The Matrix Class B ($\ell_P, \ell_{P'}$) $p \geq 1$, $p' \geq 1$

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ABSTRACT

Necessary and sufficient conditions for a matrix $A \in B(\ell_1, \ell_1)$ were determined by K. Knopp and G. G. Lorentz in 1949. Necessary and sufficient conditions for a matrix $A \in B(\ell_1, \ell_p)$, $p \geq 1$ were determined and their proof presented along classical lines in 1970 (Maddox I, 1970). A sufficient condition for a matrix $A \in B(\ell_p, \ell_{p'})$, $1 < p < \infty$ was given by (Maddox I, 1970). In 1971, Crone solved the problem for $A \in B(\ell_2, \ell_2)$ (Maddox I, 1980). In 1981 H.R. Pitt determined a necessary condition for a matrix $A \in B(\ell_p, \ell_q)$, $p > q \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $p, q \in \Re$ (Ruckle, 1981).

But never before have necessary and sufficient conditions for a matrix $A \in B(\ell_p, \ell_p)(p \geq 1, \frac{1}{p} + \frac{1}{q} = 1)$ been determined. In this paper we determine necessary and sufficient conditions for a matrix $A \in B(\ell_p, \ell_{p'})$, $(p \geq 1, p' \geq 1)$ such that $\frac{1}{p} + \frac{1}{q} = 1 = \frac{1}{p'} + \frac{1}{q'}$. From these conditions, necessary and sufficient conditions for a matrix $A \in B(\ell_p, \ell_p), p \geq 1$ are easily determined. We also present the proof for the necessary and sufficient conditions for a matrix $A \in B(\ell_1, \ell_p), p \geq 1$ along the modern functional analytic methods.

KEY WORDS: Banach spaces; Matrix, Class B

NOTATIONS

$\Re$, $\not\subset$, ℓ_1, ℓ_p, ℓ_∞, $(X.\|.\|)$, B(X), B(X,Y), $\|A\|_X$ denotes respectively the set of real numbers; the set of complex numbers; the absolutely summable sequences; the p-absolutely summable sequences; the bounded sequences; a Banach space; the algebra of bounded operators in X; the algebra of bounded linear transformations from X into Y and the norm of a bounded operator A in the space X.

$\|\hat{x}\|$ will denote the norm of a vector $\hat{x}$ in X while

$A = (a_{nk}): X \to Y$, the action of an infinite matrix A from X into Y provided

$$\sum_k a_{nk} x_k$$

converges in the Y-norm for $\hat{x} = (x_k)_{k\geq 1}$.

X and Y in our investigation will be restricted to some form of ℓ_p $(p \geq 1)$

1.0 INTRODUCTION

Let $\hat{e}_k = (0, 0, \ldots 1, 0, 0 \ldots)'$ denote a sequence with zero entries except 1 in the k[th] position, then each $\hat{x} = (x_k)_{k\geq 1}$ in X or Y has a unique representation

$$\hat{x} = \sum_{k=1}^{\infty} x_k \hat{e}_k \quad \ldots\ldots 1$$

If we denote the k[th] column of the matrix

$A = (a_{nk})$ by $\hat{v}_k = (a_{nk})_{n\geq 1} = A\hat{e}_k$ then $A = (a_{nk}): X \to Y$ has a unique representation

$$A\hat{x} = \sum_{k=1}^{\infty} x_k A\hat{e}_k = \sum_{k=1}^{\infty} x_k \hat{v}_k \quad \ldots\ldots 2$$

provided A is continuous (Franekic,1974).

Theorem 1.1

The matrix A = (a_{nk}) $\in B(\ell_1, \ell_p)$ if and only if

(I) $\hat{v}_k = (a_{nk})_{n\geq 1} \in \ell_p$

(II) $\sup_k \sum_{n=1}^{\infty} |a_{nk}|^p < \infty$

Furthermore

$$\|A\| = \sup_k \left[\sum_{n=1}^{\infty} |a_{nk}| \right]^{\frac{1}{p}}$$

The proof along classical lines was presented in 1970 (Maddox (1970) which covers the first part only.

Proof: (Necessity)

(i) $\hat{v}_k = A\hat{e}_k \in \ell_p$

(ii) $\|\hat{v}_k\|_{\ell_p} = \|A\hat{e}_k\| \leq \|A\|$. k = 1, 2, …

$$\Rightarrow \sup_k \|\hat{v}_k\| \leq \|A\|$$

$$\Rightarrow \sup_k \|\hat{v}_k\|^p < \infty$$

i.e. $\sup_k \left[\sum_{n=1}^{\infty} |a_{nk}|^p \right] < \infty$

(Sufficiency)

By the representation equation (2) i.e.

$$A\hat{x} = \sum_{k=1}^{\infty} x_k \hat{v}_k$$

$$\|A\hat{x}\| \leq \sum_{k=1}^{\infty} |x_k| \, \|\hat{v}_k\| \leq \sup_k \|\hat{v}_k\| \sum_{k=1}^{\infty} |x_k|$$

$$= \sup_k \|\hat{v}_k\|_{\ell_p} \|\hat{x}\|_{\ell_1}$$

Thus $\|A\| \le \sup_k \|\hat{v}_k\|$

This last inequality and the necessity part inequality both imply

$$\|A\| = \sup_k \|\hat{v}_k\| = \sup_k \left[\sum_{n=1}^{\infty} |a_{nk}|^p\right]^{\frac{1}{p}}$$

2.0 RESULTS

Theorem 2.1

Let $p \ge 1$, $p' \ge 1$ and q, q' be such that $\frac{1}{p}+\frac{1}{q}=1 = \frac{1}{p'}+\frac{1}{q'}$ then

$A = (a_{nk}) \in B(\ell_p, \ell_{p'})$ if and only if

(i) $\hat{v}_k = (a_{nk})_{n\ge1} \in \ell_{p'}$

(ii) $\sum_{k=1}^{\infty} \|\hat{v}_k\|_{\ell_{p'}}^q < \infty$

Furthermore

$$\|A\| = \left[\sum_{k=1}^{\infty} \|\hat{v}_k\|^q\right]^{\frac{1}{q}} = \left[\sum_{k=1}^{\infty}\left(\sum_{k=1}^{\infty} |a_{nk}|^{p'}\right)^{\frac{q}{p'}}\right]^{\frac{1}{q}}$$

Proof: (Necessity)

(i) Clearly $\hat{v}_k = A\hat{e}_k \in \ell_{p'}$

(ii) If N is an arbitrary positive integer and f be any linear functional on $\ell_{p'}$ of unit norm then we define

$$\hat{x}^{(N)} = \sum_{k=1}^{N} \operatorname{sgn} f(\hat{v}_k) \|\hat{v}_k\|_{\ell_{p'}}^{q-1} \hat{e}_k .$$

$$\|\hat{x}^{(N)}\|_{\ell_p} = \left[\sum_{k=1}^{N} \|\hat{v}_k\|^q\right]^{\frac{1}{p}}$$

If $\hat{z}^{(N)} = \dfrac{\hat{x}^{(N)}}{\|\hat{x}^{(N)}\|}$ then $\|\hat{z}^{(N)}\| = 1$ whence

$$A\hat{z}^{(N)} = \frac{1}{\|\hat{x}^{(N)}\|}\sum_{k=1}^{N} \operatorname{sgn}(\hat{v}_k)\|\hat{v}_k\|^{q-1} A\hat{e}_k$$
$$= \frac{1}{\|\hat{x}^{(N)}\|}\sum_{k=1}^{N} \operatorname{sgn} f(\hat{v}_k)\|\hat{v}_k\|^{q-1} \hat{v}_k$$

$$f(A\hat{z}^{(N)}) = \frac{1}{\|\hat{x}^{(N)}\|}\sum_{k=1}^{N} \operatorname{sgn} f(\hat{v}_k)\|\hat{v}_k\|^{q-1} f(\hat{v}_k)$$
$$= \frac{1}{\|\hat{x}^{(N)}\|}\sum_{k=1}^{N} \|\hat{v}_k\|^{q-1} |f(\hat{v}_k)|$$

$$(\mathrm{A}^*\mathrm{f})\,\hat{z}^{(N)} = \frac{1}{\|\hat{x}^{(N)}\|}\sum_{k=1}^{N} \|\hat{v}_k\|^{q-1} |\ddot{\hat{v}}(f)| \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 3$$

In the last equation, we consider $\hat{v}_k$ as an element of the second dual space and thus $\|\hat{v}_k\| = \|\ddot{\hat{v}}_k\|$. The last expression in equation (3) is non-negative so that

$$\|A^* f\| \geq |(A^* f)\hat{z}^{(N)}| = \frac{1}{\|\hat{x}^{(N)}\|}\sum_{k=1}^{N} \|\hat{v}_k\|^{q-1} |\ddot{\hat{v}}_k(f)|$$

$$\sup_{\|f\|=1}\|A^* f\| \geq \sup_{\|f\|=1}\left[\frac{1}{\|\hat{x}^{(N)}\|}\sum_{k=1}^{N} \|\hat{v}_k\|^{q-1} |\ddot{\hat{v}}_k(f)|\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ 4$$

Using the inequality (4)

$$\|A^*\| \geq \frac{1}{\|\hat{x}^{(N)}\|}\sum_{k=1}^{N} \|\hat{v}_k\|^{q-1} \sup_{\|f\|=1}|\ddot{\hat{v}}(f)|$$
$$= \frac{1}{\|\hat{x}^{(N)}\|}\sum_{k=1}^{N} \|\hat{v}_k\|^{q-1} \|\ddot{\hat{v}}_k\| \qquad Note\ \|\ddot{\hat{v}}_k\| = \|\hat{v}_k\|$$

$$= \frac{1}{\left\|\hat{x}^{(N)}\right\|}\sum_{k=1}^{N}\left\|\hat{v}_k\right\|^q$$

$\|A^*\| = \|A\|$ on a Banach space

$$\sum_{k=1}^{N}\left\|\hat{v}_k\right\|^q \le \|A\| \left\|\hat{x}^{(N)}\right\|$$

$$\sum_{k=1}^{N}\left\|\hat{v}_k\right\|^q \le \|A\| \left[\sum_{k=1}^{N}\left\|\hat{v}_k\right\|^q\right]^{\frac{1}{p}}$$

$$\left[\sum_{k=1}^{N}\left\|\hat{v}_k\right\|^q\right]^{\frac{1}{q}} \le \|A\|$$

and N being an arbitrary positive integer then

$$\left[\sum_{k=1}^{\infty}\left\|\hat{v}_k\right\|^q\right]^{\frac{1}{q}} \le \|A\| \quad \text{...} \quad 5$$

(Sufficiency) We make use of the representation equation (2) i.e.

$$A\hat{x} = \sum_{k=1}^{\infty} x_k \hat{v}_k .$$

By Hölder's inequality

$$\|A\hat{x}\| \le \sum_{k=1}^{\infty}|x_k|\,\|\hat{v}_k\| \le \left[\sum_{k=1}^{\infty}|x_k|^p\right]^{\frac{1}{p}} \left[\sum_{k=1}^{\infty}\|\hat{v}_k\|_{\ell_{p'}}^q\right]^{\frac{1}{q}}$$

Thus $\|A\| \le \left[\sum_{k=1}^{\infty}\|\hat{v}_k\|_{\ell_{p'}}^q\right]^{\frac{1}{q}}$ since we note here that $\|\hat{x}\|_{\ell_p} = \left[\sum_{k=1}^{\infty}|x_k|^p\right]^{\frac{1}{p}}$

Now the last inequality above and the inequality (5) imply that

$$\|A\| = \left[\sum_{k=1}^{\infty}\|\hat{v}_k\|_{\ell_{p'}}^q\right]^{\frac{1}{q}}$$

$$= \left[\sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} |a_{nk}|^{p'} \right)^{\frac{q}{p'}} \right]^{\frac{1}{q}}$$

Theorem 2.2

The ℓ_p - norm $\eta(p) = \left(\sum_{\kappa=1}^{\infty} |x_\kappa|^p \right)^{\frac{1}{p}}$ reduces to the supremum norm (sup norm). $\eta(p) =$

$\sup_{k \geq 1} |x_k|$ *as*

$p \to \infty$

Proof: By Jensen's inequality, if s > p then

$$\left[\sum_{k=1}^{\infty} |x_k|^s \right]^{\frac{1}{s}} \leq \left[\sum_{k=1}^{\infty} |x_k|^p \right]^{\frac{1}{p}}$$

Thus η (p) is monotonic decreasing and non-negative hence

$\lim_{p \to \infty} \eta(p) = \inf_{p > 0} \eta(p)$

If $x_k = 0$, k = 1, 2, 3, … then there is nothing to prove. If on the contrary there exists k

such that $x_k \neq 0$ then $x_k \to 0$ ($k \to \infty$) implies there is an integer k_o such that

$$|x_{k_0}| = \sup_k |x_k|.$$

$\left[\sum_{k=1}^{\infty} |x_k|^p \right]^{\frac{1}{p}} \geq \left[|x_{k_o}|^p \right]^{\frac{1}{p}} = |x_{k_0}|$ and

$|x_{k_0}| \leq \eta(p).$

$\sup_k |x_k| \leq \eta(p)$ for all values of p

hence $\sup_k |x_k| \leq \inf_{p > 0} \eta(p)$ 6

To reverse the inequality, we consider the following argument.

Let N be any positive integer. Denote

$\left[\sum_{k=1}^{N} |x_k|^p\right]^{\frac{1}{p}}$ by $\eta^{(N)}(p)$. Then $\eta^{(N)}(p) \to \eta(p)$ as $N \to \infty$.

$$\eta^{(N)}(p) = \left[\sum_{k=1}^{N} |x_k|^p\right]^{\frac{1}{p}} \le \left[\sum_{k=1}^{N} |x_{k_0}|^p\right]^{\frac{1}{p}} = |x_{k_0}| N^{\frac{1}{p}}$$

$\lim_{p\to\infty} \eta^{(N)}(p) \le |x_{k_0}|$ as $N^{\frac{1}{p}} \to 1 \ (p \to \infty)$

If $\varepsilon > 0$ there exists a positive integer N' such that

$$\eta(p) < \eta^{(N')}(p) + \varepsilon$$

$$\lim_{p\to\infty} \eta^{(N)}(p) + \varepsilon \le |x_{k_0}| + \varepsilon$$

Since N described above is arbitrary, we could set it equal to N' so

$\lim_{p\to\infty} \eta(p) \le \lim_{p\to\infty} \eta^{(N)}(p) + \varepsilon$ and

$$\lim_{p\to\infty} \eta(p) \le |x_{k_0}| + \varepsilon$$

and since $\varepsilon > 0$ is arbitrary we have

$$\lim_{p\to\infty} \eta(p) \le |x_{k_0}|$$

Hence

$\lim_{p\to\infty} \eta(p) = |x_{k_0}| = \sup_k |x_k|$ by the inequality (6).

Corollary 2.3

Let p ≥ 1 and A = (a_{nk}) ∈ B (ℓ_1, ℓ_p) then the necessary and sufficient conditions are

(i) $\hat{v}_k = (a_{nk})_{n\geq 1} \in \ell_p$. k = 1, 2, ...

(ii) $\sup_k \left[\sum_{n=1}^{\infty} |a_{nk}|^p \right] < \infty$

Proof: By theorem (2.1) set p = 1 and p' = p then p = 1 implies q → ∞ hence

$$\|A\| = \lim_{q\to\infty} \left[\sum_{k=1}^{\infty} \|\hat{v}_k\|_{\ell_p}^q \right]^{\frac{1}{q}} = \sup_k \|\hat{v}_k\|_{\ell_p}$$ by the theorem (2.2).

Then $\sup_k \left[\sum_{n=1}^{\infty} |a_{nk}|^p \right] < \infty$ which is condition (ii). Condition (i) follows from

$$\hat{v}_k = A\hat{e}_k \in \ell_p$$

The sufficiency is proved from

$$A\hat{x} = \sum_{k=1}^{\infty} x_k \hat{v}_k .$$

$$\|A\hat{x}\| \leq \sum_{k=1}^{\infty} |x_k| \, \|\hat{v}_k\| \leq \sup_k \|\hat{v}_k\| \sum_{k=1}^{\infty} |x_k|$$
$$= \sup_k \|\hat{v}_k\| \, \|\hat{x}\|.$$

Thus A is bounded.

Corollary 2.4 Our results in (5) indicate that if A = (a_{nk}) ∈ B (ℓ_p) then

$$\|A\|_{\ell_p} = \left[\sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} |a_{nk}|^p \right)^{\frac{q}{p}} \right]^{\frac{1}{q}}$$ is easily deduced from

$$\|A\|_{(\ell_p, \ell_{p'})} = \left[\sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} |a_{nk}|^{p'} \right)^{\frac{q}{p'}} \right]^{\frac{1}{q}}$$

by setting p' = p.

3.0 CONCLUSION

We have established that $A=(a_{nk})\ \varepsilon B\ (\ell_p, \ell_{p^1})$, if and only if

(1) $\hat{v}_k = (a_{nk})_{n\geq 1}\ \varepsilon\ \ell_{p^1}$

(2) $\sum_{k=1}^{\infty} \|\hat{v}_k\|^q_{\ell_{p^1}} < \infty$

where $\frac{1}{p}+\frac{1}{q}=1, \frac{1}{p^1}+\frac{1}{q^1}=1$

Furthermore

$$\|A\| = \left[\sum_{k=1}^{\infty} \|\hat{v}_k\|^q_{\ell_{p^1}}\right]^{\frac{1}{q}}$$

$$= \left[\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} |a_{nk}|^{p^1}\right)^{\frac{q}{p^1}}\right]^{\frac{1}{q}}$$

We have also proved using modern functional analytic methods that a matrix

$A=(a_{nk}) \in B(\ell_1, \ell_p)$ if and only if

(1) $\hat{v}_k = (a_{nk})_{n\geq 1} \in \ell_p$

(2) $\sup_k \sum_{n=1}^{\infty} |a_{nk}|^p < \infty$

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