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Missing Observations in Split-Plot Central Composite Designs: The Loss in Relative *A*-, *G*-, and *V*- Efficiency

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ABSTRACT: The trace (*A*), maximum average prediction variance (*G*), and integrated average prediction variance (*V*) criteria are experimental design evaluation criteria, which are based on precision of estimates of parameters and responses. Central Composite Designs (CCD) conducted within a split-plot structure (split-plot CCDs) consists of factorial (f), whole-plot axial (α), subplot axial (β), and center (c) points, each of which play different role in model estimation. This work studies relative *A*-, *G*- and *V*-efficiency losses due to missing pairs of observations in split-plot CCDs under different ratios (*d*) of whole-plot and sub-plot error variances. Three candidate designs of different sizes were considered and for each of the criteria, relative efficiency functions were formulated and used to investigate the efficiency of each of the designs when some observations were missing relative to the full one. Maximum *A*-efficiency losses of 19.1, 10.6, and 15.7% were observed at $d = 0.5$, due to missing pairs ff, $\beta\beta$, and $f\beta$, respectively, indicating a negative effect on the precision of estimates of model parameters of these designs. However, missing observations of the pairs-$cc, \alpha\alpha, \alpha c, fc,$ and $f\alpha$ did not exhibit any negative effect on these designs' relative *A*-efficiency. Maximum *G*- and *V*-efficiency losses of 10.1,16.1,0.1% and 0.1, 1.1, 0.2%, were observed, respectively, at $d = 0.5$, when the pairs-$ff, \beta\beta, cc$, were missing, indicating a significant increase in the designs' maximum and average variances of prediction. In all, the efficiency losses become insignificant as *d* increases. Thus, the study has identified the positive impact of correlated observations on efficiency of experimental designs.

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Industrial, manufacturing, and engineering and physical sciences research projects require experimentation to uncover relationships that exist between design variables and the responses of interest. Problems in response surface methodology (RSM) involves modeling and investigating an appropriate relationship between input and output variables of a process and identifying optimal operating conditions for a system under study. The assumption of a completely randomized or randomized block error control structure in statistically-designed experiments has been widely accepted in RSM research and applications. Unfortunately, while this assumption simplifies analysis and research, it may not be feasible in industrial experimental situations, which are often split-plot in nature. In such situations, some factors have levels that are difficult to change or control (termed hard-to-change (HTC) factors) due to time or cost constraints, and some with levels that are easy to control (easy-to-change (ETC) factors). For further details see Letsinger, *et al.*, (1996), Vining, *et al.*, (2005), Kowalski *et al* (2006), etc. One of the most popular response surface experimental design is the central composite design (CCD). A CCD with a split-plot structure consists of four different categories of points each of which plays different role in model estimation. These include the n_f equally-spaced factorial points that contribute to the estimation of linear and interaction terms in the model, n_a axial points (whole-plot(α) and subplot(β)), which consists of points lying on the coordinate axis of each input variable, and allow for efficient estimation of pure quadratic terms in the model, and center (c) points, which provide an internal estimate of error (i.e., the pure error), and efficiently provide information about the existence of curvature in the system. The generalized least squares (GLS) model for a split-plot response surface design is

$$\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{Z\gamma} + \boldsymbol{\epsilon} \qquad (1)$$

where **y** is the N x 1 vector of responses, $\boldsymbol{X}$ is the N x p overall model matrix, $\boldsymbol{\beta}$ is the p x 1 vector of regression coefficients, $\boldsymbol{Z}$ is an *N x b* incidence matrix assigning observations to each of the *b* whole plots; γ

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is the *N x 1* vector of whole-plot error terms, ε is the *N x 1* vector of subplot error terms. It is assumed that $\gamma_i \sim N(0, \sigma_\gamma^2)$, $\epsilon_{ij} \sim N(0, \sigma_\epsilon^2)$, $cov(\gamma_i, \epsilon_{ij}) = 0$. The four commonly-used optimality criteria are *A*-, *D*-, *G*-, and *V*- optimality criteria (See Kiefer and Wolfowitz (1959), Kiefer (1959), Box and Hunter (1957)), with the following respective goals:

D-criterion maximizes $|M| = |X'V^{-1}X|$, or equivalently, minimizes $|(X'V^{-1}X)^{-1}|$.
A-criterion minimizes $trace(X'V^{-1}X)^{-1}$

$$G \rightarrow \min_{\zeta} \left[\max_{\zeta} N f(z,x)'(X'V^{-1}X)^{-1} f(z,x) \right].$$

$$V \rightarrow \min_{\xi} \left\{ \frac{\text{N}}{\text{K}} \int_R f(z,\text{x})'(X'V^{-1}X)^{-1} f(z,\text{x}) \text{dzdx} \right\}$$
.

Where *X* is the design matrix, *x* is any point in the design region *R*, *N* is the design size, and $f(z,x) = [f_1(z,x) \ldots f_p(z,x)]$ is a vector of *p* real-valued functions based on the *p* model parameters while $K = \int_R dzdx$ is the volume of the region. The variance - covariance matrix for the observation vector **y** is

$$Var(y) = V = \sigma_\epsilon^2 I_n + \sigma_\gamma^2 ZZ' = \sigma_\epsilon^2 (I_n + dZZ')$$

where $d = \frac{\sigma_\gamma^2}{\sigma_\epsilon^2}$ gives the ratio of the two variance components. The matrix ***ZZ***' is a block diagonal matrix with diagonal matrices of $J_{n1}, J_{n2}, \ldots, J_{nz}$, where J_{ni} is an n_i x n_i matrix of 1's and n_i is the number of observations in the *i*th whole-plot. Under the assumption of normal errors, the maximum likelihood estimate (MLE) of the parameters of this model is obtained through the generalized least squares (GLS) estimation equation $\hat{\beta} = (X'V^{-1}X)^{-1}X'V^{-1}y$ With variance $Var(\hat{\beta}) = (X'V^{-1}X)^{-1}$

Even in a carefully- planned experiments, some observations may be lost during the process of data collection due to reasons that are beyond experimenter's control. Loss of experimental observations destroy desirable properties of the experiment such as orthogonality property, which enables separate and independent estimation of all model parameters. There could be a large increase in variances of predicted responses and also in the generalized variance of parameter estimates of the experiment as a result of missing observations. Thus, proper knowledge of the expected loss in efficiency of these designs when some observations are missing is highly essential so as to exercise extra caution in handling the experiment. Most published research on impact of missing observations on efficiency of response surface designs focused on the completely randomized designs, see for example, Andrews and Herzberg (1979), Ghosh (1978), Herzberg and Andrews (1975, 1976), Ahmad and Gilmour (2010), Akhtar and Prescott (1986), Akram (2002), and Yisa, *et al*. (2014). However, Angela, *et al*. (2013) and, Yakubu and Chukwu (2018) studied the robustness of split-plot CCDs to missing observations, while Angela and Yisa (2012) compared the optimality criteria of reduced models for split-plot response surface designs. Throughout RSM literature, little or no attention has been given to the effect of missing observations on efficiency of split-plot response surface designs. Thus, in this work, loss in *A*-, *G*-, and *V*-efficiency of split-plot central composite designs (CCDs) due to missing pairs of observations were investigated and compared.

MATERIALS AND METHODS

Three split-plot CCDs of different sizes were used in this study. The first design is a three –factor (one whole-plot factor and two sub-plot factors) design. The second design consists of four factors (one whole-plot factor and three sub-plot factors), while the third design is also a four-factor (two whole-plot factors and two sub-plot factors) design. These CCDs are given in TABLE 1 with *k*, *w*, and *s* denoting, respectively, numbers of design factors, whole-plot factors, and subplot factors.

Table 1. Candidate Split-plot CCDs

Number of design factors (*k*)	Number of whole plot factors (*w*)	Number of subplot factors (*s*)
3	1	2
4	1	3
	2	2

For each of these designs, there are ten possible groups of pairs that are formed from factorial *(f)*, whole-plot axial *(α)*, subplot axial *(β)*, and center points *(c)*. These groups are *ff, αα, ββ, cc, fα, fβ, fc, αβ, αc,* and *βc*. Then, *A*-, *G*-, and *V*-efficiency functions were formulated as given below. But *A*-, *G*-, and *V* criterion values for the full and corresponding reduced split-plot CCDs due to missing observations of these pairs of points were first computed using the criteria equations given above under specific values of *d = 0.5, 1.0, 1.5, 2.0,* and *2.5,* which represent the situations that the whole plot variance is half, same, one and a half, two, and, two and a half, times the subplot variance, respectively. These computed values were then tabulated and the relative efficiency values were then computed using the formulated functions in equations 2, 3, and 4.

Then, relative *A*-, *G*-, and *V*-efficiency functions were formulated as

$$RE_A = \frac{Trace\left[(X'V^{-1}X)^{-1}\right]}{Trace[(X'V^{-1}X)^{-1}]_{reduced}} \quad (2)$$

where $\mathrm{Trace}[(X'V^{-1}X)^{-1}]$ and $\mathrm{Trace}[(X'V^{-1}X)^{-1}]_{\mathrm{reduced}}$ are respectively the *A*-criterion for the full and reduced designs due to a pair of missing observations.

$$RE_G = \frac{MAX_{Z,X\epsilon R}[v(z,x)]}{MAX_{Z,X\epsilon R}v(z,x)_{reduced}} \quad (3)$$

$$RE_V = \frac{Trace\left\{(X'V^{-1}X)^{-1}B\right\}}{Trace\left\{(X'V^{-1}X)^{-1}B\right\}_{reduced}} \quad (4)$$

where $v(z,x)$ is the scaled prediction variance, $(X'V^{-1}X)^{-1}$ is the covariance matrix and $B = \left[\frac{1}{K}\int_{\Omega} f(z,x)f'(z,x)dzdx\right]$ is the moment matrix for a given split-plot response surface design; $\mathbf{f}(\mathbf{z},\mathbf{x})$is the 1 x p model vector for the selected design point.

These efficiency functions were used to investigate the relative *A*-, *G*-, and *V*-efficiency of each of the candidate split-plot central composite designs in Table 1 when some (observations) of their design points ($f, \alpha, \beta,$ and c) are missing relative to their corresponding full ones. The loss in efficiency was then computed as

$$Efficiency_{Loss} = 1 - RE \quad (5)$$

For each design, the whole-plot and subplot axial points were fixed at equal distance of 1 (i.e., α = β = 1). The computed relative *A*-, *G*-, and *V*- efficiencies for each of the candidate designs were then plotted against their corresponding ratios of error variances and given in the corresponding figures. From the formulated relative efficiencies in 2, 3, and 4, it is worth noting that: A relative *A*-, *G*-, or *V*-efficiency larger than one indicates that the reduced deign is better than the full design in terms of the trace, maximum, or integrated average prediction variance respectively. This implies that the missing observations have little or no adverse effect on the design in terms of the criterion. A relative efficiency less than one indicates that the full design is better than the reduced deign in terms of the given criterion, which implies that the missing observations have large adverse effect on the design.

RESULTS AND DISCUSSIONS

The computed criterion values due to missing observations of any of the ten possible pairs of points were given in the Tables shown for the different ratios of error variances (*d*) while the corresponding relative efficiencies generated using the formulated functions in equations 2, 3, and 4 were plotted and the efficiency curves are given in the corresponding figures.

Relative A-efficiency: (1) Three-Factor D (1,2) Split-plot CCD: The trace due to each of the possible pairs of missing observations was given in Table 2 while, using equation 1, the corresponding relative *A*-efficiency plots were in Figure 1 for the respective ratios of error variances (*d*). This figure shows that for all *d* values, A-efficiency was robust to the missing αα. However, at *d* = 0.5, missing pairs of factorial and subplot point observations ($f\beta$) has the largest effect on *A*-efficiency with efficiency loss of about 24.1%, followed by missing pairs of factorial (*ff*) point observations with about 18.8% loss in efficiency. Also at this value of *d*, efficiency losses of 5.41% and 5.52% were observed for the missing αc and cc respectively. However, these efficiency losses continue to reduce sharply as *d* increases beyond 0.5, and at about *d* = 2.5, the efficiency curve due to missing *ff* and that due to missing $f\beta$ intersect each other.

(2) Four-Factor D(1,3) Split-plot CCD: The trace due to each of the possible pairs of missing observations was given in Table 2 while the corresponding relative *A*-efficiency plots were given in Figure 2. From this figure, it was observed that efficiency curves due to missing $f\alpha$ and βc overlap each other and the same thing applies to those due to missing cc and αβ. It was also observed that *A*-efficiency was quite robust to the missing pairs of the whole plot axial points ($\alpha\alpha$) for the whole range of *d*, and slightly robust to the missing ac. At *d*= 0.5, *A*-efficiency loss of 11.49%, 8.2% and 6.4% were observed, respectively, for missing pairs of observations of $ff, f\beta,$ and$\beta\beta$. However, each of these efficiency losses continues to diminish drastically as *d* increases beyond 0.5. This result is in line with the findings of Goos and Vandebroek (2004).

(3) *Four -Factor D (2,2) Split-plot CCD:* The computed $A-$criterion values (trace) due to missing observations of any of the ten possible pairs of points in this design were given in Table 3 for the different error variance ratios (*d*). Figure 3 gives the corresponding relative *A*-efficiency plots from which it was observed that this design was adversely affected by each of the missing pairs of observations especially

at low values of *d*. However, *A*-efficiency continues to improve as *d* increases. Furthermore, the efficiency curves form distinct groups and the curves due to the missing pairs: cc, αc, and αα were slightly robust to the changes in *d*. Efficiency loss of less than 1.5% was observed at low values of *d* only when *cc* is missing, which continues to diminish as *d* increases. Among all the pairs, missing ββ caused the highest efficiency loss of about 21.1% at d = 0.5 followed by the missing fβ with about 14.15% loss in efficiency, and then the missing ff with about 9.4% efficiency loss. Thus the most influential pairs in this design were the ββ, fβ, αβ, βc, and ff. However, as d increases, the efficiency losses continue to reduce.

Relative G and V-efficiency: The scaled prediction variances (SPV), G-criterion location as well as the *V*-criterion values for the full and reduced designs were computed and presented in tables based on the given values of *d*. It can be observed from each of the Tables that the location of the maximum prediction variance (*G*) varies with the value of *d* and the category of points in the pairs. The relative *G*- and *V*-efficiencies were then obtained and presented in charts as given in the corresponding figures.

(1) Three-Factor D(1,2) CCD: TABLE 4 gives the scaled prediction variance ($v_{(z,x)}$) properties,G-criterion location and V-criterion values for this design. The relative G and V-efficiency plots were given respectively in Figures 4 and 5. From Figure 4, it can be observed that missing pairs of each of ff and $\beta\beta$ has the highest effect on *G*-efficiency for *d* = 0.5. As *d* increases, the losses continue to reduce drastically. Losing a pair of center point observations adversely affects the design efficiency for values of $d \leq 3.52$. This design was observed to be quite robust to missing whole plot axial point observations ($\alpha\alpha$) for the whole range of *d* in terms of *G*-criterion. From Figure 5, the design was observed to be quite robust to the missing pairs ($\beta\beta$) and ($\alpha\alpha$)of the subplot and wholeplot axial point observations for the whole range of *d* in terms of the *V*-efficiency. However, this criterion was highly affected by the missing pairs (*ff*) and (*cc*) respectively, of factorial and center point observations at low values of *d*, and as *d* goes beyond 3.0, the design becomes robust to these missing pairs. This agrees, to some extent, with the findings of Goos and Vandebroek (2004).

(2) Four-Factor D (1,3) Split-plot CCD: The scaled prediction variances, G-criterion locations and the V-criterion values were given in Table 5. The relative *G*- and *V*-efficiency plots were given in Figures 6 and 7 respectively. From Figure 6, we observed that the *G*-efficiency was adversely affected by missing pairs of observations of ff and $\beta\beta$ for $d \leq 0.92$ and $d \leq 3.51$ respectively. The effect continues to reduce as d increases. We also observed that this criterion was robust to missing pairs of whole plot axial and center point observations for the whole range of d. Figure 7 shows that the highest adverse effect on the relative V-efficiency was due to missing pairs of the subplot axial observations ($\beta\beta$) and the center observations (*cc*) for small values of *d* and the efficiency improves as *d* increases. This criterion was observed to be quite robust to the missing pairs of observations of the factorial points (ff) and the whole plot axial points (αα) for the whole range of *d*.

(3) Four-Factor D (2,2) Split-plot CCD: The scaled prediction variances, G-criterion locations and V-criterion values were given in Table 6. The relative *G*- and *V*-efficiency plots were given in Figures 8 and 9, respectively. Figure 8 shows that the *G*-efficiency was adversely affected by missing pairs of observations of the subplot axial ($\beta\beta$), factorial (*ff*), and center (*cc*) points, for $d \leq 0.91, d \leq 2.51,$ and $d \leq 4.93$ respectively. However, as *d* increases beyond these values, these effects diminished. The efficiency was quite robust to the missing pairs of the whole plot axial point observations (αα) for the whole range of *d*. From Figure 9, we observed that the relative *V*-efficiency was adversely affected by missing pairs of center points (*cc*) at values of *d* below 5.0. However, as *d* increases this effect diminishes. This relative efficiency was quite robust to the missing factorial, subplot and whole plot axial points: ff, ββ and αα for the whole range of *d*.

Table 2. $A-$ criterion valuesfor complete design and for reduced designs due to a pair of missing observations in **D (1,2)** CCD for different *d* values

d	trace($(X'V^{-1}X)^{-1}$) due to missing										
	None	ff	αα	Bβ	Cc	Fα	fβ	Fc	Aβ	αc	βc
0.5	1.799	2.223	1.816	2.016	2.132	2.016	2.267	2.120	1.909	1.916	2.014
1	2.616	3.094	2.634	2.839	2.949	2.849	3.111	2.953	2.731	2.733	2.836
5	9.153	9.738	9.171	9.385	9.486	9.408	9.683	9.513	9.275	9.270	9.379
10	17.32	17.93	17.34	17.55	17.65	17.58	17.86	17.68	17.44	17.44	17.552

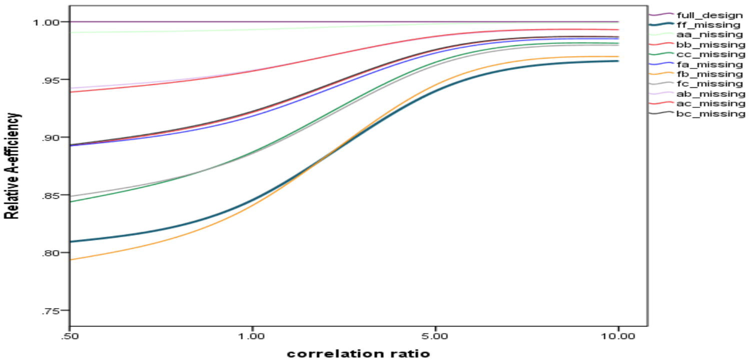

Fig. 1. Relative A-efficiency curves for the reduced and full split-plot D(1,2) CCDs under different variance ratios of error variances (correlation ratio)

Table 2. A − criterion valuefor complete design and for reduced designs due to a pair of missing observations in **D (1,3)** CCD

D	$tr((X'V^{-1}X)^{-1})$ due to missing									
	None	ff	αα	Bβ	Cc	Fα	fβ	Fc	Aβ	αc	βc
0.5	1.483	1.665	1.486	1.589	1.536	1.559	1.625	1.581	1.537	1.507	1.558
1	2.219	2.408	2.222	2.328	2.272	2.297	2.364	2.318	2.274	2.243	2.296
5	8.105	8.303	8.108	8.216	8.158	8.184	8.252	8.205	8.162	8.128	8.183
10	15.46	15.66	15.46	15.57	15.51	15.54	15.61	15.56	15.52	15.49	15.54

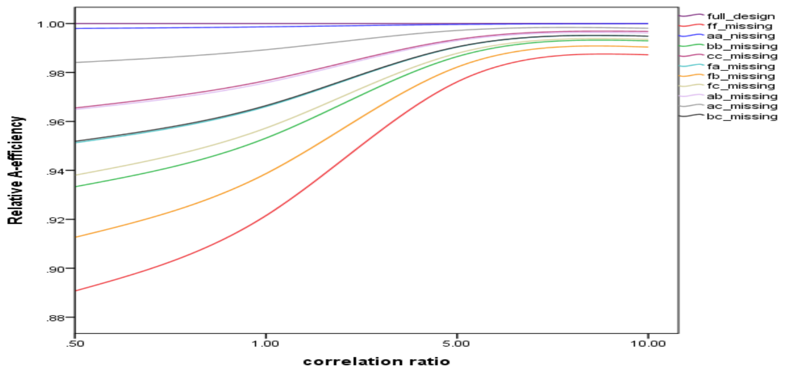

Fig. 2. Relative A-efficiency curves for the reduced and full split-plot D(1,3) CCDs under different variance ratios

Table 3: A − criterion values for full design and for reduced designs due to a pair of missing observations in D(2,2) under different *d*

D	$trace(M^{-1}((X'V^{-1}X)^{-1}))$ due to missing									
	None	ff	αα	Bβ	Cc	Fα	fβ	Fc	Aβ	αc	βc
0.5	2.724	3.016	2.785	3.412	2.743	2.847	3.163	2.832	3.072	2.753	3.058
1	3.925	4.234	3.989	4.62	3.945	4.05	4.377	4.035	4.284	3.955	4.269
5	13.53	13.86	13.59	14.23	13.55	13.65	13.99	13.64	13.9	13.56	13.88
10	25.53	25.87	25.61	26.24	25.56	25.66	26.01	25.65	25.91	25.56	25.89

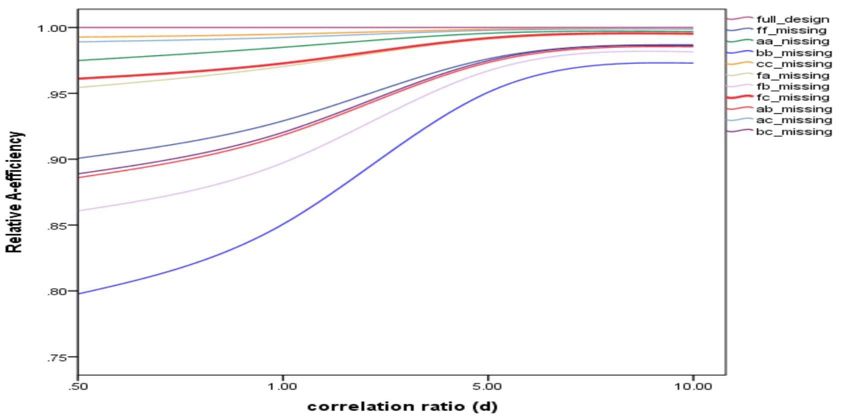

Fig. 3. Relative A-efficiency curves for the reduced and full split-plot D(2,2) CCDs under different correlation ratios

Table 4. SPV properties and G-criterion location for the full design and for the design with a pair of missing observations for **D(1,2)** split-plot CCD

D	$v_{(z,x)}$ due to missing	Design point ±1	$\alpha(1.732)$	$\beta(1.732)$	0	G – location z_1	x_1	x_2	V
0.5									
	Full	12.786	10.166	14.094	11.999	0.000	1.732	0.000	8.217
	ff	15.338	9.473	15.141	10.999	1.000	1.000	1.000	8.559
	αα	11.782	9.352	12.963	10.999	0.000	0.000	1.732	7.598
	ββ	11.853	9.382	15.097	10.999	0.000	1.732	0.000	8.223
	cc	11.72	9.319	12.92	14.666	0.000	0.000	0.000	9.27
	ff								
1	-	11.839	12.708	13.904	14.999	0.000	0.000	0.000	9.517
	αα	13.75	11.792	14.589	13.749	0.000	1.732	0.000	9.606
	ββ	10.9	11.674	12.778	13.749	0.000	0.000	0.000	8.775
	cc	10.961	11.7	14.384	13.749	0.000	0.000	1.732	9.263
	ff	10.852	11.649	12.745	16.499	0.000	0.000	0.000	10.028
	αα								
5	-	9.946	13.523	17.791	20.999	0.000	0.000	0.000	12.119
	cc	10.205	16.375	13.127	19.249	0.000	0.000	0.000	11.476
	ff	9.134	16.317	12.407	19.249	0.000	0.000	0.000	11.126
	αα	9.158	16.327	12.945	19.249	0.000	0.000	0.000	11.298
	ββ	9.117	16.308	12.396	20.166	0.000	0.000	0.000	11.543
	cc								
10	-	9.516	18.946	13.436	22.363	0.000	0.000	0.000	12.71
	ff	9.331	17.406	12.73	20.499	0.000	0.000	0.000	11.86
	αα	8.732	17.372	12.323	20.499	0.000	0.000	0.000	11.66
	ββ	8.746	17.378	12.616	20.499	0.000	0.000	0.000	11.755
	cc	8.723	17.367	12.316	20.999	0.000	0.000	0.000	11.888

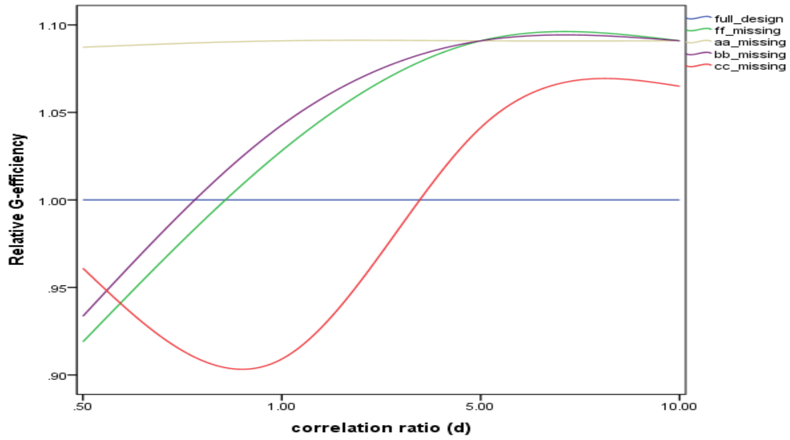

Fig. 4. Relative G-efficiency curves of the reduced split-plot CCDs for the full quadratic model in **one** whole plot and **two** subplot variables under different correlation ratios

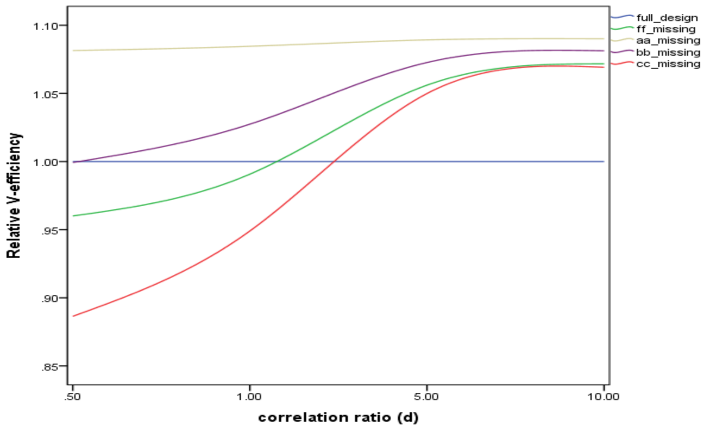

Fig. 5. Relative V-efficiency curves of the reduced split-plot CCDs for the full quadratic model in **one** whole plot and **two** subplot variables for various degrees of correlation

Table 5. Scaled prediction variance ($v_{(z,x)}$) properties and G-criterion location for the full design and for the design with a pair of missing observations for **D (1,3)** split-plot CCD

d	$v_{(z,x)}$ due to missing	Design point ±1	α(2.00)	β(2.00)	0	G – location				V
0.5						z_1	x_1	x_2	x_3	
	Full	22.597	16.985	24.934	19.167	0.000	2.000	0.000	0.000	11.837
ff		27.842	16.327	25.081	18.333	1.000	1.000	1.000	1.000	11.723
αα		21.640	16.255	23.859	18.333	0.000	2.000	0.000	0.000	11.341
ββ		21.737	16.276	29.691	18.333	0.000	0.000	2.000	0.000	11.933
cc		21.615	16.246	23.850	19.555	0.000	0.000	0.000	2.000	11.893
1	-	21.221	22.924	24.289	25.875	0.000	0.000	0.000	0.000	15.099
ff		25.127	21.991	24.168	24.750	1.000	1.000	1.000	1.000	14.757
αα		20.317	21.934	23.240	24.750	0.000	0.000	0.000	0.000	14.456
ββ		20.389	21.952	27.675	24.750	0.000	2.000	0.000	0.000	14.913
cc		20.298	21.928	23.233	25.666	0.000	0.000	0.000	0.000	14.870
5	-	18.464	34.803	22.992	39.291	0.000	0.000	0.000	0.000	21.622
ff		19.330	33.312	22.309	37.583	0.000	0.000	0.000	0.000	20.792
αα		17.668	33.292	21.995	37.583	0.000	0.000	0.000	0.000	20.686
ββ		17.692	33.299	23.496	37.583	0.000	0.000	0.000	0.000	20.844
cc		17.661	33.290	21.993	37.888	0.000	0.000	0.000	0.000	20.824
10	-	17.837	37.503	22.697	42.340	0.000	0.000	0.000	0.000	23.104
ff		17.977	35.884	21.883	40.500	0.000	0.000	0.000	0.000	22.160
αα		17.065	35.873	21.711	40.500	0.000	0.000	0.000	0.000	22.102
ββ		17.078	35.877	22.532	40.500	0.000	0.000	0.000	0.000	22.189
cc		17.062	35.872	21.710	40.666	0.000	0.000	0.000	0.000	22.178

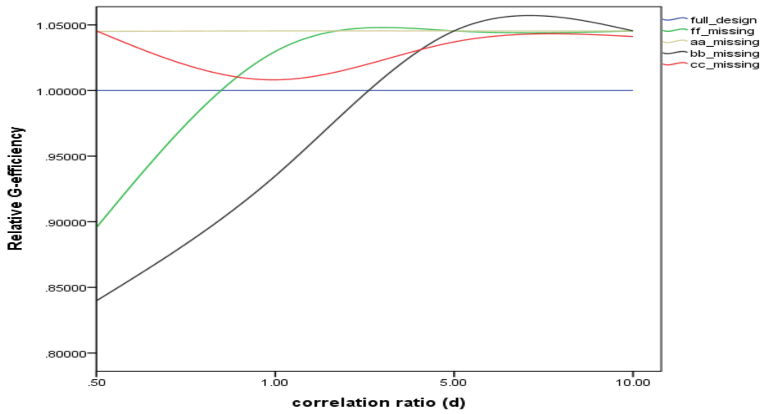

Fig. 6. Relative G-Efficiency curves of the reduced split-plot CCDs for the full quadratic model in D(1,3) CCD under different variance ratios

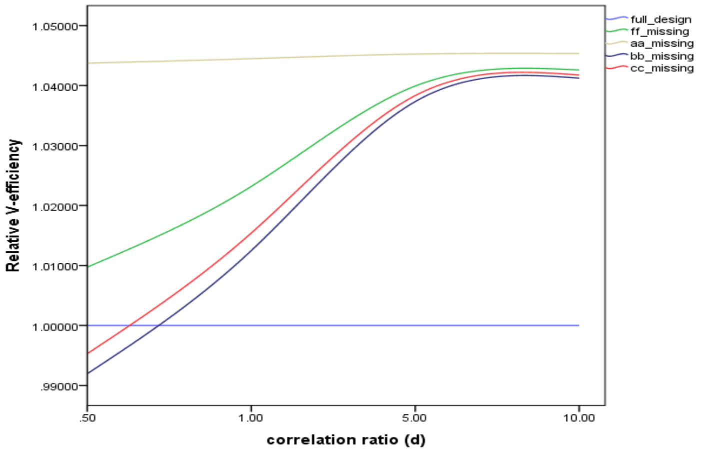

Fig. 7. Relative V-efficiency curves of the reduced split-plot CCDs for the full quadratic model in D(1,3) CCD under different variance ratios

Table 6. Scaled prediction variance ($v_{(z,x)}$) properties and G-criterion location for the full design and for the design with a pair of missing observations for **D(2,2)** split-plot CCD

d	$v_{(z,x)}$ due to missing	Design point ±1	α(2.00)	β(2.00)	0	G – location z_1	z_2	x_1	x_2	V
0.5										
	Full	21.666	16.111	22.222	20.000	0.000	0.000	2.000	0.000	**11.796**
ff		26.632	15.467	22.410	19.000	1.000	1.000	1.000	1.000	11.479
αα		20.656	15.420	21.184	19.000	0.000	0.000	2.000	0.000	11.271
ββ		20.679	15.401	25.717	19.000	0.000	0.000	0.000	2.000	11.702
cc		20.583	15.305	21.111	25.333	0.000	0.000	0.000	0.000	14.162
1	-	21.805	20.138	22.222	25.000	0.000	0.000	0.000	0.000	**14.337**
ff		25.679	19.270	22.153	19.000	1.000	1.000	1.000	1.000	13.848
αα		20.771	19.220	21.167	23.750	0.000	0.000	0.000	0.000	13.671
ββ		20.792	19.209	24.588	23.750	0.000	0.000	2.000	0.000	14.005
cc		20.715	19.131	21.111	28.500	0.000	0.000	0.000	0.000	15.837
5	-	22.083	28.194	22.222	35.000	0.000	0.000	0.000	0.000	**19.421**
ff		22.855	26.839	21.494	33.250	0.000	0.000	0.000	0.000	18.537
αα		20.998	26.785	21.130	33.250	0.000	0.000	0.000	0.000	18.467
ββ		21.007	26.813	22.280	33.250	0.000	0.000	0.000	0.000	18.584
cc		20.979	26.784	21.111	34.833	0.000	0.000	0.000	0.000	19.189
10	-	22.146	30.025	22.222	37.272	0.000	0.000	0.000	0.000	**20.576**
ff		22.084	28.554	21.323	19.000	2.000	0.000	0.000	0.000	19.596
αα		21.049	28.524	21.121	35.409	0.000	0.000	0.000	0.000	19.557
ββ		21.054	28.539	21.749	35.409	0.000	0.000	0.000	0.000	19.621
cc		21.039	28.523	21.111	36.272	0.000	0.000	0.000	0.000	19.950

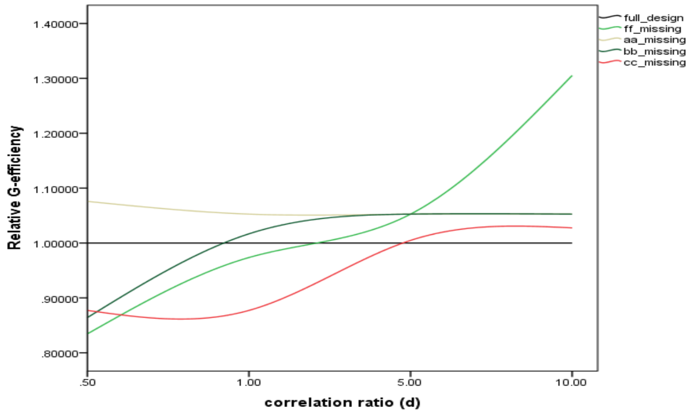

Fig. 8. Relative G-efficiency curves of the reduced split-plot CCDs for the full quadratic model D(2,2) CCD under different variance ratios

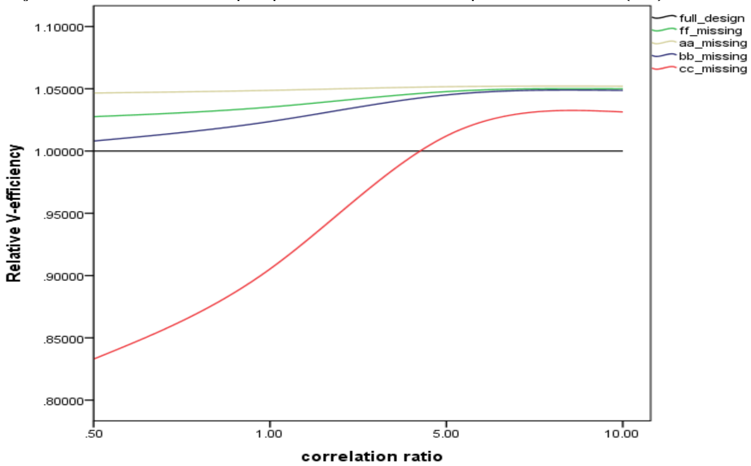

Fig. 9. Relative V-efficiency curves of the reduced split-plot CCDs for the full quadratic model in **two** whole plot and **two** subplot variables under different variance ratios

Conclusion: This study has established the robustness potentials of estimates of the model parameters and the predicted responses of split-plot CCDs at low values of the ratio (d) of the whole-plot and subplot error variances when pairs of observations of the various design points (factorial (f), whole-plot axial (α), subplot axial (β), and center (c) points) are missing.

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