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Superiority of Legendre Polynomials to Chebyshev Polynomial in Solving Ordinary Differential Equation

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ABSTRACT: In this paper, we proved the superiority of Legendre polynomial to Chebyshev polynomial in solving first order ordinary differential equation with rational coefficient. We generated shifted polynomial of Chebyshev, Legendre and Canonical polynomials which deal with solving differential equation by first choosing Chebyshev polynomial $T^*_n(X)$, defined with the help of hypergeometric series $T^*_n(x) = F(-n, n, ½; X)$ and later choosing Legendre polynomial $P^*_n(x)$ define by the series $P^*_n(x) = F(-n, n+1, 1; X)$; with the help of an auxiliary set of Canonical polynomials Q_k in order to find the superiority between the two polynomials. Numerical examples are given which show the superiority of Legendre polynomials to Chebyshev polynomials. @JASEM

The so-called Canonical polynomials introduced by Lanczos(A) have hitherto been used in application to the Tau method for the solution of ordinary differential equation via Legendre polynomials and Chebyshev polynomials.

In this paper, we described how canonical polynomials can easily be constructed as basis to the solution of first order differential equations. From a computational point of view, the canonical polynomials are attractive, easily generated, using a simple recursive relation and its associated conditional of the given problem via Legendre and Chebyshev polynomials is of great importance.

The paper of Oritz(B) gives an account of the theory of the Tau method which it subsequently uses in the problems considered to illustrate the effectiveness and superiority of Legendre polynomials to Chebyshev polynomials.

THE METHOD USED

IN THIS SECTION, WE GENERATE CANONICAL POLYNOMIALS FOLLOWING LANCZOS(A), WE DEFINE CANONICAL POLYNOMIALS $Q_K(X)$; K = 0 WHICH ARE UNIQUELY ASSOCIATED WITH THE OPERATOR. CONSIDER A LINEAR DIFFERENTIAL EQUATION

$Y'-Y=0, Y(0)=1$ 1

GENERATING THE CANONICAL POLYNOMIALS,

$L=D/DX - I$

$LX^K = KX^{K-1} - X^K$

THUS,

$LX^K - KLQ_{K-1}(X) - LQ_K(X) - LQ_K(X)=0$

FROM THE LINEARITY OF L, AND THE EXISTENCE OFL- ', WE HAVE

$Q_K(X) + X^K = KQ_{K-1}(X)$

SINCE $DQ_K(X) = X^K$

FROM THE BOUNDARY CONDITION

$DY(X) = 0 \Rightarrow X^K = 0$

$Q_K(X) = KQ_{K-1}(X)$

IT FOLLOWS THAT

$Q_K(X) = K!S_K(X)$

FOR THE DIFFERENTIAL EQUATION CONSIDER

CHEBYSHEV POLYNOMIALS

WE RECALL SOME WELL-KNOWN PROPERTIES OF THE CHEBYSHEVPOLYNOMIALS:

$T_N^*(X) = F(-N,N,1/2,X)$ $T_N^*(X) = COS N(COS-1(X)$, $-1 = X = 1$ WHERE X = COS 0. TO EVALUATE THE FIRST FEW POLYNOMIALS, WE FOLLOW

$T_0(X) = T_0(COS 0) = 1$

$T_1(X) = T_1(COS 0) = X$

WE NOW MAKE USE OF THE RECURSIVE RELATION

$T_{N+1}(X) = 2XT_N(X) - T_{N-1}(X)$

TO GENERATE OTHERS FOR N=1,2,3,...

LEGENDRE POLYNOMIALS

LEGENDRE POLYNOMIALS $P_N^*(X)$, DEFINED BY THE HYPERGEOMETRIC SERIES

$P_N^*(X) = F(-N, N+1, 1:X) = F(\alpha,\beta,\delta: X)$

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$\Rightarrow P_N^*(X) = 1 + \frac{\alpha\beta X}{\delta} + \frac{\alpha(\alpha+1)\beta(\beta+1)X^2}{\delta(\delta+1)} + + \frac{\alpha(\alpha+1)(\alpha+2)\ldots+(\alpha+N)\beta(\beta+1)\ldots(\beta+N)X}{\delta(\delta+1)(\delta+2)\quad +\ldots+ \quad(\delta+N)}$

WHERE N=0

$P_0^*(X) = 1$

WHEN N=1 $\Rightarrow P_1^*(X) = 1-2X$

WHEN N=2 $\Rightarrow 1 - 6X + 12X^2$ ETC.

THE TAU METHOD

ORITZ (B) GIVES AN ACCOUNT OF THE THEORY OF TAU METHOD; SUCH IS APPLIED TO THE FOLLOWING BASIC PROBLEM.

$LY(X)=P_M(X)Y^{(M)}(X)+\ldots\ldots+P_0(X)Y(X) = F(X);$ $A=X=B$; $Y^M(X)$ STANDS FOR THE DERIVATIVE OF ORDER M OF Y(X) AND $Y(X)=Y_N(X)= \Sigma A X = \Sigma A Q(X)$

WHERE $Q_K(X)$ IS THE CANONICAL POLYNOMIAL. HERE, WE NEED A SMALL PERTURB TERM WHICH LEADS TO THE CHOICE OF CHEBYSHEV POLYNOMIALS WHICH OSCILLATES WITH EQUAL AMPLITUDE IN THE RANGE CONSIDERED.

$P_N(X) = \tau T_N^*(X)$ WHERE $T_N^*(X)$ IS THE SHIFTED CHEBYSHEV POLYNOMIAL WHICH ARE OFTEN USED WITH THE TAU METHOD AND

$T_N^*(X) = \sum_{K=0}^{N} C_K^N X^K$ WHERE C_K^N ARE COEFFICIENTS OF X^K. WE ASSUME HERE THAT A TRANSFORMATION HAS BEEN MADE SUCH THAT A=0 AND B=1 TO SIMPLIFY MATTER FURTHER IN ORDER TO GET THE SHIFTED CHEBYSHEV POLYNOMIAL I.E

$T_0^*(X)=1$, $T_1^*(X)=X=(1-2\theta)$, $T_2^*(X)=1-8\theta+8\theta^2=1-8X+8X^2$

RESULT AND DISCUSSION

CONSIDER THE DIFFERENTIAL EQUATION

$Y'-Y = 0,\ Y(0) = 1$ 1

WHICH DEFINES THE EXPONENTIAL FUNCTION.

$Y(X) = E^X = 1+X+X^2/2+X^3/3+..$2

WHICH CONVERGES IN THE ENTIRE COMPLEX PLAIN. IF WE TRUNCATE THE TAYLOR SERIES

$Y_N(X) = 1+X+X^2/2!+\ldots\ldots\ldots+X^N/N!+\ldots\ldots$3

THIS FUNCTION SATISFIES THE DIFFERENTIAL EQUATION

$Y'_N-Y_N = X^N/N!$ 4

SUPPOSE WE ARE SOLVING 1 IN THE RANGE OF (0,1).

NOW BY CHOSING CHEBYSHEV POLYNOMIALS $T_N^*(X)$ DEFINED WITH THE HELP OF THE HYPERGEOMETRIC SERIES $T_N^*(X) = F(-N, N, 1/2; X)$ AS THE ERROR TERM ON THE RIGHT HAND SIDE OF (1) WE THEREFORE SOLVE THE DIFFERNTIAL EQUATION

$Y'_N-Y_N = \delta T_N^*(X)$ 5

BY INTRODUCING CANONICAL POLYNOMIAL QK.QK(X) IS DEFINED BY

$Q'_K - Q_K(X) = X^K$

$\Rightarrow Q_K(X) = -K!S_K(X)$ 6

IF WE DENOTE ITS PARTIAL SUM OF THE FIRST K+1 TERMS OF THE TAYLOR SERIES BY SK(X) SUCH THAT

$S_K(X) = 1+X+X^2/2!+\ldots\ldots.+X^N/N!\ldots$ 7

WRITING OUT POLYNOMIALS T*N(X) EXPLICITLY AS

$T_N^*(X) = C_N^0+C_N^1X+C_N^2X^2+\ldots\ldots\ldots+C_N^NX^N = \sum_{K=0}^{N} C_K^NX^K$ 8

BY SUPERPOSITION OF LINEAR OPERATION WE HAVE

$Y_N(X) = -\tau\sum_{K=0}^{N} C_K^N K!S_K(X)$ 9

SATISFY THE BOUNDARY CONDITION $Y_N(0) = 1$, WILL YIELDS

$-\tau\sum_{K=0}^{N} C_K^N K!S_K(0) = 1$

$-\tau = \frac{1}{\sum_{K=0}^{N} C_K^N K!}$

THE FINAL SOLUTION BECOMES

$$Y_N(X) = \frac{\sum_{K=0}^{N} C_K^N K!S_K(X)}{\sum_{K=0}^{N} C_K^N K!} \quad .....10$$

WHEN N = 4

$T_4^*(X) = 1-32X+160X^2-256X^3+128X^4$

$$Y_4(X) = \frac{\sum_{K=0}^{N} C_K^4 K!S_K(X)}{\sum_{K=0}^{N} C_K^4 K!}$$

WHERE

$\sum^{4} C_K^4 K!SK(X) =$

$K=0$

$C_0^4 0!S_0(X)+C_1^4 1!S_1(X)+C_2^4 2!S_2(X)+C_3^4 3!S_3(X)+C_4^4 4!S_4(X)$

$\sum_{K=0}^{4} C_K^4 K!SK(X) = C_0^4 0!+C_1^4 1!+C_2^4 2!+C_3^4 3!+C_4^4 4!$

$S_K(X) = 1+X+X^2/2!+..........+X^N/N! = \sum_{K=0}^{N} X^K/K!$

$\Rightarrow S_0(X) = 1,\ S_1(X) = 1+X,\ S_2(X) = 1+X+X^2/2!$
$S_3(X) = 1+X+X^2/2!+X^3/3!,\ S_4(X) = 1+X+X^2/2!+X^3/3!+X^4/4!$

HENCE

$Y_4(X) = \dfrac{1325+1824X+928X^2+256X^3+128X^4}{1825}$ .11

THE ABOVE SOLUTION LOOKS LIKE WEIGHTED AVERAGE OF THE PARTIAL SUMS $S_K(X)$. THIS WEIGHTING IS VERY EFFICIENT IF X = 1 WE OBTAIN

$Y_4(1) = 4961/1825 = 2.718356$..12

THE EXACT VALUE

$Y_4(1) = E^1 = 2.7182818284$ .13

HENCE ERROR = EXACT VALUE - APPROXIMATE VALUE.

$\text{ERROR} = -7.4*10^{-5}$

WHEREAS THE UNWEIGHTED PARTIAL SUM $S_4(1)$ GIVES

$65/24 = 2.70832$

WITH ERROR $= 1.0 * 10^{-2}$

HERE, WE SEE THE GREAT INCREASED CONVERGENCE THUS OBTAINED.

HOWEVER, THE RANGE (0, 1) IS ACCIDENTAL NOW TESTING WITH ANALYTIC FUNCTIONS WHICH ARE DEFINED AT ALL POINTS OF THE COMPLEX PLANE EXCEPT FOR SINGULAR POINTS. HENCE,OUR AIM WILL BE TO OBTAIN Y(Z) WHERE Z MAY BE CHOSEN AS ANY NON-SINGULAR COMPLEX POINT.

IN VIEW OF THIS, WE CHOOSE OUR ERROR POLYNOMIAL IN THE FORM $T_N^*(X/Z)$ AND SOLVE THE GIVEN DIFFERENTIAL EQUATION ALONG THE COMPLEX RAY WHICH CONNECTS THE POINT X=0 WITH THE POINT X=Z. THEN SOLVING THE DIFFERENTIAL EQUATION

$DY_N(X) = \varphi T_N^*(X/Z)$ 14

BY CONSIDERING Z MERELY AS A GIVEN CONSTANT, WE FINALLY SUBSTITUTE FOR X THE END-POINT X=Z OF THE RANGE IN WHICH $T_N^*(X/Z)$ IS USABLE.

HENCE,

$T_N^*(X/Z) = \sum_{K=0}^{N} C_K^N X^K / Z^K$ .... .15

WE OBTAIN

$Y_N(Z) = \dfrac{\sum_{K=0}^{N} C_K^N S_K(Z) K! / Z^K}{T_N^*(-1/Z)}$ 16

THE PREVIOUS APPROXIMATIONS HAVE NOW TURNED INTO RATIONAL APPROXIMATIONS GIVING THE SUCCESSIVE APPROXIMATES AS THE RATIO OF TWO POLYNOMIALS OF ORDER N.

WHEN N=4, WE HAVE

$Y_4(X) = \dfrac{\sum_{K=0}^{N} C_K^4 S_K(Z) K! / Z^K}{\sum_{K=0}^{N} C_K^4 K! / Z^K}$ 17

$= \dfrac{3072+1536Z+320Z^2+32Z^3+Z^4}{3072-1536Z+320Z^2-32Z^3+Z^4}$ ..17A

NOW REPLACING THE COEFFICIENT C_K^N OF THE CHEBYSHEV POLYNOMIAL BY THE CORRESPONDING COEFFICIENT OF THE LEGENDRE POLYNMIAL $P_N^*(X)$ DEFINED THE HYPERGEOMETRIC SERIES $P_N^*(X) = F(-N, N+1, 1; X)$

HENCE

$Y^{NP}(Z) = \dfrac{\sum_{K=0}^{N} P_K^N S_K(Z) / Z^K}{\sum_{K=0}^{N} P_K^N / Z^K}$ 18

WHEN N = 4

$Y_4^P(Z) = \dfrac{\sum_{K=0}^{N} P_K^4 S_K(Z) / Z^K}{\sum_{K=0}^{N} P_K^4 / Z^K}$ .18A

$P_4^*(X) = 1-20X+180X^2-840X^3+1680X^4$

WE NOW HAVE $Y_4^P(Z) = \dfrac{1680+840Z+180Z^2+20Z^3+Z^4}{1680-840Z+180Z^2-20Z^3+Z^4}$

PUTTING Z = 1, WE OBTAIN

$Y_4^P(1) = 2721/1001 = 2.71828172$...........19

WHEREAS THE EXACT VALUE = 2.7182818284

HENCE THE ERROR $=\varsigma = 1.1*10^{-7}$

COMPARING THE RESULT OF CHEBYSHEV WITH LEGENDRE WE DISCOVER THAT

LEGENDRE SOLUTION GIVE MUCH CLOSER E-VALUE THEN THE VALUES OBTAINED BY THE CHEBYSHEV WEIGHTING.
IF WE PROCEED BY PUTTING Z = I, WE OBTAIN SUCCESSIVE APPROXIMATIONS OF $E^I = COS1+ISIN1 = 0.54030231+0.841470981$
IN THE CASE N = 4 CONSIDERED

$Y_4^P(I) = 1501+820I$
1501-820I
= 1580601+2461640I
2925401
$Y_4^P(I) = 0.540302338+0.8414709641$
ERROR $\eta = -3*10^{-8}+2*10^{-8}I$
WHEREAS THE WEIGHTING BY CHEBYSHEV COEFFICIENT YIELDS
$Y_4^C(I) = 2753+1504I$
2753-1504I
= 5316993+8281024I
9841025
$Y_4^C(I) = 0.5402885+0.8414798I$
ERROR $\eta = 1.4*10^{-5}-0.9*10^{-5}I$
SEE TABLE 1 FOR SOME NUMERICAL RESULTS FOR THE ERROR ESTIMATES BASED ON THE EXAMPLE 1, WHEN X = 1

EXAMPLE 2
$Y'(1+X) = 1, Y(0) = 0.$
THE EXACT VALUE (SOLUTION) => $Y(X) = LOG(1+X)$
=> $Y(X) = X-X^2/2+X^3/3$
FOLLOWING THE ILLUSTRATION OF EXAMPLE 1 WE HAVE CANONICAL POLYNOMIAL BECOMES
$Q_K(X) = (-1)_{K-1}S_K(X)$
THE PERTURBED TERM BECOMES
$Y'(1+X) = 1+\varphi T^*N(X)$

$$Y^N(X) = \varphi\sum_{K=1}^{N} C_K{}^N(-1)_K S_K(X)$$

WHERE

$$S_K(X) = \sum_{K=0}^{N} \frac{(-1)^{K+1}X^K}{K}$$

$$T_N{}^*(X) = \sum_{K=0}^{N} C_K{}^N X^K$$

HENCE WE HAVE THE TABLE FOR THE RESULT OF EXAMPLE

CONCLUSIONS: The polynomials of legendre and chebyshev has been described. The two method is shown to be accurate efficient and general in application for sufficiently solution y(x) and for tau polynomial approximation $y_n(x)$.

the result obtained in the present work demonstrate the effectiveness and superiority of legendre polynomials to chebyshev polynomials for the solution of order linear differential equation. The variants of the error estimated described the case of reciprocal radii in which the point x = 0 becomes a singular point of our domain legendre polynomial fail to give better value than the chebyshev polynomials even of the end point x = 1. By excluding, however the point x = 0 by defining our range as (e,1) which by a simple linear transformation can then be changed back to the standard range (0,1). The condition that our domain shall contain no singular points is now satisfied.

in the vicinity of singularity $p_n{}^*(x)$ (i.e. the legendre polynomials) gives larger errors than the $t_n{}^*(x)$ (i.e. chebyshev) for small values of n. As n increases, the polynomials $p_n{}^*(x)$ compete with $t_n{}^*(x)$ with increasing accuracy to the $t_n{}^*(x)$ for the purpose of end point approximation.

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