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An Implicit Collocation Method for Direct Solution of Fourth Order Ordinary Differential Equations

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ABSTRACT: This paper presented a linear multistep method for solving fourth order initial value problems of ordinary differential equations. Collocation and interpolation methods are adopted in the derivation of the new numerical scheme which is further applied to finding direct solution of fourth order ordinary differentiation equations. This implementation strategy is more accurate and efficient than Adams–Bashforth Method solution. The newly derive scheme have better stabilities properties than that of the Adams-Bashforth Method. Numerical examples are included to illustrate the reliability and accuracy of the new methods.

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Higher order (linear and non –linear) ordinary differential equations of the form as presented in equation 1 are often encountered by scientists and engineers .The solutions of such equations have engaged the attention of many applied mathematicians, both the theorists and numerical analysts. Many of such empirical results yielding higher order differential equations are not solvable analytically. The older numerical methods adopted for such higher order differential equations are only capable of handling first order equations of the type as in equation 2:

$$y^{m} = f(t, y, y', y'', \dots, y^{m-1}), \quad y^{m-1}(t_0) = \mu_{m-1}, \qquad m = 1, 2, \quad (1)$$

$$y' = f(t, y), \quad y(t_0) = \mu, \qquad f \in C[a, b] \times \Re^{m} \qquad (2)$$

This implies that such problems will be reduced to system of first order equations. The approach of reducing such equations to a system of first order equations leads to serious computational steps that seemingly a vicious circle in the computer age. Eminent scholars have contributed significantly in their works in this area of research to solving problem (1) using different numerical methods, scholars viz: (Lambart, 1973); (Jacques and Judd, 1987); (Adee et al; 2005); (Awoyemi, 2005); (Kayode and Awoyemi, 2005); (Awoyemi and Idowul, 2005), (Fatunla, 1988); (Kayode, 2008a); (Jator, 2007); Owolabi et al. (2010). Attempts have been made by some researchers to solve directly problem (1) for m = 4 by developing methods of step number k = 4 with varying order of accuracy, Kayode (2008b). But none of these could handle problem (1) directly when m > 4 without reducing it to a system of lower other problems. However, researches keep improving on the direct solution for solving ordinary differential equations (ODEs) using different approaches. Awoyemi and Kayode (2010); adopted a zero – stable optimal order method for direct solution of second order differential equation. Method for solving special equations of problem (1) directly without the first derivative of the form;

$$y^{(m)} = f(t, y), \; m = 2 \qquad (3)$$

The equation (3) has been considered by (Awoyemi and Kayode, 2002) and (Badmus and Yahaya, 2009). In this article, problem (1) is directly solved by developing a 4 – step derivative for m = 4. This section presents derivation of the new method, applications of order of accuracy and error constants of the new discrete scheme, and region of absolute stabilities of the new scheme.

Derivation of the Method: The proposed numerical method for direct solution of general fourth order differential equations is of the form of a continuous linear multistep

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MATERIALS AND METHODS

$y_k(x) = \sum_{j=o}^{k-1} \propto_j(x) y_{n+j} + h^4 \sum_{j=2}^{k} \beta_j(x) f_{n+j}$ (4)

Let the approximate solution y(x) to Problem (1) be taken to be a partial sum of a power series $\varphi_j(x)$ of a single variable x in the form

$$\varphi_j(x) = \sum_{j=o}^{2(k-1)} a_j x^j \quad (5)$$

where $a_j's, j = 0, 1, \cdots 2(k-1)$ are real coefficients.

The first, second, third and fourth derivative of Equation (5) are given as follows

$$\varphi^{(1)}{}_j(x) = \sum_{j=1}^{2(k-1)} j a_j x^{j-1} \quad (6)$$

$$\varphi^{(2)}{}_j(x) = \sum_{j=2}^{2(k-1)} j(j-1) a_j x^{j-2} \quad (7)$$

$$\varphi^{(3)}{}_j(x) = \sum_{j=3}^{2(k-1)} j(j-1)(j-2) a_j x^{j-3} \quad (8)$$

$$\varphi^{(4)}{}_j(x) = \sum_{j=4}^{2(k-1)} j(j-1)(j-2)(j-3) a_j x^{j-4} \quad (9)$$

$$\varphi^{(4)}{}_j(x) = \sum_{j=4}^{2(k-1)} j(j-1)(j-2)(j-3) a_j x^{j-4} = f(x, y', y'', y''') \quad (10)$$

Equation (10) was collocated at some selected grid points $x = x_{n+i}, i = 1, 3, \cdots, k$ while equation (5) was interpolated at grid points $x = x_{n+i}, i = 0, 1, \cdots k-1$ to have the system of linear equations.

$$\sum_{j=4}^{2(k-1)} j(j-1)(j-2)(j-3)\, a_j \Phi_{j-4}(x_{n+i}) = f_{n+i}, \quad i = 1, 3, \cdots k \quad (11)$$

$$\sum_{j=0}^{2(k-1)} a_j\, \Phi_j(x_{n+i}) = y_{n+i}, \quad i = 0, 1, 2, \cdots k-1 \quad (12)$$

Where $f_{n+i} = f(x_{n+i}, y_{n+i}, y'_{n+i}, y''_{n+i}, y'''_{n+i}), \quad i = 0, 1, 2, \cdots k; \quad y_{n+i} = y(x_{n+1})$ and $y^{(r)} = \frac{d^r y}{dx^r}$

Solving the system in Equation (11) and (12) for $a_j's$, where $j = 0, 1, \cdots, k+2$, yields a system of non – linear equation of the form:

AX= U (13)

$A = X^{-1}U$

Solving (13) for $a_j s$ using MATLAB and by letting $x = th + x_{n+4}$, where $t = \frac{x - x_{n+k-1}}{h}$ such that $t \in (0, 1]$ and substituting back into (5) gives a continuous multistep method in the form:

$$y_k(x) = \sum_{j=0}^{k-1} \propto_J(t) y_{n+j} + \sum_{j=0}^{k} \beta_j(t) f_{n+j} \quad (14)$$

The coefficient $\propto_j(t)$ and $\beta_j(t)$ are obtained as follows:

$$\propto_0(t) = \frac{1}{1680}(123t + 98t^2 - 126t^3 - 105t^4 - 7t^5 + 7t^6 + t^7)$$

$$\propto_1(t) = \frac{-1}{420}(272t + 308t^2 - 56t^3 - 105t^4 - 7t^5 + 7t^6 + t^7)$$

$$\propto_2(t) = \frac{1}{280}(552t + 658t^2 + 14t^3 - 105t^4 - 7t^5 + 7t^6 + t^7)$$

$$\propto_3(t) = \frac{-1}{420}(1392t + 1148t^2 + 84t^3 - 105t^4 - 7t^5 + 7t^6 + t^7)$$

$$\propto_4(t) = \frac{1}{1680}(1680 + 3212t + 1778t^2 + 154t^3 - 105t^4 - 7t^5 + 7t^6 + t^7)$$

$$\beta_1(t) = \frac{1}{20160h}(-240t + 284t^2 + 630t^3 + 420t^4 + 35t^5 - 28t^6 - 5t^7)$$

$\beta_3(t)=\frac{1}{10080h}$ (1824t + 3920t^2+ 2814t^3+ 735t^4- 14t^5- 35t^6-4t^7)
$\beta_5(t)=\frac{1}{20160h}$ (48t + 196t^2+ 294t^3+ 210t^4+ 77t^5+ 14t^6+t^7) (15)

The first, second, and third derivatives of equation (15) is given as follows:
$\alpha'_0(t) = \frac{1}{1680h}$ (132 + 196t - 378t^2- 420t^3- 35t^4+ 42t^5+ 7t^6)
$\alpha'_1(t) = \frac{-1}{420h}$ (272 + 616t - 168t^2- 420t^3- 35t^4+ 42t^5+ 7t^6)
$\alpha'_2(t)= \frac{1}{280h}$ (552 + 1316t + 42t^2- 420t^3- 35t^4+ 42t^5+ 7t^6)
$\alpha'_3(t)= \frac{-1}{420h}$ (1392 + 2296t + 252t^2- 420t^3- 35t^4+ 42t^5+ 7t^6)
$\alpha'_4(t) = \frac{1}{1680h}$ (3212 + 3556t + 462t^2- 420t^3- 35t^4+ 42t^5+ 7t^6)
$\beta'_1(t)= \frac{1}{20160h^2}$ (-240 + 568t +1890t^2+1680t^3+ 175t^4- 168t^5- 35t^6)
$\beta'_3(t)= \frac{1}{10080h^2}$ (1824 + 7820t +8442t^2+2940t^3- 70t^4- 210t^5- 28t^6)
$\beta'_5(t)= \frac{1}{20160h^2}$ (48 + 392t +882t^2+840t^3+385t^4+ 84t^5+ 7t^6) (16)
$\alpha''_0(t)= \frac{1}{1680h^2}$ (196 - 756t -1260t^2-140t^3+ 210t^4+42t^5)
$\alpha''_1(t) = \frac{-1}{420h^2}$ (616 - 336t -1260t^2-140t^3+ 210t^4+42t^5)
$\alpha''_2(t) = \frac{1}{280h^2}$ (1316 + 84t -1260t^2-140t^3+ 210t^4+42t^5)
$\alpha''_3\ (t) = \frac{-1}{420h^2}$ (2296 + 504t - 1260t^2-140t^3+ 210t^4+42t^5)
$\alpha''_4(t) = \frac{1}{1680h^2}$ (3556 + 924t - 1260t^2-140t^3+ 210t^4+42t^5)
$\beta''_1(t) = \frac{1}{20160h^3}$ (568 + 3780t +5040t^2+700t^3- 840t^4-210t^5)
$\beta''_3(t) = \frac{1}{10080h^3}$ (7820 + 16884t +8820t^2-280t^3- 1050t^4-168t^5)
$\beta''_5(t) = \frac{1}{20160h^3}$ (392 + 1764t + 2520t^2+1540t^3+ 420t^4+42t^5) (17)
$\alpha'''_0(t) = \frac{1}{1680h^3}$ (-756 - 2520t - 420t^2+840t^3+ 210t^4)
$\alpha'''_1(t) = \frac{-1}{420h^3}$ (-336 - 2520t - 420t^2+840t^3+ 210t^4)
$\alpha'''_2(t) = \frac{1}{280h^3}$ (84 - 2520t - 420t^2+840t^3+ 210t^4)
$\alpha'''_3(t)= \frac{-1}{420h^3}$ (5044 - 2520t - 420t^2+840t^3+ 210t^4)
$\alpha'''_4(t)= \frac{1}{1680h^3}$ (924 - 2520t - 420t^2+840t^3+ 210t^4)
$\beta'''_1(t)= \frac{1}{20160h^4}$ (3780 + 10080t +2100t^2- 3360t^3- 1050t^4)
$\beta'''_3(t)= \frac{1}{10080h^4}$ (16884 + 17640t - 2100t^2-3360t^3- 1050t^4)
$\beta'''_5(t) = \frac{1}{20160h^4}$ (1764 + 5040t +4620t^2+1680t^3+ 2100t^4) (18)

For any sample discrete scheme to be determined from the continuous linear multistep method; Equation (15) and its first, second and third derivatives arising from Equations (15) , (16), (17) and (18) are substituted in Equation (19) to (22) when t = 1, we obtain discrete scheme and its derivatives as follows:
Substituting $t = 1$ in Equations (19) to (22), we obtain discrete scheme as follows:

$y_{n+5} = 4y_{n+4} - 6y_{n+3} + 4y_{n+2} - y_{n+1} + \frac{h^4}{24}[f_{n+5} + 22f_{n+3} + f_{n+1}]$ (19)

$y_{n+5}^{(1)} = \frac{1}{210h}$ (853y_{n+4} − 1767y_{n+3} + 1128y_{n+2} − 157y_{n+1} − 57y_n +
$\frac{h^4}{48}$ [1319f_{n+5}+20738f_{n+3}+1679f_{n+1}]) (20)

$y_{n+5}^{(2)} = \frac{1}{60\,h^2}$ (119y_{n+4} − 236y_{n+3} + 54y_{n+2} +124y_{n+1} − 61y_n +
$\frac{h^4}{8}$ [159f_{n+5} +1526f_{n+3}+203f_{n+1}]) (21)

$y_{n+5}^{(3)} = \frac{1}{40h^3}$ (-23y_{n+4}+132y_{n+3}-258y_{n+2}+212y_{n+1}-63y_n +
$\frac{h^4}{12}$ ([317f_{n+5}+1364f_{n+3}+275f_{n+1}]) (22)

Application of Order of Accuracy and Error Constant of the New Discrete Schemes: The order of method (15) when; $t = 1$, we have equation (19) as:

$$y_{n+5} - 4y_{n+4} + 6y_{n+3} - 4y_{n+2} + y_{n+1} = \frac{h^4}{24}(f_{n+5} + 22f_{n+3} + f_{n+1})$$

With order $p = 6$ and error constant given by $c_{p+2} = \frac{-3360}{720}$ or $\frac{-14}{3}$, the discrete method (19) is consistent and zero stable. This satisfies the necessary and sufficient condition for the convergence of linear multistep methods and internal of absolute stability is $x\,(\theta) = (0{,}16)$ for $0 < x < 16$.

Region of Absolute Stability of the New Scheme (19)*:* To find the region of absolute stability we use the known boundary locus method. Consequently, we utilize the boundary locus curve, which is obtained by setting

$$\bar{h} = \frac{\rho(r)}{\delta(r)},\ r = e^{i\theta},\ \text{and}\ 0^0 \leq \theta \leq 180^0$$

The curve is normally symmetric about the real axis. The upper half is obtained for $0^0 \leq \theta \leq 180^0$ inclusive and a mirror image of the curve through the real axis completes the region of absolute stability.

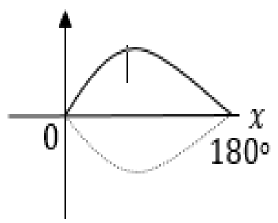

Table 1: Region of Absolute Stability for equation (19)

θ	0^0	30^0	60^0	90^0	120^0	150^0	180^0
$x(\theta)$	0	0.031	-1.561	4.800	3.765	-19.854	16.000
$y(\theta)$	0	0.048	2.704	0.000	0.725	34.380	0.000

RESULTS AND DISCUSSION

Two none linear numerical examples are solved to demonstrate the accuracy and convergence of the derived discrete method (19) and their results were compared with that of the existing Adams – Bashforth Method of the same order.

Problem 1: The system of equations $y^{iv} = x,\ y_0 = 0,\ y_0' = 1,\ y_0'' = 0, y_0''' = 0$ has

Theoretical solution as: $y_{(x)} = \frac{x^5}{120} + x,\ h = 0.1, 0 \leq x \leq 1$.

Problem 2: The system of equations

$$y^{(4)} = (y')^2 - yy''' - 4x^2 + e^x(1 - 4x + x^2), 0 \leq x \leq 1,$$

$y(0) = 1, y'(0) = 1,\ \ y''(0) = 3, y'''(0) = 1$ has theoretical solution as $y(x) = x^2 + e^x,\ h = 0.1$.

Hence the region of absolute stability of equation (19) is [0, 16].

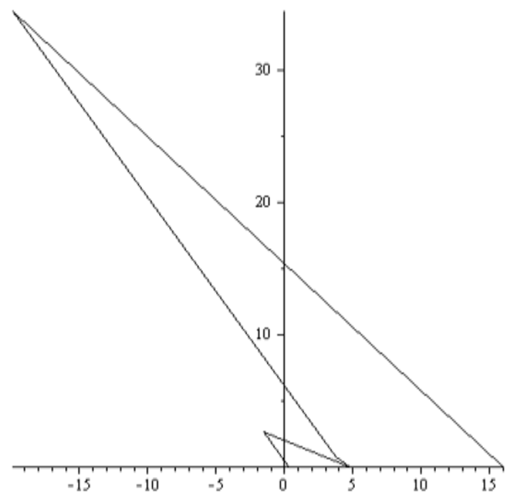

Fig 1. Graph of Absolute Stability of the Derived Method (19)

The accuracy of our method (19) was tested on fourth order numerical problems 1 and 2 and their results were compared with the existing Adams-Bashforth Method as shown in Tables 4, and 7 in order to ascertain the performance of our new numerical method which is in line with (Kayode, 2008); (Olabode, 2009); (Awoyemi and Kayode, 2005); (Fatokun, 2006); Taiwo and Ogunlaran (2008).

It is observed that solutions of the new scheme for problem one is closer to that of the exact solution of the same problem and the error is small and it can be neglected with order of absolute error 10^{-9}, solving problem 1 with the existing Adams- Bashforth Method. The result obtained is not consistent with the exact solutions and the error is large with order of absolute error 10^{-2}as shown in table 4.

Table 2: Solution to Problem 1 of Absolute Errors of the New Scheme

Theoretical/Approximate Solution		$h = 0.1$	
x	**Exact Solution**	**New Scheme Solution**	**Errors**
0.1	0.100000083	0.100000083	0.000000000
0.2	0.200002666	0.200002666	0.000000000
0.3	0.30002025	0.30002025	0.000000000
0.4	0.400085333	0.400085333	0.000000000
0.5	0.500260416	0.500260414	-2.0×10^{-09}
0.6	0.600648000	0.600647994	-6.0×10^{-09}
0.7	0.701400583	0.701400579	-4.0×10^{-09}
0.8	0.802730666	0.802730686	2.0×10^{-08}
0.9	0.904920750	0.904920852	1.02×10^{-07}
1.0	1.008333333	1.008333649	3.0×10^{-07}

Table 3: Solution to Problem 1 of Absolute Errors of Adams- Bashforth Method

Theoretical/Approximate Solution $h = 0.1$			
x	**Exact Solution**	**Adams- Bashforth Scheme**	**Errors**
0.1	0.100000083	0.100000083	0.000000000
0.2	0.200002666	0.200002666	0.000000000
0.3	0.30002025	0.30002025	0.000000000
0.4	0.400085333	0.400085333	0.000000000
0.5	0.500260416	0.445085333	-5.518×10^{-02}
0.6	0.600648000	0.500085333	-1.006×10^{-01}
0.7	0.701400583	0.565085333	-1.363×10^{-01}
0.8	0.802730666	0.640085333	-1.626×10^{-01}
0.9	0.904920750	0.725085333	-1.798×10^{-01}
1.0	1.008333333	0.889529777	-1.188×10^{-01}

Table 4: Comparison of Errors Arising from the New Method and Adams- Bashforth Method for Problem 1

x	New Scheme	Adams-Bashforth Scheme	Errors in New Scheme	Errors in Adams-Bassforth
0.1	0.100000083	0.100000083	0.000000000	0.000000000
0.2	0.200002666	0.200002666	0.000000000	0.000000000
0.3	0.30002025	0.30002025	0.000000000	0.000000000
0.4	0.400085333	0.400085333	0.000000000	0.000000000
0.5	0.500260414	0.445085333	-2.0×10^{-09}	-5.518×10^{-02}
0.6	0.600647994	0.500085333	-6.0×10^{-09}	-1.006×10^{-01}
0.7	0.701400579	0.565085333	-4.0×10^{-09}	-1.363×10^{-01}
0.8	0.802730686	0.6405333	2.0×10^{-08}	-1.626×10^{-01}
0.9	0.904920852	0.725085333	1.02×10^{-07}	-1.798×10^{-01}
1.0	1.008333649	0.889529777	3.0×10^{-07}	-1.188×10^{-01}

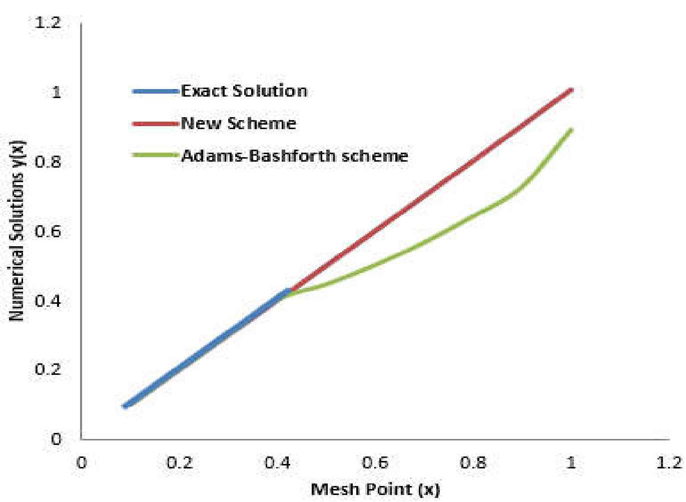

Fig 2. Graph of Exact Solution versus New Derived Scheme and Adams-Bashforth Method for Problem 1.

Table 5: Solution to Problem 2 of Absolute Errors of the New Scheme

Theoretical/Approximate Solution $h = 0.1$			
x	**Exact Solution**	**New Scheme**	**Errors**
0.1	1.115170918	1.115170919	0.000000000
0.2	1.261402758	1.261402758	0.000000000
0.3	1.439858808	1.439858808	0.000000000
0.4	1.651824698	1.651824698	0.000000000
0.5	1.898721271	1.898550000	-1.713×10^{-05}
0.6	2.182118800	2.181175092	-9.437×10^{-05}
0.7	2.503752707	2.500643637	-3.109×10^{-04}
0.8	2.865540928	2.857600094	-7.941×10^{-04}
0.9	3.269603111	3.252315271	-1.729×10^{-03}
1.0	3.718281828	3.684479444	-3.380×10^{-03}

Table 6: Solution to Problem 2 of Absolute Errors of Adams-Bashforth Method

Theoretical/Approximate Solution $h = 0.1$			
x	**Exact solution**	**Adams- Bashforth Method**	**Errors**
0.1	1.115170918	1.115170919	0.000000000
0.2	1.261402758	1.261402758	0.000000000
0.3	1.439858808	1.439858808	0.000000000
0.4	1.651824698	1.651824698	0.000000000
0.5	1.898721271	1.645447899	-2.533×10^{-01}
0.6	2.182118800	1.463264349	-7.189×10^{-01}
0.7	2.503752707	1.068008719	-1.436×10^{0}
0.8	2.865540928	0.537962589	-2.326×10^{0}
0.9	3.269603111	-0.144266840	-3.414×10^{0}
1.0	3.718281828	-0.996684237	-4.715×10^{0}

Table 7: Comparison of Errors Arising from the New Method and Adams- Bashforth Method for Problem 2

x	**New Scheme**	**Adams-Bashforth Scheme**	**Errors in New Scheme**	**Errors in Adams-Bassforth**
0.1	1.115170919	1.115170919	0.000000000	0.000000000
0.2	1.261402758	1.261402758	0.000000000	0.000000000
0.3	1.439858808	1.439858808	0.000000000	0.000000000
0.4	1.651824698	1.651824698	0.000000000	0.000000000
0.5	1.898550000	1.645447899	-1.713×10^{-05}	-2.533×10^{-01}
0.6	2.181175092	1.463264349	-9.437×10^{-05}	-7.189×10^{-01}
0.7	2.500643637	1.068008719	-3.109×10^{-04}	-1.436×10^{0}
0.8	2.857600094	0.537962589	-7.941×10^{-04}	-2.326×10^{0}
0.9	3.252315271	-0.144266840	-1.729×10^{-03}	-3.414×10^{0}
1.0	3.684479444	-0.996684237	-3.380×10^{-03}	-4.715×10^{0}

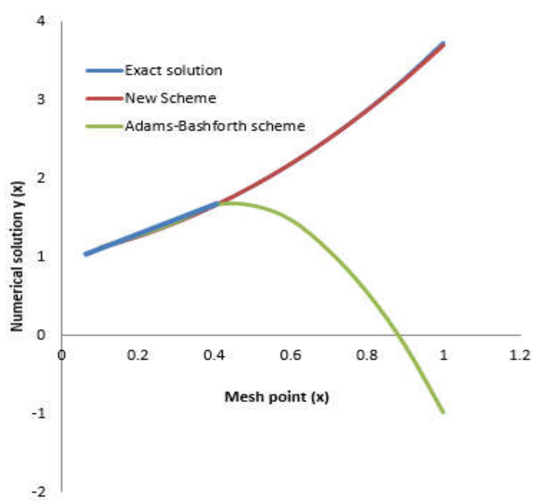

Fig 3: The Graph of Exact Solution Compared to the New Derived Scheme and Adams-Bashforth Method for Problem 2

It is also observed in table 5 that the solutions of problem 2 when solve using the newly derived scheme is consistent with the exact solutions of the same problem implies that they are the approximate solutions of the problem with order of absolute error10^{-5}, while the solutions of the existing method for same problem is not approaching the exact solutions with order of absolute error 10^{-1} which is too large as shown in table 7.

However, from the analysis above, the new scheme when solved with the two (ODE) problems gives a better results and is more accurate and efficient than the Adams- Bashforth method when apply to solved fourth order ordinary differential equations.

Conclusion: The order six method developed through collocation approach is capable of solving linear and non- linear general fourth order ordinary differential equations directly without reduction to system of first order equations. This reduced the computational burden and its inevitable effects on computer time. The new scheme (19) was tested on two numerical problems and the result obtained from the new scheme was compared with the result of the same problems using Adams-Bashforth Method. The zero stability property of the new scheme (19) serve as advantages over the existing method (Adams- Bashforth).

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