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Controllability and Null Controllability of Linear Systems

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ABSTRACT: This paper establishes sufficient conditions for the controllability and null controllability of linear systems. The aim is to use the variation of constant formula to deduce our controllability grammiam, by exploiting the properties of the grammiam and the asymptotic stability of the free system, we achieved our results. @JASEM

Differential equations, in general, are an important tool for harnessing into single system and analyzing the inter-relationship between different components which otherwise may continue to remain independent on each other. It is known in Sebakhy and Bayoumi (1973) that, in the study of economics, biology and physiological systems as well as electromagnetic systems composed of such subsystems interconnected by hydraulic, mechanical and various other linkages, one encounters phenomena which cannot be readily modeled unless relations involving time delays are admitted. Models for such systems can be controlled. A delayed control on such systems will affect the evolution of the system in an indirect manner (Artstein and Tadmore, 1982), where the decisions in the control function are shifted, twisted or combined before affecting the evolution. Models for systems with delay in the control occur in the study of gas-pressures bipropellant rocket systems, in population models and in some complex economic systems etc. The controllability of systems with delays in the control has been studied by several authors see (Balachandran, 1987; Balachandran and Dauer, 1996; Chukwu, 1979; Onwuatu, 1989). In particular Manitius and Olbrot (1972) studied the system

$$\dot{x}(t) = \sum_{i=0}^{p} A_i(t)x(t-h) + \sum_{i=0}^{p} B_i(t)u(t-h_i) \quad (1)$$

and gave sufficient conditions for the relative controllability of (1). Our interest, is to integrate the concept of null controllability into a generalized system with delay in state and control given by

$$\dot{x}(t) = L(t, x_t) + C(t)u(t-h) \qquad (2)$$

We shall give sufficient conditions for the null controllability with constraint of (2) when relative controllability is assumed. Our results complement and extend known results.

BASIC NOTATIONS AND PRELIMINARIES

Let n and m be positive integers, E the real line $(-\infty,\infty)$. We denote by E^n the space of real n- tupples with the Euclidean norm denoted by $|\cdot|$. If J is any interval of E the usual Lebesgue space of square integrable (equivalent class of) functions from J to E^n will be denoted by $L_2(J, E^n)$. $L_1([t_0,t_1], E^n)$ denotes the space of integrable functions from $[t_0,t_1]$ to E^n. Let $h>0$ be given, for functions $x:[t_0-h,t_1]\to E^n$, $t\in[t_0,t_1]$, we use x_t to denote the functions on $[-h,0]$ defined by $x_t(s)=x(t+s)$ for $s\in[-h,0]$.

Consider the system

$$\dot{x}(t) = L(t, x_t) + C(t)u(t-h) \qquad (3)$$

where

$$L(t,\phi) = \int_{-h}^{0} d\eta(t,s)\phi(s) \qquad (4)$$

satisfied almost everywhere on $[t_0,t_1]$. The integral is in the Lebesgue- Stieltjes sense with respect to s. $L(t,\phi)$ is continuous in t, linear in ϕ. $\eta(t,s)$ is an $n\times n$ matrix function measurable in t and of

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bounded variation in ϕ on $[-h,0]$ for each $t \in [t_0,t_1]$. $C(t)$ is an $n \times m$ matrix assumed to be bounded and measurable on $[t_0,t_1]$. The control function $u(t) \subset E^m$, is assumed to be measurable and bounded on every finite interval. Throughout the sequel, the controls of interest are, $B = L_2([t_0,t_1],E^m), U \subseteq L_2([t_0,t_1],E^m)$ a closed and bounded subset of B with zero in the interior relative to B.

If X and Y are linear spaces is a mapping, we shall use the symbols $D(T), \Re(T)$ and $N(T)$ to denote the domain, range and null spaces of T respectively.

Definition 1 - The complete state of system (3) at time t is given by $z(t) = \{x(t), x_t, u_t\}$

Definition 2 - System (3) is relatively controllable on $[t_0,t_1]$, if for every $z(t_0)$ and every vector $x_1 \in E^n$, there exist a control $u \in B$, such that the corresponding trajectory of system (3) satisfies $x(t) = x_1$. If system (3) is relatively controllable on each interval $[t_0,t_1]$, $t_1 > t_0$, we say, system (3) is relatively controllable.

Definition 3 - System (3) is said to be null controllable at $t = t_1$, if for any initial state $\{x_0, x_{t_0}, u_{t_0}\}$ on $[t_0 - h, t_0]$, there exists an admissible control $u(t) \in B$ defined on $[t_0, t_1 - h]$ such that the response is brought to the origin of E^n at $t = t_1$, using the control effort

$$u(t) = \begin{cases} u(t), & on\ [t_0, t_1 - h] \\ 0, & on\ [t_1 - h, t_1] \end{cases}$$

see Sebakhy and Bayoumi (1973). It is null controllable with constraints at $t = t_1$, if for any initial state $\{x_0, x_{t_0}, u_{t_0}\}$ on $[t_0 - h, t_0]$, there exists an admissible control $u(t) \in U$, defined on $[t_0, t_1 - h]$ such that the response $x(t)$ of system (3) satisfies $x(t) = 0$, using the control effort

$$u(t) = \begin{cases} u(t) \in U, & on\ [t_0, t_1 - h] \\ 0, & on\ [t_1 - h, t_1] \end{cases}$$

Definition 4 - The domain D of null controllability of system (3) is the set of all initial points $x_0 \in E^n$ for which the solution $x(t)$ of system (3) with $x(t_0) = x_0$ satisfies $x(t_1) = 0 \in E^n$ at some t_1 using $u \in U$

Definition 5 - An operator $T : X \to Y$, where X and Y are linear spaces, is said to be closed if for any sequence $u_n \in D(T)$ such that $u_n \to u$ and $Tu_n \to v$, u belongs to $D(T)$ and $Tu = v$.

The variation of parameter of system (3) for $t > t_0 + h$ imply the existence of a unique absolutely continuous solution $x(t)$ of system (3), with initial complete state $z(t_0)$ of the form

$$x(t) = X(t,t_0)x(t_0) + \int_{t_0}^{t} X(t,s)C(s)u(s+h)ds \quad (5)$$

where $X(t,s)$ satisfies the equation

$$\frac{\partial X(t,s)}{\partial t} = L(t, X_t(\cdot,s)),\ t > s$$

almost everywhere

$$X(t,s)=\begin{cases}0, & s-h<t<s\\ I, & t=s \ \ (I\equiv Identity\,matrix)\end{cases}$$

$X(t,s)$ is called the fundamental matrix solution of the system

$x(t)=L(t,x_t)$ (6)

We now obtain a more convenient form of the solution (3) by expressing (5) as

$$x(t)=X(t,t_0)\left[x(t_0)+\int_{t_0-h}^{t_0}X(t_0,s+h)C(s+h)u_{t_0}(s)ds\right]+\int_{t_0}^{t-h}X(t,s+h)C(s+h)u(s)ds \qquad (7)$$

The solution $x(t)$ of system (3) at $t=t_1$ (Klamka, 1980) becomes

$$x(t_1)=X(t_1,t_0)\left[x(t_0)+\int_{t_0-h}^{t_0}X(t_0,s+h)C(s+h)u_{t_0}(s)ds\right]+\int_{t_0}^{t_1-h}X(t_1,s+h)C(s+h)u(s)ds \qquad (8)$$

We now define the $n\times n$ controllability matrix of system (3), given by

$$W(t_0,t_1)=\int_{t_0}^{t_1-h}[X(t_1,s+h)C(s+h)][X(t_1,s+h)C(s+h)]^T \qquad (9)$$

Where T denotes transpose

Definition 6 - The reachable set of system (3) at time t_1 using L_2 controls is the subset of E^n given by

$$P(t_1,t_0)=\left\{\int_{t_0}^{t_1-h}X(t_1,s+h)C(s+h)u(s)ds : u\in L_2\right\}$$

and the constraint reachable set of system (3) is given by

$$R(t_1,t_0)=\left\{\int_{t_0}^{t_1-h}X(t_1,s+h)C(s+h)u(s)ds : u\in U\right\}$$

The constraint reachable set with unspecified end time is given by

$$R(t_0)=\underset{t_1>t_0}{U}R(t_1,t_0)$$

Definition 7 - System (3) is said to be proper in E^n On $[t_0,t_1]$ if

$$c^T\left[X(t_1,s+h)C(s+h)\right]\equiv 0$$

almost everywhere $t\in[t_0,t_1]$, $c\in E^n$ implies $c=0$. If system (3) is proper on $[t_0,t_0+\delta]$ for each $\delta>0$, we say system (3) is proper at time t_0. If system (3) is proper on each interval $[t_0,t_1]$, $t_1>t_0>0$, we say the system is proper in E^n

NULL CONTROLLABILITY WITH CONSTRAINED CONTROLS

Lemma 1 - The following are equivalent

(i) $W(t_0,t_1)$ is non singular for each t

(ii) System (3) is proper in E^n for each interval $[t_0,t_1]$

(iii) System (3) is relatively controllable on each interval $[t_0,t_1]$

Proof - $W(t_0,t_1)$ is non singular implies $W(t_0,t_1)$ is positive definite, that is

$$c^T\left[X(t_1,s+h)C(s+h)\right]=0$$

almost everywhere on $[t_0,t_1]$ implies $c=0$. Therefore, (i) implies (ii).

To prove the equivalence of (ii) and (iii) let $c \in E^n$, and assume that

$$c^T\left[X(t_1, s+h)C(s+h)\right] \equiv 0$$

almost everywhere $t \in [t_0, t_1]$ for each t_1, then

$$\int_{t_0}^{t_1-h} c^T X(t_1, s+h)C(s+h)u(s)ds = c^T \int_{t_0}^{t_1-h} X(t_1, s+h)C(s+h)u(s)ds = 0$$

for $u \in L_2$. It follows from this that c is orthogononal to the set $P(t_1, t_0)$. We assume that system (3) is relatively controllable, then $P(t_1, t_0) = E^n$, so that $c = 0$, meaning that (iii) implies (ii). Conversely, assume that system (3) is not controllable, so that $P(t_1, t_0) \subseteq E^n$, for $t_1 > t_0$. Then, there exists $c \neq 0$, $c \in E^n$, such that $c^T P(t_1, t_0) = 0$.

It follows that for all admissible control $u \in L_2$

$$0 = c^T \int_{t_0}^{t_1-h} \left[X(t_1, s+h)C(s+h)\right]u(s)ds = \int_{t_0}^{t_1-h} c^T\left[X(t_1, s+h)C(s+h)\right]u(s)ds$$

Hence

$$c^T\left[X(t_1, s+h)C(s+h)\right] \equiv 0, \text{ almost everywhere } t \in [t_0, t_1],\ c \neq 0.$$

To show that (i) implies (ii)

We define the operator $K : L_2\left([t_0, t_1],\ E^n\right) \to E^n$ by $K(u) = \int_{t_0}^{t_1-h} X(t_1, s+h)C(s+h)u(s)ds$

If we assume that $W(t_0, t_1)$ is singular. Then the symmetric operator $KK^T = W(t_0, t_1)$ is positive definite. But this holds, if and only if $rank\ W(t_0, t_1) = n$.

Theorem 1 - System (3) is proper on $[t_0, t_1]$ if and only if $0 \in \operatorname{int} R(t_1, t_0)$.

Proof - If $R(t_1, t_0)$ is a closed and convex subset of E^n (Klamka, 1976), then a point y_1 on the boundary of $R(t_1, t_0)$ implies that, there is a support plane Π of $R(t_1, t_0)$ through y_1, that is $c^T(y - y_1) \leq 0$ for each $y \in R(t_1, t_0)$ where $c \neq 0$ is an outward normal to Π. If u_1 is the control corresponding to y_1, we have

$$c^T \int_{t_0}^{t_1-h} X(t_1, s+h)C(s+h)u(s)ds \leq c^T \int_{t_0}^{t_1-h} X(t_1, s+h)C(s+h)u_1 ds$$

for each $u \in U$. Since U is a unit sphere, this last inequality holds for each $u \in U$, if and only if

$$c^T \int_{t_0}^{t_1-h} X(t_1, s+h)C(s+h)u(s)ds \leq \int_{t_0}^{t_1-h} c^T X(t_1, s+h)C(s+h)u_1 ds$$

$$\leq \int_{t_0}^{t_1-h} \left|c^T X(t_1, s+h)C(s+h)\right| ds$$

and $u_1(t) = \operatorname{sgn} c^T\left(X(t_1, s+h)C(s+h)\right)$ as y_1 is on the boundary. Since we always have $0 \in R(t_1, t_0)$. If 0 were not in the interior of $R(t_1, t_0)$ then 0 is on the boundary, hence, from the foregoing, this implies $0 = \int_{t_0}^{t_1-h} \left|c^T X(t_1, s+h)C(s+h)\right| ds$ so that

$c^T\left[X(t_1, s+h)C(s+h)\right] = 0$ almost everywhere $t \in [t_0, t_1]$. This by our definition implies that the system is not proper since $c^T \neq 0$. This completes the proof.

T

heorem 2 - System (3) is relatively controllable if and only if $0 \in \operatorname{int} R(t_1, t_0)$ for each $t_1 > t_0$

Proof - By lemma 1, system (3) is relatively controllable on $[t_0, t_1], t_1 > t_0$ if and only if, it is proper on $[t_0, t_1]$. Therefore, by theorem 1, system (3) is relatively controllable if and only if $0 \in \operatorname{int} R(t_1, t_0)$.

Theorem 3 - If system (3) is relatively controllable on $[t_0, t_1]$ for each $t_1 > t_0$, then the domain null controllability of system (3) contains zero in its interior

Proof - Assume that system (3) is relatively controllable on $[t_0, t_1]$, $t_1 > t_0$ then by theorem 2, $0 \in \text{int}\, R(t_1, t_0)$, for each $t_1 > t_0$. Since $x = 0$ is a solution of system (3) with $u = 0$, we have $0 \in D$. Hence, If $0 \notin \text{int}\, D$, then there exists a sequence $x_m \subseteq E^n$ such that $x_m \to 0$ as $m \to \infty$ and no x_m is in D, that is $x_m \neq 0$. From (5), we have

$$0 \neq x(t_1) = X(t_1, t_0)x_m(t_0) + \int_{t_0}^{t_1} X(t_1, s)C(s)u(s+h)ds$$

for any $t_1 > t_0$ and any $u \in U$. Hence, for $u = 0$, $z_m \overset{def}{=} x(t_1) = X(t_1, t_0)x_m(t_0)$,

is not in $R(t_1, t_0)$ for any $t_1 > t_0$. Therefore the sequence $z_m \subseteq E^n$ is such that $z_m \notin R(t_1, t_0), z_m \neq 0$, but $z_m \to 0$ as $m \to \infty$. Therefore, $0 \notin R(t_1, t_0)$ a contradiction. Hence, $0 \in \text{int}\, D$.

Theorem 4 - Assume

(i) System (3) is relatively controllable on $[t_0, t_1]$ for each $t_1 > t_0$

(ii) The zero solution of system (6) is uniformly asymptotically stable, so that the solution of (6) satisfies $|x(t)| \leq k|x_0| e^{-a(t-t_0)}$, $a > 0, k > 0$ are constants. Then system (3) is null controllable with constraints.

Proof - By (i) and theorem 3, the domain D of null controllability of system (3) contains zero in its interior. Therefore there exists a ball B_1 such that $0 \in B_1 \subset D$. By (ii), every solution of system (3) (with $u = 0$) satisfies $x(t) \to 0$ as $t \to \infty$. Hence at some $t_1 < \infty$ (with $u = 0$) $x(t_1) \in B_1 \subseteq D$, for $t_1 > t_0$. Therefore, using t_1 and $x_1 = x(t_1)$ at $u = 0$ as initial data there exists a $u \in U$ and some $t_2 > t_1$ such that the solution $x(t)$ of system (3) satisfies $x(t_2) = 0$. Thus proving the theorem.

Lemma 2

System (3) is relatively controllable on $[t_0, t_1]$ if and only if rank $W(t_0, t_1) = n$.

The proof is quite standard and simple, for similar proof see Klamka (1976).

Theorem 5 - System (3) is null controllable with constraints if

(i) Rank $W(t_0, t_1) = n$, for $t_1 > t_0$

(ii) The zero solution of (6) is uniformly asymptotically stable

Proof - Immediately from theorem 4 and lemma 1

Theorem 6 - The system (6) is null controllable with constraints if

(i) Rank $W(t_0, t_1) = n$, for $t_1 > t_0$

(ii) If all the characteristics roots have negative real part

Proof : Immediately.

Conclusion: In this paper we have developed and proved sufficient conditions for the controllability and null controllability of liner systems with delay in the state and control. That is, if the uncontrolled system is asymptotically stable and the controlled system is relatively controllable then, the system is null controllable with constraints.

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