



Mathematical Modeling of Effect of Pumping Rate on Contaminant Transport in Riverbank Filtration System

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ABSTRACT: Riverbank filtration (RBF) is a natural technology that is used for river water treatment. This research seeks to investigate the effect of pumping rate on the transport of colloids in RBF. However, this work considered Dissolved Organic Matter (DOM) as a nutrient for bacteria. The mathematical model consists of groundwater flow equation and colloids concentration equations. The equations were solved analytically using parameter expanding method and Eigen function expansion techniques. The results obtained are presented graphically and discussed. It was observed that increase in pumping rate value enhance both the hydraulic head and concentration of colloids which slightly reduces the quality of pumped water from RBF.

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Riverbank filtration (RBF) can be defined as the natural or induced transport of water from the river to the aquifer, through the riverbed. In 1800, this system was initially reported in United Kingdom, and is now widely accepted in Europe. In Germany, 63% of fresh water sources are from groundwater while 15.3% are from RBF and groundwater recharge. In Netherlands, 39% of surface water is treated using RBF and/or dune recharge (Ray, 2011). The effectiveness and efficiency of riverbank filtration largely depend on local conditions which include the geochemistry of water (from the aquifer and river), the hydrogeology and hydrology of the site, the geochemistry of microbial populations, and associated metabolic activities (Lee and Lee, 2010). Other factors are soil texture and quality (Mustafa *et al.*, 2014). The overall design of RBF systems requires detailed hydrogeological site investigation, knowledge about the hydrological characteristics of the catchment as well as defining the catchment area (Grischek *et al.*, 2002). The effect of hydraulic conductivity on one dimensional contaminant transport in RBF system was investigated by Mustafa *et al.* (2018) using Green's function approach. The result obtained show that increase in hydraulic conductivity value led to an increase in contaminant concentration in pumping well area. Kim *et al.* (2003) investigated numerically the effect of dissolve organic matter and bacteria on contaminant

transport in riverbank filtration. The results show that contaminant transport is enhanced markedly in the presence of DOM and bacteria, and the impact of DOM on contaminant mobility is greater than that of bacteria under examined conditions. Mustafa *et al.* (2014) reviewed that there are few analytical models of RBF system. From their investigation they categorized the available models into two: the first category describes the flow of groundwater induced from wells near the river. While the second category describes the pollutants transportation from river to the well. They also concluded that the analytical model that simulate the potential of microorganism in RBF systems are very rear. Shamsuddin *et al.* (2014) presented a case study of bank infiltration (BI) method which evaluates the effects of groundwater pumping and BI operation on the installation wells as well determine the effects of pumping rate on flow paths, travel time, the size of the pumping and capture zone delineation. The results indicate that the migration of river water into the aquifer is generally slow and depends on the pumping rate and distance from well to the river. Singh *et al.* (2011) conducted a comparative study between Laplace Transform Technique and Fourier Transform Technique in solving one-dimensional transport equation along unsteady groundwater flow in semi-infinite aquifer. Their result indicates that Fourier Transform Technique is better

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than Laplace Transform Technique in predicting contaminants concentration along groundwater flow in semi homogenous aquifer. Aiyesimi and Jimoh (2012) used computational analysis to study one-dimensional non-linear reactive contaminant flow with an initial continuous point source. They discovered that from the origin, the concentration decreases with increase in time and distance. Chattopadhyay and Deo (2018) discussed various mathematical models involving water pollutant transport equation in Damoda River, Bermo Region. The implicit central scheme in space and a forward difference method in time was also given to evaluate the generalized transport equation. Different parameters were varied and they observed a general decrease in contaminant concentration. Bohaienko and Bulavatsky (2019) developed a mathematical model of solutes migration under the conditions of groundwater filtration with $k -$ Caputo fractional derivatives. They use finite difference scheme to simulate the dynamics of anomalous soluble substance migration under the conditions of two-dimensional steady-state plane-vertical filtration with a free surface. The results of numerical experiments on modeling the dynamics of the considered processes were presented. Gothwal and Thatikonda (2017) studied a mathematical model for the transport of fluoroquinolone and its resistant bacteria in aquatic environment. The simulation results of the model for different cases show that the concentration of

antibiotic, organic matter, segregation rate, and horizontal transfer rate are the governing factors in the variation of population density of resistant bacteria. In this present work, we seek to investigate the effect of variable dependent pumping rate on the concentration of the colloids using parameter expanding method and Eigen function expansion techniques.

Mathematical Formulation: We consider a situation where contaminants are present together with dissolve organic matter (DOM), bacteria and virus in an aquifer under simplistic river bank filtration conditions. The aquifer is conceptualized as a five-phase system: three mobile colloidal phases, an aqueous phase and a stationary solid phase. An analytical approach is used to describe the interactions of contaminants with DOM, bacteria, virus and solid matrix. It is assumed that the aquifer is saturated with dissolve organic matter and DOM is utilized as a nutrient for bacteria. The aquifer is unconfined, homogeneous and isotropic. The pumping rate of the pumping well is assumed to depend on the space variable X . Based on the above assumptions, the equations governing the phenomenon are as follows:

The one-dimensional groundwater flow equation in aquifer is given as:

$$S_s \frac{\partial h}{\partial t} = K \frac{\partial^2 h}{\partial x^2} - \frac{q}{Q_o \left(1 - \frac{x}{L+x}\right)} S_s e^{-\frac{\beta}{R_c} t} + \lambda_0 C_c \tag{1}$$

The mass balance equation for bacteria (captured and suspended) in the aqueous phase of saturated porous media may be describe as:

$$\begin{aligned} \varepsilon \frac{\partial C_b}{\partial t} + \rho_b K_b \frac{\partial C_b}{\partial t} - \varepsilon K \frac{\partial h}{\partial x} \frac{\partial C_b}{\partial x} = \varepsilon \frac{\partial}{\partial x} \left(D_b \frac{\partial C_b}{\partial x} \right) + \frac{\mu_{\max}}{K_s} \left(\begin{aligned} &C_c + \frac{\rho_s K_1 C_c}{\varepsilon} + C_o K_2 C_c + \\ &C_b K_3 C_c + C_v K_5 C_c \end{aligned} \right) \varepsilon C_b - \\ K_{dmb} \varepsilon C_b + K_o Y_b \varepsilon C_o + \frac{\mu_{\max}}{K_s} \left(\begin{aligned} &C_c + \frac{\rho_s K_1 C_c}{\varepsilon} + \frac{\rho_b K_b C_b K_4 C_c}{\varepsilon} + \\ &\frac{\rho_v K_v C_v K_6 C_c}{\varepsilon} \end{aligned} \right) \rho_b K_b C_b - K_{dib} \rho_b K_b C_b \end{aligned} \tag{2}$$

The mass balance equation for virus (captured and suspended) in the aqueous phase of saturated porous media may be describe as:

$$\begin{aligned} \varepsilon \frac{\partial C_v}{\partial t} + \rho_v K_v \frac{\partial C_v}{\partial t} - \varepsilon K \frac{\partial h}{\partial x} \frac{\partial C_v}{\partial x} = \varepsilon \frac{\partial}{\partial x} \left(D_v \frac{\partial C_v}{\partial x} \right) + \frac{\mu_{\max}}{K_s} \left(\begin{aligned} &C_c + \frac{\rho_s K_1 C_c}{\varepsilon} + C_o K_2 C_c + \\ &C_b K_3 C_c + C_v K_5 C_c \end{aligned} \right) \varepsilon C_v \\ - K_{dmv} \varepsilon C_v + \frac{\mu_{\max}}{K_s} \left(\begin{aligned} &C_c + \frac{\rho_s K_1 C_c}{\varepsilon} + \frac{\rho_b K_b C_b K_4 C_c}{\varepsilon} + \frac{\rho_v K_v C_v K_6 C_c}{\varepsilon} \end{aligned} \right) \rho_v K_v C_v - K_{div} \rho_v K_v C_v \end{aligned} \tag{3}$$

The mass balance equation for contaminant may be expressed as:

$$R_c \frac{\partial C_c}{\partial t} - K \frac{\partial h}{\partial x} \frac{\partial C_c}{\partial x} = \frac{\partial}{\partial x} \left(D_c \frac{\partial C_c}{\partial x} \right) - \frac{\mu_{\max} C_b}{K_s Y_b} (1 + \rho_s K_1 + \rho_b K_b) C_c - \frac{\mu_{\max} C_v}{K_s Y_v} (1 + \rho_s K_1 + \rho_v K_v) C_c + \frac{q}{Q_o \left(1 - \frac{x}{L+x} \right)} S_s e^{-\frac{\beta}{R_c} t} \tag{4}$$

The mass balance equation for DOM in the aqueous phase may be expressed as:

$$\varepsilon \frac{\partial C_o}{\partial t} + \rho_s K_o \frac{\partial C_o}{\partial t} - \varepsilon K \frac{\partial h}{\partial x} \frac{\partial C_o}{\partial x} = \varepsilon \frac{\partial}{\partial x} \left(D_o \frac{\partial C_o}{\partial x} \right) - \frac{\mu_{\max} C_o K_2 C_{co}}{K_s Y_b} \varepsilon C_b \tag{5}$$

Where:

$h(x, t)$ is the hydraulic head, H is the head value at $t = 0$, $Q = \lambda_o C_c$ is the volumetric flux per unit volume representing source or sink terms/reaction rate (concentration of contaminant) in the source or sink, β is the biological processes at the riverbed, R_c is the retardation factor, q is the stream depletion flow rate, S_s is the specific storage coefficient, K is the hydraulic conductivities, $m_o(x, t)$ is the concentration of contaminant measured at the pumping well, Q_o is the pumping rate, L is the distance of the pumping well from the river, C_b, C_v, C_c & C_o are the concentration of the bacteria, virus, contaminant and DOM suspended in the aqueous phase respectively, ε is the water content, V_p is the pore water velocity, D_b, D_v, D_c & D_o are the hydrodynamic dispersion coefficient of bacteria, virus, contaminant and DOM respectively, ρ_b & ρ_v are the density of bacteria and virus respectively, σ_b & σ_v are the volumetric fraction of bacteria and virus respectively, K_b, K_v are the linear equilibrium distribution coefficient of bacteria and virus between the aqueous phase and the solid phase respectively, K_o is the first-order decay rate coefficient of DOM, K_s is the half saturation constant, Y_b and Y_v is the yield coefficient of bacteria and

virus, μ_{\max} is the maximum growth rate, K_1 is the linear equilibrium distribution coefficient of contaminants between the aqueous phase and the solid matrix, K_2 is the linear equilibrium distribution coefficient of contaminants between the aqueous phase and DOM, K_3 & K_4 are the linear equilibrium distribution coefficient of contaminants between the aqueous phase, mobile and immobile bacteria respectively, K_5 & K_6 are the linear equilibrium distribution coefficient of contaminants between the aqueous phase, mobile and immobile virus respectively, K_{dmb} & K_{dib} are the decay rate coefficient of mobile and immobile bacteria respectively, K_{dmv} & K_{div} are the decay rate coefficient of mobile and immobile virus respectively, ρ_s is the dry bulk density of solid matrix, σ_{cs} is the mass fraction of contaminants sorbed on solid matrix, Q_{oc} is the utilization rate of contaminants sorbed on DOM, σ_{co} is the mass fraction of contaminants attached to DOM, The dependence of diffusion coefficients on the concentration of contaminant is taken in to account by the mathematical expressions:
 $D_b = D_{bo} e^{\sigma C_c}, D_v = D_{vo} e^{\eta C_c}, D_c = D_{co} e^{\nu C_c}$
 and $D_o = D_{oo} e^{\mu C_c}$ (6)
 The initial and boundary conditions associated with the equations are formulated as:

$$\left. \begin{aligned} h(x, 0) = H, \quad h(0, t) = H, \quad \left. \frac{\partial h}{\partial x} \right|_{x=L} = 0 \\ C_b(x, 0) = \frac{C_{b0}x}{L}, \quad -D_b^* \frac{\partial C_b}{\partial x} \Big|_{x=0} + C_b(0, t) = 0, \quad \left. \frac{\partial C_b}{\partial x} \right|_{x=L} = 0 \\ C_v(x, 0) = \frac{C_{v0}x}{L}, \quad -D_v^* \frac{\partial C_v}{\partial x} \Big|_{x=0} + C_v(0, t) = 0, \quad \left. \frac{\partial C_v}{\partial x} \right|_{x=L} = 0 \\ C_c(x, 0) = \frac{C_{c0}x}{L}, \quad -D_c^* \frac{\partial C_c}{\partial x} \Big|_{x=0} + C_c(0, t) = 0, \quad \left. \frac{\partial C_c}{\partial x} \right|_{x=L} = 0 \\ C_o(x, 0) = \frac{C_{o0}x}{L}, \quad -D_o^* \frac{\partial C_o}{\partial x} \Big|_{x=0} + C_o(0, t) = 0, \quad \left. \frac{\partial C_o}{\partial x} \right|_{x=L} = 0 \end{aligned} \right\} \quad (7)$$

Method of Solution: Dimensional analysis: Equation (1) - (7) were non-dimensionalized using the following dimensionless variables

$$t' = \frac{Kt}{S_s L^2}, \quad h' = \frac{h}{H}, \quad x' = \frac{x}{L}, \quad \phi = \frac{C_b}{C_{b0}}, \quad \psi = \frac{C_v}{C_{v0}}, \quad \theta = \frac{C_c}{C_{c0}}, \quad \varphi = \frac{C_o}{C_{o0}} \quad (8)$$

Then we obtain

$$\frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial x^2} - \frac{\lambda_1}{\left(1 - \frac{x}{1+x}\right)} e^{-\lambda_3 t} + \lambda_2 \theta \quad (9)$$

$$R_b \frac{\partial \phi}{\partial t} - \alpha_0 \frac{\partial h}{\partial x} \frac{\partial \phi}{\partial x} = D_1 \frac{\partial}{\partial x} \left((1 + \sigma_1 \theta) \frac{\partial \phi}{\partial x} \right) - \alpha_1 \phi + \alpha_4 (\alpha_5 \theta + \alpha_6 \varphi \theta + \alpha_7 \phi \theta + \alpha_8 \psi \theta) \phi + \alpha_9 (\alpha_5 \theta + \beta_1 \phi \theta + \beta_2 \psi \theta) \phi + \alpha_3 \varphi \quad (10)$$

$$R_v \frac{\partial \psi}{\partial t} - \alpha_0 \frac{\partial h}{\partial x} \frac{\partial \psi}{\partial x} = D_2 \frac{\partial}{\partial x} \left((1 + \eta_1 \theta) \frac{\partial \psi}{\partial x} \right) - r_0 \psi + r_1 (\alpha_5 \theta + \alpha_6 \varphi \theta + \alpha_7 \phi \theta + \alpha_8 \psi \theta) \psi + r_2 (\alpha_5 \theta + \beta_1 \phi \theta + \beta_2 \psi \theta) \psi \quad (11)$$

$$R_c \frac{\partial \theta}{\partial t} - \alpha_0 \frac{\partial h}{\partial x} \frac{\partial \theta}{\partial x} = D_3 \frac{\partial}{\partial x} \left((1 + \nu_1 \theta) \frac{\partial \theta}{\partial x} \right) - m_1 \phi \theta - m_2 \psi \theta + \frac{m_3}{\left(1 - \frac{x}{1+x}\right)} e^{-\lambda_3 t} \quad (12)$$

$$R_o \frac{\partial \varphi}{\partial t} - \alpha_0 \frac{\partial h}{\partial x} \frac{\partial \varphi}{\partial x} = D_4 \frac{\partial}{\partial x} \left((1 + \mu_1 \theta) \frac{\partial \varphi}{\partial x} \right) - g_1 \phi \varphi \theta \quad (13)$$

$$\left. \begin{aligned} h(x, 0) = 1, \quad h(0, t) = 1, \quad \left. \frac{\partial h}{\partial x} \right|_{x=1} = 0 \\ \phi(x, 0) = x, \quad -D_b^* \frac{\partial \phi}{\partial x} \Big|_{x=0} + \phi(0, t) = 0, \quad \left. \frac{\partial \phi}{\partial x} \right|_{x=1} = 0 \\ \psi(x, 0) = x, \quad -D_v^* \frac{\partial \psi}{\partial x} \Big|_{x=0} + \psi(0, t) = 0, \quad \left. \frac{\partial \psi}{\partial x} \right|_{x=1} = 0 \\ \theta(x, 0) = x, \quad -D_c^* \frac{\partial \theta}{\partial x} \Big|_{x=0} + \theta(0, t) = 0, \quad \left. \frac{\partial \theta}{\partial x} \right|_{x=1} = 0 \\ \varphi(x, 0) = x, \quad -D_o^* \frac{\partial \varphi}{\partial x} \Big|_{x=0} + \varphi(0, t) = 0, \quad \left. \frac{\partial \varphi}{\partial x} \right|_{x=1} = 0 \end{aligned} \right\} \quad (14)$$

Analytical Solution: Equations (9) – (13) satisfies (14) were solved analytically using parameter expanding method and Eigen function expansion techniques and we obtained

$$h(x, t) = 1 + \sum_{n=1}^{\infty} q_0 e^{-q_1 t} \sin \frac{(2n-1)\pi x}{2} + \sum_{n=1}^{\infty} h_n(t) \sin \frac{(2n-1)\pi x}{2} \quad (15)$$

$$\begin{aligned} \phi(x, t) = e^{\rho_{10} t} + \sum_{n=1}^{\infty} p_8 e^{\rho_{11} t} \cos n \pi x + \frac{f_o}{D_b^*} \left(e^{\rho_{10} t} + \sum_{n=1}^{\infty} p_8 e^{\rho_{11} t} \right) \left(x - \frac{x^2}{2} \right) + \\ L_o(t) + \sum_{n=1}^{\infty} L_n(t) \cos n \pi x \end{aligned} \quad (16)$$

$$\psi(x, t) = e^{p_{10}t} + \sum_{n=1}^{\infty} p_8 e^{p_{11}t} \cos n\pi x + \frac{f_o}{D_v^*} \left(e^{p_7t} + \sum_{n=1}^{\infty} p_8 e^{p_9t} \right) \left(x - \frac{x^2}{2} \right) + \tag{17}$$

$$Q_o(t) + \sum_{n=1}^{\infty} Q_n(t) \cos n\pi x$$

$$\theta(x, t) = p_2 - p_3 e^{-\lambda_3 t} + \sum_{n=1}^{\infty} (p_4 e^{-\lambda_3 t} + p_5 e^{-p_6 t}) \cos n\pi x + \tag{18}$$

$$\frac{f_o}{D_c^*} \left(p_2 - p_3 e^{-\lambda_3 t} + \sum_{n=1}^{\infty} (p_4 e^{-\lambda_3 t} - p_5 e^{-p_6 t}) \right) \left(x - \frac{x^2}{2} \right) + N_o(t) + \sum_{n=1}^{\infty} N_n(t) \cos n\pi x$$

$$\varphi(x, t) = e^{p_{10}t} + \sum_{n=1}^{\infty} p_8 e^{p_{11}t} \cos n\pi x + \frac{f_o}{D_o^*} \left(1 + \sum_{n=1}^{\infty} p e^{-p_1 t} \right) \left(x - \frac{x^2}{2} \right) + \tag{19}$$

$$M_o(t) + \sum_{n=1}^{\infty} M_n(t) \cos n\pi x$$

Where,

$$h_n(t) = \left(2b_3 e^{-p_{12}t} \left(\frac{2}{(2n-1)\pi} \left(\frac{p_2}{p_{12}} (e^{p_{12}t} - 1) - \frac{p_3}{p_{12} - \lambda_3} (e^{(p_{12} - \lambda_3)t} - 1) \right) - \right) + \frac{4b_3 e^{-p_{12}t}}{p_{12} - \lambda_3} \left(\frac{(2(-1))^n + \pi - 2n\pi}{\pi^2 (4n^2 - 4n + 1)} \right) (e^{(p_{12} - \lambda_3)t} - 1) \right)$$

$$M_o(t) = -\frac{2f_o D_4}{R_o D_o^*} \left(t - \sum_{n=1}^{\infty} \frac{p(e^{-p_1 t} - 1)}{p_1} \right) - \frac{2f_o}{3D_o^*} \sum_{n=1}^{\infty} \frac{pp_1(e^{-p_1 t} - 1)}{p_1} - \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{8n(n\pi) p q_0 (2n-1) (e^{-(p_1+q_1)t} - 1)}{R_o (4n-1)(p_1+q_1)} + \frac{2n_1}{3} -$$

$$\left(\frac{p_2(e^{p_{10}t} - 1)}{p_{10}} - \frac{p_3(e^{(p_{10} + \lambda_3)t} - 1)}{p_{10} - \lambda_3} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{pp_4(e^{(p_{10} - p_1 - \lambda_3)t} - 1)}{2(p_{10} - p_1 - \lambda_3)} + \frac{2b_{16}}{R_o} \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{pp_5(e^{(p_{10} - p_1 - p_6)t} - 1)}{2(p_{10} - p_1 - p_6)} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{pp_4(e^{(p_{11} - \lambda_3)t} - 1)}{2(p_{11} - \lambda_3)} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{pp_5(e^{(p_{11} - p_6)t} - 1)}{2(p_{11} - p_6)} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p^2 p_2 (e^{(p_{11} - p_1)t} - 1)}{2(p_{11} - p_1)} - \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p^2 p_3 (e^{(p_{11} - p_1 - \lambda_3)t} - 1)}{2(p_{11} - p_1 - \lambda_3)} \right)$$

$$M_n(t) = -\frac{2f_o e^{-p_1 t}}{(n\pi)^2 D_o^*} \sum_{n=1}^{\infty} pp_1 t + \frac{2D_4 b_{15} e^{-p_1 t}}{R_o} \left(\sum_{n=1}^{\infty} \frac{pp_2 (n\pi)^2 t}{2} - \sum_{n=1}^{\infty} \frac{pp_3 (n\pi)^2 (e^{-\lambda_3 t} - 1)}{2\lambda_3} \right) - \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{8n(n\pi) p q_0 (2n-1) (e^{-(p_1+q_1)t} - e^{-p_1 t})}{2R_o (12n^2 + 4n - 1) q_1} - \frac{2n_1 e^{-p_1 t}}{(n\pi)^2} -$$

$$\frac{2b_{16} e^{-p_1 t}}{R_o} \left(\sum_{n=1}^{\infty} \frac{p_4 (e^{(p_{10} - \lambda_3)t} - 1)}{2(p_{10} - \lambda_3)} + \sum_{n=1}^{\infty} \frac{p_5 (e^{(p_{10} - p_6)t} - 1)}{2(p_{10} - p_6)} + \sum_{n=1}^{\infty} \frac{pp_2 (e^{p_{10}t} - 1)}{2p_{10}} - \sum_{n=1}^{\infty} \frac{pp_3 (e^{(p_{10} - \lambda_3)t} - 1)}{2(p_{10} - \lambda_3)} + \sum_{n=1}^{\infty} \frac{pp_2 (e^{p_{11}t} - 1)}{2p_{11}} + \sum_{n=1}^{\infty} \frac{pp_3 (e^{-\lambda_3 t} - 1)}{2\lambda_3} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{3p^2 p_4 (e^{(p_{11} - \lambda_3)t} - 1)}{8(p_{11} - \lambda_3)} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{3p^2 p_5 (e^{(p_{11} - p_6)t} - 1)}{8(p_{11} - p_6)} \right)$$

$$L_o(t) = e^{p_{10}t} \left(\frac{2f_o \alpha_1}{3R_b D_b^*} - \frac{2f_o D_1}{R_b D_b^*} \right) \left(t + \sum_{n=1}^{\infty} \frac{p_8 (e^{(p_{11} - p_{10})t} - 1)}{p_{11} - p_{10}} \right) - \frac{2f_o e^{p_{10}t}}{3D_b} \left(p_{10}t + \sum_{n=1}^{\infty} \frac{p_8 p_{11} (e^{(p_{11} - p_{10})t} - 1)}{p_{11} - p_{10}} \right)$$

$$+ \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{8(n\pi)^2 (2n-1) q_0 p_8 (e^{(p_{11} - q_1)t} - e^{p_{10}t})}{2R_b \pi (2n-1) (p_{11} - p_{10} - q_1)}$$

$$+ \frac{2e_1 e^{p_{10}t}}{R_b} \left(p_2 t + \frac{p_3 (e^{-\lambda_3 t} - 1)}{\lambda_3} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_4 p_8 (e^{(p_{11} - p_{10} - \lambda_3)t} - 1)}{2(p_{11} - p_{10} - \lambda_3)} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_5 p_8 (e^{(p_{11} - p_{10} - p_6)t} - 1)}{2(p_{11} - p_{10} - p_6)} \right)$$

$$\begin{aligned}
 & \left(\frac{p_2(e^{p_{10}t} - 1)}{p_{10}} - \frac{p_3(e^{(p_{10}-\lambda_3)t} - 1)}{p_{10} - \lambda_3} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_2 p_8^2 (e^{(2p_{11}-p_{10})t} - 1)}{2(2p_{11} - p_{10})} \right. \\
 & + \frac{2e_2 e^{p_{10}t}}{R_v} \left(\sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_3 p_8^2 (e^{(2p_{11}-p_{10}-\lambda_3)t} - 1)}{2(2p_{11} - p_{10} - \lambda_3)} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_4 p_8 (e^{(p_{11}-\lambda_3)t} - 1)}{2(p_{11} - \lambda_3)} \right. \\
 & \left. \left. + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_5 p_8 (e^{(p_{11}+p_7-p_{10}-p_6)t} - 1)}{2(p_{11} + p_7 - p_{10} - p_6)} \right) \right) \\
 & + \frac{2e_3 e^{p_{10}t}}{R_b} \left(\frac{p_2(e^{p_7t} - 1)}{p_7} - \frac{p_3(e^{(p_7-\lambda_3)t} - 1)}{p_7 - \lambda_3} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_4 p_8 (e^{(p_{11}+p_7-p_{10}-\lambda_3)t} - 1)}{2(p_{11} + p_7 - p_{10} - \lambda_3)} + \right. \\
 & \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_5 p_8 (e^{(p_{11}+p_7-p_{10}-p_6)t} - 1)}{2(p_{11} + p_7 - p_{10} - p_6)} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_4 p_8 (e^{(p_9-\lambda_3)t} - 1)}{2(p_9 - \lambda_3)} \\
 & + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_5 p_8 (e^{(p_9-p_6)t} - 1)}{2(p_9 - p_6)} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_2 p_8^2 (e^{(p_{11}+p_9-p_{10}-\lambda_3)t} - 1)}{2(p_{11} + p_9 - p_{10} - \lambda_3)} - \\
 & \left. \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_3 p_8^2 (e^{(p_{11}+p_9-p_{10}-p_6)t} - 1)}{2(p_{11} + p_9 - p_{10} - p_6)} \right) \\
 & + \frac{2b_3 \alpha_6 e^{p_{10}t}}{R_b} \left(p_2 t + \frac{p_3(e^{-\lambda_3 t} - 1)}{\lambda_3} - \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_4 p_8 (e^{-(p_1+\lambda_3)t} - 1)}{2(p_1 + \lambda_3)} - \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_5 p_8 (e^{-(p_1+p_6)t} - 1)}{2(p_1 + p_6)} \right. \\
 & + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_4 p_8 (e^{(p_{11}-p_{10}-\lambda_3)t} - 1)}{2(p_{11} - p_{10} - \lambda_3)} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_5 p_8 (e^{(p_{11}-p_{10}-p_6)t} - 1)}{2(p_{11} - p_{10} - p_6)} \\
 & \left. + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_2 p_8^2 (e^{(p_{11}-p_{10}-p_1)t} - 1)}{2(p_{11} - p_{10} - p_1)} - \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_3 p_8^2 (e^{(p_{11}-p_{10}-p_1-\lambda_3)t} - 1)}{2(p_{11} - p_{10} - p_1 - \lambda_3)} \right) \\
 & - \frac{2b_7(1 - e^{p_{10}t})}{R_b p_{10}} + \frac{2n_4 e^{p_{10}t}}{3} \\
 L_n(t) &= -\frac{2f_0 \alpha_1 e^{p_{11}t}}{R_b D_b^3 (n\pi)^2} \left(\frac{e^{(p_{10}-p_{11})t}}{p_{10} - p_{11}} + \sum_{n=1}^{\infty} p_8 t \right) - \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{8(n\pi)^2 (2n-1) q_0 p_8 (e^{(p_{11}-q_1)t} - e^{p_{11}t})}{2R_b \pi (12n^2 + 4n - 1) q_1} \\
 & - \frac{D_1 b_4 e^{p_{11}t}}{R_b} \left(\sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_3 p_8 (n\pi)^2 (e^{-\lambda_3 t} - 1)}{\lambda_3} + \sum_{n=1}^{\infty} (n\pi)^2 p_2 p_8 t \right) + \frac{2f_0 e^{p_{10}t}}{D_b^3 (n\pi)^2} \left(\sum_{n=1}^{\infty} p_8 p_{11} t + \frac{p_{10} (e^{(p_{10}-p_{11})t} - 1)}{p_{10} - p_{11}} \right) \\
 & + \frac{e_1 e^{p_{11}t}}{R_b} \left(\sum_{n=1}^{\infty} p_2 p_8 t + \sum_{n=1}^{\infty} \frac{p_3 p_8 (e^{-\lambda_3 t} - 1)}{\lambda_3} + \sum_{n=1}^{\infty} \frac{p_4 (e^{(p_{10}-p_{11}-\lambda_3)t} - 1)}{(p_{10} - p_{11} - \lambda_3)} + \sum_{n=1}^{\infty} \frac{p_5 (e^{(p_{10}-p_{11}-p_6)t} - 1)}{(p_{10} - p_{11} - p_6)} \right) \\
 & + \frac{2e_2 e^{p_{11}t}}{R_b} \left(\sum_{n=1}^{\infty} \frac{p_2 p_8 (e^{p_7 t} - 1)}{p_7} - \sum_{n=1}^{\infty} \frac{p_3 p_8 (e^{(p_{10}-\lambda_3)t} - 1)}{p_{10} - \lambda_3} + \sum_{n=1}^{\infty} \frac{p_4 (e^{(2p_{10}-p_{11}-\lambda_3)t} - 1)}{2(2p_{10} - p_{11} - \lambda_3)} + \right. \\
 & \sum_{n=1}^{\infty} \frac{p_4 (e^{(2p_{10}-p_{11}-\lambda_3)t} - 1)}{2(2p_{10} - p_{11} - \lambda_3)} + \sum_{n=1}^{\infty} \frac{p_5 (e^{(2p_{10}-p_{11}-p_6)t} - 1)}{2(2p_{10} - p_{11} - p_6)} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{3p_4 p_8^2 (e^{(p_{11}-\lambda_3)t} - 1)}{8(p_{11} - \lambda_3)} \\
 & \left. + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{3p_5 p_8^2 (e^{(p_{11}-p_6)t} - 1)}{8(p_{11} - p_6)} \right) \\
 & + \frac{2e_3 e^{p_{11}t}}{R_b} \left(\sum_{n=1}^{\infty} \frac{p_2 p_8 (e^{p_7 t} - 1)}{2p_7} - \sum_{n=1}^{\infty} \frac{p_3 p_8 (e^{(p_7-\lambda_3)t} - 1)}{2(p_7 - \lambda_3)} + \sum_{n=1}^{\infty} \frac{p_4 (e^{(p_{10}+p_7-p_{11}-\lambda_3)t} - 1)}{2(p_{10} + p_7 - p_{11} - \lambda_3)} \right. \\
 & + \sum_{n=1}^{\infty} \frac{p_5 (e^{(p_{10}+p_7-p_{11}-p_6)t} - 1)}{2(p_{10} + p_7 - p_{11} - p_6)} + \sum_{n=1}^{\infty} \frac{p_2 p_8 (e^{(p_{10}+p_9-p_6)t} - 1)}{2(p_{10} + p_9 - p_6)} - \\
 & \left. \sum_{n=1}^{\infty} \frac{p_3 p_8 (e^{(p_{10}+p_9-p_6-\lambda_3)t} - 1)}{2(p_{10} + p_9 - p_6 - \lambda_3)} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{3p_4 p_8^2 (e^{(p_9-\lambda_3)t} - 1)}{8(p_9 - \lambda_3)} \right) \\
 & + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{3p_5 p_8^2 (e^{(p_9-p_6)t} - 1)}{8(p_9 - p_6)} \left. \right)
 \end{aligned}$$

$$\begin{aligned}
 & \left. \frac{2b_5\alpha_6 e^{p_{11}t}}{R_b} \left(\sum_{n=1}^{\infty} \frac{p_4(e^{(p_{10}-p_{11}-\lambda_3)t} - 1)}{2(p_{10}-p_{11}-\lambda_3)} + \sum_{n=1}^{\infty} \frac{p_5(e^{(p_{10}-p_{11}-p_6)t} - 1)}{2(p_{10}-p_{11}-p_6)} - \sum_{n=1}^{\infty} \frac{p_3 p_8(e^{(p_1-p_{11}-\lambda_3)t} - 1)}{2(p_1-p_{11}-\lambda_3)} \right) \right. \\
 & \left. + \sum_{n=1}^{\infty} \frac{p_2 p_8(e^{(p_{10}-p_{11}-p_1)t} - 1)}{2(p_{10}-p_{11}-p_1)} - \sum_{n=1}^{\infty} \frac{p_3 p_8(e^{(p_{10}-p_{11}-p_1-\lambda_3)t} - 1)}{2(p_{10}-p_{11}-p_1-\lambda_3)} + \sum_{n=1}^{\infty} \frac{p_2 p_8 t}{2} \right. \\
 & \left. - \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{3p_4 p_8^2(e^{-(p_1+\lambda_3)t} - 1)}{8(p_1+\lambda_3)} - \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{3p_5 p_8^2(e^{-(p_1+p_6)t} - 1)}{8(p_1+p_6)} \right) \\
 & - \sum_{n=1}^{\infty} \frac{b_7 p_8(e^{-(p_1+2p_{11})t} - e^{p_{11}t})}{R_b(p_{11}+p_1)} - \frac{2n_4 e^{p_{11}t}}{(n\pi)^2} \\
 N_o(t) = & - \frac{2f_0 D_3}{R_c D_c^*} \left(p_2 t + \frac{p_3(e^{-\lambda_3 t} - 1)}{\lambda_3} - \sum_{n=1}^{\infty} \left(\frac{p_4(e^{-\lambda_3 t} - 1)}{\lambda_3} + \frac{p_5(e^{-p_6 t} - 1)}{p_6} \right) \right) - \\
 & \frac{2f_0}{3D_c^*} \left(-p_3(e^{-\lambda_3 t} - 1) + \sum_{n=1}^{\infty} (p_4(e^{-\lambda_3 t} - 1) + p_5(e^{-p_6 t} - 1)) \right) - \\
 & \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{8n(n\pi)q_0(2n-1)}{R_o(4n-1)} \left(\frac{p_4(e^{-(p_1+q_1)t} - 1)}{2(q_1+\lambda_3)} + \frac{p_5(e^{-(q_1+p_6)t} - 1)}{2(q_1+p_6)} \right) + \frac{2n_2}{3} - \\
 & \frac{2b_{12}}{R_c} \left(\frac{p_2(e^{p_{10}t} - 1)}{p_{10}} - \frac{p_3(e^{(p_{10}-\lambda_3)t} - 1)}{p_{10}-\lambda_3} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{pp_4(e^{(p_{11}-\lambda_3)t} - 1)}{2(p_{11}-\lambda_3)} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{pp_5(e^{(p_{11}-p_6)t} - 1)}{2(p_{11}-p_6)} \right) - \\
 & \frac{2b_{13}}{R_c} \left(\frac{p_2(e^{p_7 t} - 1)}{p_7} - \frac{p_3(e^{(p_7-\lambda_3)t} - 1)}{p_7-\lambda_3} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{pp_4(e^{(p_9-\lambda_3)t} - 1)}{2(p_9-\lambda_3)} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{pp_5(e^{(p_9-p_6)t} - 1)}{2(p_9-p_6)} \right) \\
 N_n(t) = & - \frac{2f_0 e^{-p_6 t}}{(n\pi)^2 D_c} \left(\frac{p_3 \lambda_3 (e^{(p_6-\lambda_3)t} - 1)}{(p_6-\lambda_3)} - \sum_{n=1}^{\infty} \left(\frac{p_3 \lambda_3 (e^{(p_6-\lambda_3)t} - 1)}{(p_6-\lambda_3)} + p_3 p_6 t \right) \right) + \\
 & \frac{D_3 b_{11}}{R_c} \left(\sum_{n=1}^{\infty} (n\pi)^2 e^{-p_6 t} \left(\frac{p_3 p_4 (e^{(p_6-2\lambda_3)t} - 1)}{(p_6-2\lambda_3)} - \frac{p_3 p_5 (e^{-\lambda_3 t} - 1)}{\lambda_3} - \frac{p_2 p_4 (e^{(p_6-\lambda_3)t} - 1)}{(p_6-\lambda_3)} - p_2 p_5 t \right) \right) - \\
 & \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{8n(n\pi)q_0(2n-1)e^{-p_6 t}}{R_c(12n^2+4n-1)} \left(\frac{p_5(e^{-q_1 t} - 1)}{2q_1} - \frac{p_4(e^{(p_6-\lambda_3-q_1)t} - 1)}{2(p_6-\lambda_3-q_1)} \right) - \frac{2n_2 e^{-p_6 t}}{(n\pi)^2} - \\
 & \frac{b_{12} e^{-p_6 t}}{R_c} \left(\sum_{n=1}^{\infty} \frac{pp_2(e^{(p_6+p_1)t} - 1)}{p_6-p_{11}} - \sum_{n=1}^{\infty} \frac{pp_3(e^{(p_6+p_{11}-\lambda_3)t} - 1)}{(p_6+p_{11}-\lambda_3)} + \sum_{n=1}^{\infty} \frac{p_4(e^{(p_{10}+p_6-\lambda_3)t} - 1)}{(p_{10}+p_6-\lambda_3)} + \sum_{n=1}^{\infty} \frac{p_5(e^{p_{10}t} - 1)}{p_{10}} \right) - \\
 & \frac{b_{13} e^{-p_6 t}}{R_c} \left(\sum_{n=1}^{\infty} \frac{pp_2(e^{(p_6+p_9)t} - 1)}{p_6+p_2} - \sum_{n=1}^{\infty} \frac{pp_3(e^{(p_6+p_9-\lambda_3)t} - 1)}{(p_6+p_9-\lambda_3)} + \sum_{n=1}^{\infty} \frac{p_4(e^{(p_6+p_7-\lambda_3)t} - 1)}{(p_6+p_7-\lambda_3)} + \sum_{n=1}^{\infty} \frac{p_5(e^{p_7 t} - 1)}{p_7} \right) \\
 Q_o(t) = & e^{p_7 t} \left(\frac{2f_0 r_0}{3R_v D_v^*} - \frac{2f_0 D_2}{R_v D_v^*} \right) \left(t + \sum_{n=1}^{\infty} \frac{p_8(e^{(p_9-p_7)t} - 1)}{p_9-p_7} \right) - \frac{2f_0 e^{p_7 t}}{3D_v^*} \left(p_7 t + \sum_{n=1}^{\infty} \frac{p_8 p_9(e^{(p_9-p_7)t} - 1)}{p_9-p_7} \right) \\
 & + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{8(n\pi)^2(2n-1)q_0 p_8(e^{(p_9-q_1)t} - e^{p_7 t})}{2R_v \pi(2n-1)(p_9-p_7-q_1)} \\
 & + \frac{2e_4 e^{p_7 t}}{R_v} \left(p_2 t + \frac{p_3(e^{-\lambda_3 t} - 1)}{\lambda_3} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{pp_4(e^{(p_9-p_7-\lambda_3)t} - 1)}{2(p_9-p_7-\lambda_3)} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{pp_5(e^{(p_9-p_7-p_6)t} - 1)}{2(p_9-p_7-p_6)} \right) \\
 & + \frac{2e_5 e^{p_7 t}}{R_v} \left(\frac{p_2(e^{p_{10}t} - 1)}{p_{10}} - \frac{p_3(e^{(p_{10}-\lambda_3)t} - 1)}{p_{10}-\lambda_3} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_4 p_8(e^{(p_{10}+p_9-p_7-\lambda_3)t} - 1)}{2(p_{10}+p_9-p_7-\lambda_3)} + \right. \\
 & \left. \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_5 p_8(e^{(p_{10}+p_9-p_7-p_6)t} - 1)}{2(p_{10}+p_9-p_7-p_6)} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_4 p_8(e^{(p_{11}-\lambda_3)t} - 1)}{2(p_{11}-\lambda_3)} \right. \\
 & \left. + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_5 p_8(e^{(p_{11}-p_6)t} - 1)}{2(p_{11}-p_6)} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_2 p_8^2(e^{(p_{11}+p_9-p_7)t} - 1)}{2(p_{11}+p_9-p_7)} + \right. \\
 & \left. - \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_3 p_8^2(e^{(p_{11}+p_9-p_7-\lambda_3)t} - 1)}{2(p_{11}+p_9-p_7-\lambda_3)} \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{2e_6 e^{p_9 t}}{R_v} \left(\sum_{n=1}^{\infty} \frac{p_2 p_8 (e^{p_7 t} - 1)}{p_7} - \sum_{n=1}^{\infty} \frac{p_3 p_8 (e^{(p_7 - \lambda_3)t} - 1)}{p_7 - \lambda_3} + \sum_{n=1}^{\infty} \frac{p_4 (e^{(2p_7 - p_9 - \lambda_3)t} - 1)}{2(2p_7 - p_9 - \lambda_3)} \right. \\
 & \left. + \sum_{n=1}^{\infty} \frac{p_5 (e^{(2p_7 - p_9 - p_6)t} - 1)}{2(2p_7 - p_9 - p_6)} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{3p_4 p_8^2 (e^{(p_9 - \lambda_3)t} - 1)}{8(p_9 - \lambda_3)} \right. \\
 & \left. + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{3p_5 p_8^2 (e^{(p_9 - p_6)t} - 1)}{8(p_9 - p_6)} \right) \\
 & + \frac{2b_9 \alpha_6 e^{p_9 t}}{R_v} \left(\sum_{n=1}^{\infty} \frac{p_4 (e^{(p_7 - p_9 - \lambda_3)t} - 1)}{2(p_7 - p_9 - \lambda_3)} + \sum_{n=1}^{\infty} \frac{p_5 (e^{(p_7 - p_9 - p_6)t} - 1)}{2(p_7 - p_9 - p_6)} + \sum_{n=1}^{\infty} \frac{p_2 p_8 t}{2} + \sum_{n=1}^{\infty} \frac{p_3 p_8 (e^{-\lambda_3 t} - 1)}{2\lambda_3} \right. \\
 & \left. + \sum_{n=1}^{\infty} \frac{p_2 p_8 (e^{(p_7 - p_9 - p_1)t} - 1)}{2(p_7 - p_9 - p_1)} - \sum_{n=1}^{\infty} \frac{p_3 p_8 (e^{(p_7 - p_9 - p_1 - \lambda_3)t} - 1)}{2(p_7 - p_9 - p_1 - \lambda_3)} \right. \\
 & \left. - \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{3p_4 p_8^2 (e^{-(p_1 + \lambda_3)t} - 1)}{8(p_1 + \lambda_3)} - \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{3p_5 p_8^2 (e^{-(p_1 + p_6)t} - 1)}{8(p_1 + p_6)} \right) \\
 & - \frac{2n_3 e^{p_9 t}}{(n\pi)^2} \\
 Q_n(t) = & - \frac{2f_0 r_0 e^{p_9 t}}{R_v D_v^* (n\pi)^2} \left(\frac{e^{(p_9 - p_7)t}}{p_9 - p_7} + \sum_{n=1}^{\infty} p_8 t \right) + \frac{2f_0 e^{p_9 t}}{D_v^* (n\pi)^2} \left(\sum_{n=1}^{\infty} p_8 p_9 t + \frac{p_7 (e^{(p_7 - p_9)t} - 1)}{p_7 - p_9} \right) \\
 & + \frac{D_2 b_8 e^{p_9 t}}{R_v} \left(- \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_3 p_8 (n\pi)^2 (e^{-\lambda_3 t} - 1)}{\lambda_3} - \sum_{n=1}^{\infty} (n\pi)^2 p_2 p_8 t \right) \\
 & - \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{8(n\pi)^2 (2n-1) q_0 p_8 (e^{(p_9 - q_1)t} - e^{p_9 t})}{2R_v \pi (12n^2 + 4n - 1) q_1} \\
 & + \frac{e_4 e^{p_9 t}}{R_v} \left(\sum_{n=1}^{\infty} p_2 p t + \sum_{n=1}^{\infty} \frac{p p_3 (e^{-\lambda_3 t} - 1)}{\lambda_3} + \sum_{n=1}^{\infty} \frac{p_4 (e^{(p_7 - p_9 - \lambda_3)t} - 1)}{(p_7 - p_9 - \lambda_3)} + \sum_{n=1}^{\infty} \frac{p_5 (e^{(p_7 - p_9 - p_6)t} - 1)}{(p_7 - p_9 - p_6)} \right) \\
 & + \frac{2e_5 e^{p_9 t}}{R_v} \left(\frac{p_2 (e^{p_{10} t} - 1)}{p_{10}} - \frac{p_3 (e^{(p_{10} - \lambda_3)t} - 1)}{p_{10} - \lambda_3} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_4 p_8 (e^{(p_{10} + p_9 - p_7 - \lambda_3)t} - 1)}{2(p_{10} + p_9 - p_7 - \lambda_3)} + \right. \\
 & \left. \sum_{n=1}^{\infty} \frac{p_4 (e^{(p_{10} - p_9 + p_7 - \lambda_3)t} - 1)}{2(p_{10} - p_9 + p_7 - \lambda_3)} + \sum_{n=1}^{\infty} \frac{p_5 (e^{(p_{10} - p_9 + p_7 - p_6)t} - 1)}{2(p_{10} - p_9 + p_7 - p_6)} \right. \\
 & \left. + \sum_{n=1}^{\infty} \frac{p_2 p_8 (e^{p_{10} t} - 1)}{2p_{10}} - \sum_{n=1}^{\infty} \frac{p_3 p_8 (e^{(p_{10} - \lambda_3)t} - 1)}{2(p_{10} - \lambda_3)} + \sum_{n=1}^{\infty} \frac{p_2 p_8 (e^{(p_{11} + p_7 - p_9)t} - 1)}{2(p_{11} + p_7 - p_9)} \right. \\
 & \left. - \sum_{n=1}^{\infty} \frac{p_3 p_8 (e^{(p_{11} + p_7 - p_9 - \lambda_3)t} - 1)}{2(p_{11} + p_7 - p_9 - \lambda_3)} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_4 p_8^2 (e^{(p_{11} - \lambda_3)t} - 1)}{2(p_{11} - \lambda_3)} + \right. \\
 & \left. \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{3p_5 p_8^2 (e^{(p_{11} - p_6)t} - 1)}{8(p_{11} - p_6)} \right) \\
 & + \frac{2e_6 e^{p_9 t}}{R_v} \left(\sum_{n=1}^{\infty} \frac{p_2 p_8 (e^{p_7 t} - 1)}{p_7} - \sum_{n=1}^{\infty} \frac{p_3 p_8 (e^{(p_7 - \lambda_3)t} - 1)}{p_7 - \lambda_3} + \sum_{n=1}^{\infty} \frac{p_4 (e^{(2p_7 - p_9 - \lambda_3)t} - 1)}{2(2p_7 - p_9 - \lambda_3)} \right. \\
 & \left. + \sum_{n=1}^{\infty} \frac{p_5 (e^{(2p_7 - p_9 - p_6)t} - 1)}{2(2p_7 - p_9 - p_6)} + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{3p_4 p_8^2 (e^{(p_9 - \lambda_3)t} - 1)}{8(p_9 - \lambda_3)} \right. \\
 & \left. + \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{3p_5 p_8^2 (e^{(p_9 - p_6)t} - 1)}{8(p_9 - p_6)} \right) \\
 & + \frac{2b_9 \alpha_6 e^{p_9 t}}{R_v} \left(\sum_{n=1}^{\infty} \frac{p_4 (e^{(p_7 - p_9 - \lambda_3)t} - 1)}{2(p_7 - p_9 - \lambda_3)} + \sum_{n=1}^{\infty} \frac{p_5 (e^{(p_7 - p_9 - p_6)t} - 1)}{2(p_7 - p_9 - p_6)} + \sum_{n=1}^{\infty} \frac{p_2 p_8 t}{2} + \sum_{n=1}^{\infty} \frac{p_3 p_8 (e^{-\lambda_3 t} - 1)}{2\lambda_3} \right. \\
 & \left. + \sum_{n=1}^{\infty} \frac{p_2 p_8 (e^{(p_7 - p_9 - p_1)t} - 1)}{2(p_7 - p_9 - p_1)} - \sum_{n=1}^{\infty} \frac{p_3 p_8 (e^{(p_7 - p_9 - p_1 - \lambda_3)t} - 1)}{2(p_7 - p_9 - p_1 - \lambda_3)} \right. \\
 & \left. - \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{3p_4 p_8^2 (e^{-(p_1 + \lambda_3)t} - 1)}{8(p_1 + \lambda_3)} - \sum_{n=1}^{\infty} \sum_{n=1}^{\infty} \frac{3p_5 p_8^2 (e^{-(p_1 + p_6)t} - 1)}{8(p_1 + p_6)} \right) - \frac{2n_3 e^{p_9 t}}{(n\pi)^2}
 \end{aligned}$$

$$b_2 = \frac{\lambda_1}{\alpha_0}, \quad b_3 = \frac{\lambda_2}{\alpha_0}, \quad b_4 = \frac{\sigma_1}{\alpha_0}, \quad b_5 = \frac{\alpha_4}{\alpha_0}, \quad b_6 = \frac{\alpha_9}{\alpha_0}, \quad b_7 = \frac{\alpha_3}{\alpha_0} b_8 = \frac{\gamma_1}{\alpha_0}, \quad b_9 = \frac{r_1}{\alpha_0},$$

$$b_{10} = \frac{r_2}{\alpha_0}, \quad b_{11} = \frac{V_1}{\alpha_0}, \quad b_{12} = \frac{m_1}{\alpha_0}, \quad b_{13} = \frac{m_2}{\alpha_0}, \quad b_{15} = \frac{\mu_1}{\alpha_0}, \quad b_{16} = \frac{g_1}{\alpha_0}, \quad f_0 = \frac{1}{\alpha_0}, \quad \lambda_1 = \frac{1}{\beta_p \lambda_c},$$

$$m_3 = \frac{\alpha_s}{\beta_p}, \quad e_1 = b_5 \alpha_5 + b_6 \alpha_5, \quad e_2 = b_5 \alpha_7 + b_6 \beta_1, \quad e_3 = b_5 \alpha_8 + b_6 \beta_2, \quad e_4 = b_9 \alpha_5 + b_{10} \alpha_5,$$

$$e_5 = b_9 \alpha_7 + b_{10} \beta_1, \quad e_6 = b_9 \alpha_8 + b_{10} \beta_2,$$

$$p = p_8 = \frac{2((-1)^n - 1)}{(n\pi)^2}, \quad p_1 = \frac{D_4(n\pi)^2}{R_o}, \quad p_2 = 1 + \frac{2m_3}{\lambda_3 R_c}, \quad p_3 = \frac{2m_3}{\lambda_3 R_c}, \quad p_4 = \frac{2m_3((-1)^n - 1)}{R_c(n\pi)^2 \left(\frac{D_3(n\pi)^2}{R_c} - \lambda_3 \right)},$$

$$p_5 = \frac{2((-1)^n - 1)}{(n\pi)^2} \left(1 - \frac{m_3}{R_c \left(\frac{D_3(n\pi)^2}{R_c} - \lambda_3 \right)} \right), \quad p_6 = \frac{D_3(n\pi)^2}{R_c}, \quad p_7 = \frac{r_0}{R_v}, \quad p_9 = \frac{(r_0 - D_2(n\pi)^2)}{R_v}, \quad p_{10} = \frac{\alpha_1}{R_b},$$

$$p_{11} = \frac{(\alpha_1 - D_1(n\pi)^2)}{R_b}, \quad p_{12} = \left(\frac{(2n-1)\pi}{2} \right)^2, \quad n_1 = -\frac{f_0 \left(1 + \sum_{n=1}^{\infty} p \right)}{D_o^*}, \quad n_2 = -\frac{f_0 \left(p_2 - p_3 + \sum_{n=1}^{\infty} (p_4 + p_5) \right)}{D_c^*},$$

$$n_3 = -\frac{f_0 \left(1 + \sum_{n=1}^{\infty} p_8 \right)}{D_v^*}, \quad n_4 = -\frac{f_0 \left(1 + \sum_{n=1}^{\infty} p_8 \right)}{D_b^*}$$

RESULTS AND DISCUSSION

In this section, the governing equations (9) - (14) were solved analytically using parameter expanding method and Eigen functions expansions techniques, the solutions are discussed with the help of input data. From Figures 2 and 3 we observed that the hydraulic head h increases along distance and decreases with time, but increases with increase in pumping rate β_p .

Figure 4 shows a sinusoidal form along distance and later decrease, but increases with increase in Pumping rate β_p . In a similar manner in Figures 5 and 6 the pumping rate increases the concentration of contaminant θ along distance and later decreases.

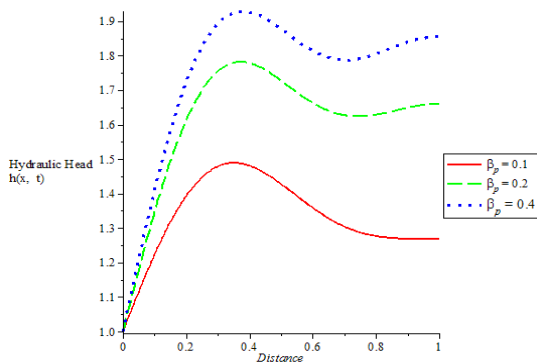


Fig 2: Relation between Hydraulic head against distance at various values of pumping rate.

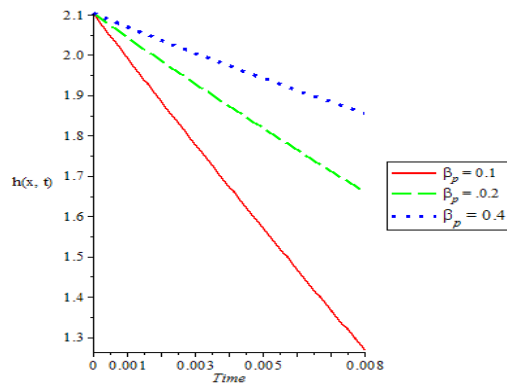


Fig 3: Relation between Hydraulic head against time at various values of pumping rate

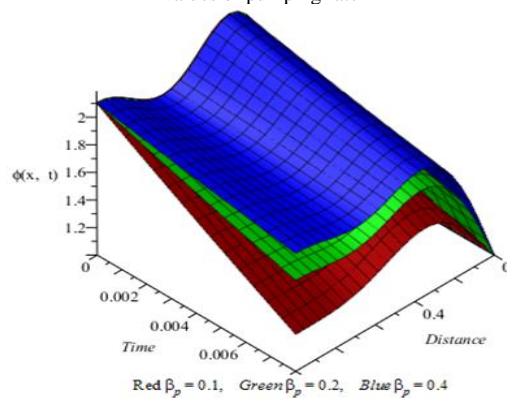


Fig 4: Relationships among bacteria concentration, distance and time at various values of pumping rate

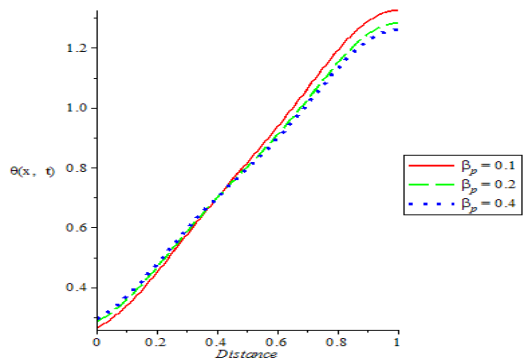


Fig 5: Relation between contaminant concentrations against distance at various values of pumping rate

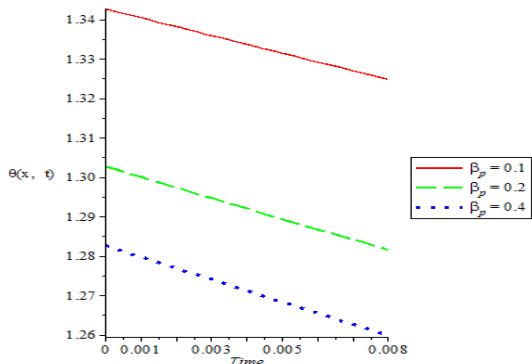


Fig 6: Relation between contaminant concentrations against time at various values of pumping rate

Conclusion: We developed unsteady one - dimensional analytical model that incorporates variable dependent pumping rates on a contaminant transport in riverbank filtration system. Based on the above results we can conclude that: Pumping rate values increases both the hydraulic head and concentration of colloids. The quality of pumped water from RBF will slightly reduce due to the increase in pumping rate. The bacteria concentrations of the pumped water will be high if the pumping well is closed to the river.

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