ll+\frac{2kn}{(n-k-1)}$ and

$HQIC=-2ll+2k\log\left[\log(n)\right]$ wherell denotes the log-likelihood value evaluated with the maximum likelihood estimates, k is the number of model parameters and n is the sample size. The model with the lowest values of these statistics would be chosen as the best model to fit the dataset.

Dataset: The dataset represents the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli reported by Bjerkedal (1960) and used by Umar *et al* (2019) and Eraikhuemen *et al* (2020b). It is given as follows: 10, 33, 44, 56, 59, 72, 74, 77, 92, 93, 96, 100, 100, 102, 105, 107, 107, 108, 108, 108, 109, 112, 113, 115, 116,120, 121, 122, 122, 124, 130, 134, 136, 139, 144, 146, 153, 159, 160, 163, 163, 168, 171, 172, 176,183, 195, 196, 197, 202, 213, 215, 216, 222, 230,231, 240, 245, 251, 253, 254, 255, 278, 293, 327,342, 347, 361, 402, 432, 458, 555.

The descriptive statistics for this data are presented in Table 2.

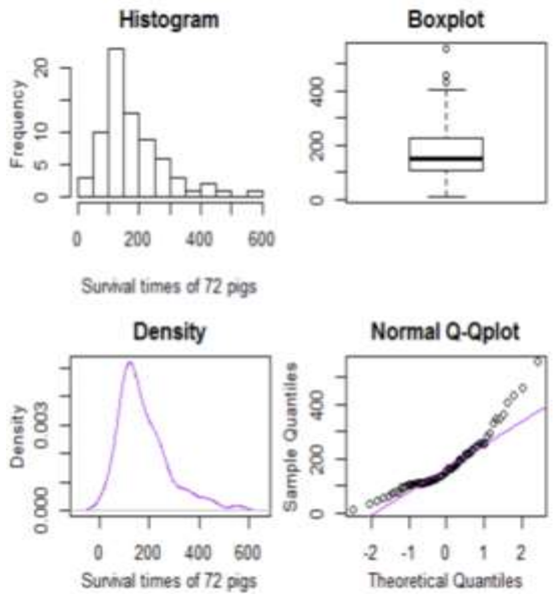

Fig 3: A graphical summary of the survival times (in days) of 72 guinea pigs data

From the descriptive statistics in Table 2 and the graphical display shown in Figures 3 above, it is clear that the dataset is positively skewed, and therefore would be flexible for skewed distributions just like the proposed model. Table 3 clearly shows that the ECKEP distribution has the smallest values of *-ll*, *AIC*, *BIC*, *CAIC* and *HQIC* compared to the other four distributions using the real life dataset. The results in Table 3 also provide evidence for us to agree that the ECKEP distribution fits the real life data better than the other four models. The histogram of the data with fitted densities and estimated cumulative distribution functions displayed in Figure 4 for the dataset also support the claim that the ECKEPD performs better than the KWPD, KMWD, KWD and the KEWD in fitting the dataset.

Table 2: Descriptive Statistics for the survival times (in days) of 72 guinea pigs data

No	Minimum	Q_1	Median	Q_3	Mean	Maximum	Variance	Skewness	Kurtosis
72	10.00	108.0	149.5	224.0	176.8	555.0	10705.1	1.34128	1.98852

Table 3: Performance Evaluation of the Distributions Using the MSIC Based on the Dataset

Distributions	Parameter estimates	-ll=(-log-likelihood value)	*AIC*	*CAIC*	*BIC*	*HQIC*
ECKEPD	$\hat{\alpha}$ =0.9724245 $\hat{\beta}$ =0.9724251 $\hat{\theta}$=0.9724244 $\hat{a}$ =0.9724247 $\hat{b}$ =0.9724244	-3.2417	3.5165	4.4256	14.8999	8.0483
KWPD	$\hat{\alpha}$ =0.9527246 $\hat{\beta}$ =0.9527250 $\hat{\theta}$=0.9527245 $\hat{a}$ =0.9527248 $\hat{b}$ =0.9527243	8.5247	27.0494	27.9585	38.4327	31.5811
KEWD	$\hat{\alpha}$ =1.5449044 $\hat{\beta}$ =4.5867910 $\hat{\theta}$=0.6931186 $\hat{a}$ =5.6917160 $\hat{b}$ =0.0027483	7.2857	24.5714	25.4805	35.9548	29.1032
KMWD	$\hat{\alpha}$ =0.2755403 $\hat{\beta}$ =1.4274148 $\hat{\theta}$=0.3175745 $\hat{a}$ =0.0043696 $\hat{b}$ =0.1260376	10.6803	31.3606	32.2697	42.7440	35.8924
KWD	$\hat{\alpha}$ =8.869616 $\hat{\beta}$ =0.002776 $\hat{a}$ =0.167300 $\hat{b}$ =1.038767	9.0052	26.0104	26.6074	35.1170	29.6358

Conclusion: In this paper, we have proposed a family of distributions called the Exponentiated Complementary Kumaraswamy-G Power Series (ECK-GPS) family of distributions. In particular, we study the statistical properties of the Exponentiated Complementary Kumaraswamy Exponential Poisson (ECKEP) distribution, a member of the ECK-GPS family of distributions. An application of the ECKEP distribution to a real life dataset shows that it performs better in fitting the given dataset when compared to the KWPD, KMWD, KWD and the KEWD.

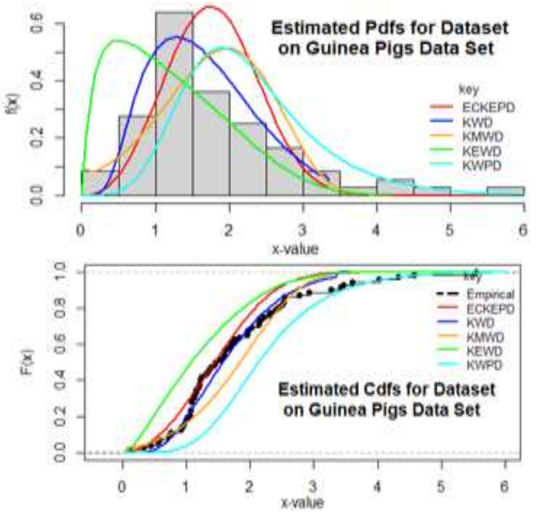

Fig 4: Histogram and plots of the estimated densities (pdfs) and cdfs of the ECKEPD, KWD, KMWD, KEWD and KWPD fitted to the dataset.

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