Generalised Recurrent Finsler Space of Second Order II

S.M. Uppal

Department of Mathematics and Computer Science, Jomo Kenyatta University of Agriculture and Technology, P.O. Box 62 000, Nairobi, KENYA.

ABSTRACT

Sinha and Singh (1971 a, b; 1973) have studied recurrent Finsler spaces of second order and discussed the properties of recurrent curvature tensor and recurrence tensor fields in them. Singh (1981) has defined generalised recurrent Finsler space of second order and denoted it by $G(2\text{-}F_n)$. This paper defines G-2 recurrent projective tensor fields and the properties of associated recurrence vector and tensor fields in $G(2\text{-}F_n)$. The notations of Rund (1971 a) have been followed in this paper.

1.0 INTRODUCTION

In a Finsler space F_n, the projective curvature tensor field, the projective tensor field and the deviation tensor field are defined by

$$W^i_{jkh} = H^i_{jkh} + \frac{\delta^i_j}{n+1}\left(H_{kh} - H_{hk}\right) + \frac{\dot{x}^i}{n+1}\left(\dot{\partial}_j H_{kh} - \dot{\partial}_j H_{hk}\right)$$

$$+ \frac{\delta^i_k}{n^2-1}\left(nH_{jh} + H_{hj} + \dot{x}^r \dot{\partial}_j H_{hr}\right) - \frac{\delta^i_h}{n^2-1}\left(nH_{jk} + H_{kj} + \dot{x}^r \dot{\partial}_j H_{kr}\right) \quad \text{..............(1.1)}$$

$$W^i_{jk} = H^i_{jk} + \frac{\dot{x}^i}{n+1}\left(H_{jk} - H_{kj}\right) + \frac{\delta^i_j}{n^2-1}\left(nH_k + \dot{x}^r H_{kr}\right) - \frac{\delta^i_k}{n^2-1}\left(nH_j + \dot{x}^r H_{jr}\right) \quad \text{.....(1.2)}$$

and

$$W^i_j = H^i_j - H\delta^i_j - \frac{1}{n+1}\left(\dot{\partial}_k H^k_j - \dot{\partial}_j H\right)\dot{x}^i \quad \text{..................(1.3)}$$

respectively. Rund (1971 a)

The projective curvature tensor field satisfies the following identities.

$$W^i_{jkh}\dot{x}^j = W^i_{kh}, \quad W^i_{jkh}\dot{x}^j\dot{x}^k = W^i_h \quad \text{............................(1.4)}$$

$$W^i_{jk}\dot{x}^j = W^i_k, \quad W^i_k\dot{x}^k = 0, \quad \dot{\partial}_h W^i_k \dot{x}^k = -W^i_h \quad \text{............................(1.5)}$$

and

$$W^i_{jkh} + W^i_{khj} + W^i_{hjk} = 0 \quad \text{............................(1.6)}$$

Here W^i_k is homogeneous of the second degree in its directional arguments.

The tensor fields satisfy the following commutation formulae

$$T_{(h)(k)} - T_{(k)(h)} = -\dot{\partial}_r T H^r_{hk}, \qquad \text{.........................(1.7)}$$

$$T^i_{j(h)(k)} - T^i_{j(k)(h)} = -\dot{\partial}_r T^i_j H^r_{hk} - T^i_r H^r_{jhk} + T^r_j H^i_{rhk}, \qquad \text{.........................(1.8)}$$

$$\left(\dot{\partial}_k T\right)_{(h)} - \left(\dot{\partial}_k T_{(h)}\right) = 0 \qquad \text{.........................(1.9)}$$

and

$$\left(\dot{\partial}_k T^i_j\right)_{(h)} - \dot{\partial}_k T^i_{j(h)} = T^i_r G^r_{jkh} - T^r_j G^i_{rkh} \qquad \text{.........................(1.10)}$$

where H^r_{hk} and H^r_{jhk} are the Berwald's curvature tensor fields. The Connection coefficient G^i_{jk} is positively homogeneous of degree zero in $\dot{x}^i$ We also have $G^i_{jhk} = \dot{\partial}_j G^i_{kh}$, where G^i_{jkh} is symmetric with respect to its lower indices. Rund(1971 a)

In the recurrent Finsler space F_n, the projective tensor field and the deviation tensor field satisfy the recurrence relations:

$$W^i_{jk(\ell)(m)} = a_{\ell m} W^i_{jk}, \qquad \text{.......................(1.11)}$$

and

$$W^i_{k(\ell)(m)} = a_{\ell m} W^i_k, \qquad \text{.......................(1.12)}$$

where $a_{\ell m} \neq 0$ is recurrence tensor field (Sinha and Singh, 1973).

It is observed that the projective curvature tensor field W^i_{jkh} is birecurrent projective curvature tensor field under the condition $G^i_{jkh} = 0$.

Singh (1981) has defined generalised Finsler space of second order as follows: A Finsler space F_n, in which the Berwald curvature tensor field H^i_{jkh} satisfies the relation

$$H^i_{jkh(\ell)(m)} = H^i_{jkh(\ell)} K_m + H^i_{jkh} a_{\ell m}, H^i_{jkh} \neq 0, \qquad \text{.......................(1.13)}$$

where K_m and $a_{\ell m}$ are non zero associated recurrence vector and tensor fields, is called generalised recurrent Finsler space of second order. It is denoted by $G(2\text{-}F_n)$. Also the curvature tensor field is called G-2 recurrent tensor field.

Transvecting (1.13) successively by $\dot{x}^j$ and $\dot{x}^k$, we have

$$H^i_{kh(\ell)(m)} = H^i_{kh(\ell)} K_m + K^i_{kh} a_{\ell m} \qquad \text{.......................(1.14)}$$

and

$$H^i_{h(\ell)(m)} = H^i_{h(\ell)} K_m + H^i_h a_{\ell m} \qquad \text{.......................(1.15)}$$

Contracting with respect to the indices i and n in (1.13), (1.14) and (1.15), we get

$$H_{jk(\ell)(m)} = H_{jk(\ell)} K_m + H_{jk} a_{\ell m} \qquad \text{.......................(1.16)}$$

$$H_{k(\ell)(m)} = H_{k(\ell)} K_m + H_k a_{\ell m} \qquad \text{.......................(1.17)}$$

and

$$H_{(\ell)(m)} = H_{(\ell)} K_m + H \; a_{\ell m} \qquad \text{.......................(1.18)}$$

respectively.

2.0 G-2 - Recurrent Projective Tensor Fields

We consider a generalised Finsler space of second order $G(2\text{-}F_n)$ and study the recurrence properties of the projective curvature tensor fields, projective tensor field and the projective deviation tensor field in it. We discuss the properties of associated recurrence vector and tensor fields also in this section.

Theorem 2.1

In $G(2\text{-}F_n)$, the projective tensor field and the projective deviation tensor field satisfy the recurrence relations $W^i_{jk(\ell)(m)} = W^i_{jk(\ell)} K_m + W^i_{jk} a_{\ell m}$ and $W^i_{k(\ell)(m)} = W^i_{k(\ell)} K_m + W^i_k a_{\ell m}$ respectively.

Proof

Differentiating (1.2) twice covariantly and using the equations (1.14), (1.16) and (1.17), we get

$$\begin{aligned} W^i_{jk(\ell)(m)} &= H^i_{jk(\ell)} K_m + H^i_{jk} a_{\ell m} \\ &+ \frac{\dot{x}^i}{n+1}\left[H_{jk(\ell)} K_m + a_{\ell m} H_{jk} - H_{kj(\ell)} K_m - H_{kj} a_{\ell m}\right] \\ &+ \frac{\delta^i_j}{n^2-1}\left[nH_{k(\ell)} K_m + nH_k a_{\ell m} + \dot{x}^r H_{kr(\ell)} K_m + \dot{x}^r H_{kr(\ell)} a_{\ell m}\right] \qquad \text{................(2.1)} \\ &- \frac{\delta^i_k}{n^2-1}\left[nH_{j(\ell)} K_m + nH_j a_{\ell m} + \dot{x}^r H_{jr(\ell)} K_m + \dot{x}^r H_{jr} a_{\ell m}\right] \end{aligned}$$

From (1.2), the equation (2.1) yields

$$W^i_{jk(\ell)(m)} = W^i_{jk(\ell)} K_m + W^i_{jk} a_{\ell m} \qquad \text{................(2.2)}$$

Transvecting (2.2) by $\dot{x}^j$ and using the equation (1.5). We obtain

$$W^i_{k(\ell)(m)} = W^i_{k(\ell)} K_m + W^i_k a_{\ell m} \qquad \text{................(2.3)}$$

Definition 2.1

The projective tensor field W^i_{jk} and the deviation tensor field W^i_k satisfying the conditions (2.2) and (2.3) respectively are called generalised birecurrent projective deviation tensor fields. In brief, we write them as G-2 - recurrent projective deviation tensor fields.

Theorem 2.2

In $G(2\text{-}F_n)$, the necessary and sufficient condition for the relation $W^r{}_{[k} H^i_{r]} = W^i_{k\,[(\ell)} K_{m]} \dot{x}^\ell \dot{x}^m$ to be true is that $a_{[\ell m]} \dot{x}^\ell \dot{x}^m = 0$

Proof

Commuting the indices ℓ and m in (2.3) and subtracting the obtained result from it, we obtain

$$W^i_{k[(\ell)(m)]} - W^i_{k\,[(\ell)} K_{m]} + W^i_k a_{[\ell m]} \qquad \text{.....................(2.4)}$$

By virtue of the commutation formula (1.8), it yields

$$-W^i_{r\;k} H^r_{\ell m} - W^i_r H^r_{k\ell m} + W^r_k H^i_{r\ell m} = 2W^i_{k\,[(\ell)} K_{m]} + 2W^i_k a_{[\ell m]} \qquad \text{....................(2.5)}$$

Transvecting (2.5) by $\dot{x}^\ell \dot{x}^m$ and using properties of the curvature tensor H^i_{jk} and the deviation tensor H^i_j (Rund (1971 a)), we get

$$W^r{}_{[k} H^i_{r]} = \left(W^i_{k\,[(\ell)} K_{m]} + W^i_k a_{[\ell m]}\right) \dot{x}^\ell \dot{x}^m \qquad \text{....................(2.6)}$$

Let us assume that $a_{[\ell m]} \dot{x}^\ell \dot{x}^m = 0$, , then (2.6) reduces to

$$W^r{}_{[k} H^i_{r]} = W^i_{k\,[(\ell)} K_{m]} \dot{x}^\ell \dot{x}^m. \qquad \text{....................(2.7)}$$

Conversely, if the relation (2.7) is true, then (2.6) gives

$$W^i_k a_{[\ell m]} \dot{x}^\ell \dot{x}^m = 0. \qquad \text{....................(2.8)}$$

Since $W^i_k \neq 0$, , therefore the equation (2.8) implies

$$a_{[\ell m]} \dot{x}^\ell \dot{x}^m = 0. \qquad \text{....................(2.9)}$$

Remark 2.1

The associated recurrence tensor field $a_{\ell m}$ is not symmetric in general.

Theorem 2.3

In $G(2\text{-}F_n)$, the associated recurrence tensor $a_{\ell m}$ is homogeneous of degree zero in $\dot{x}^i$ when the associated recurrence vector K_m is independent of the line elements.

Proof

Applying partial derivative to the equation (2.3) with respect to $\dot{x}^j$, we get

$$\dot{\partial}_j W^i_{k(\ell)(m)} = \dot{\partial}_j W^i_{k(\ell)} K_m + W^i_k \dot{\partial}_j a_{\ell m} + W^i_{jk} a_{\ell m} \quad \text{.. (2.10)}$$

in view of (1.5) with the assumption that K_m is independent of $\dot{x}$.

From the commutation formula (1.10), the equations (1.5) and (2.2), the above equation becomes

$$W^r_{k(\ell)} G^i_{rjm} - W^i_{r(\ell)} G^r_{kjm} - W^i_{k(r)} G^r_{\ell jm} = W^r_k G^i_{rj\ell} - W^i_r G^r_{kj\ell} K_m + W^i_k \dot{\partial}_j a_{\ell m} \quad \text{................(2.11)}$$

Transvecting (2.11) by $\dot{x}^j$ and noting $G^i_{kjm}\dot{x}^j = 0$, we obtain

$$W^i_k \dot{\partial}_j a_{\ell m} \dot{x}^j = 0 \quad \text{................(2.12)}$$

since $W^i_k \neq 0$, the equation (2.12) establishes the theorem.

Cor. 2.1

In an affinely connected G(2-F_n), if the associated recurrence vector K_m is independent of the line elements and $G^i_{jkh} = 0$, then the associated recurrence tensor $a_{\ell m}$ is also independent of the line elements.

Proof

It is direct consequence of the equation (2.11) with the assumption $G^i_{jkh} = 0$, since $W^i_k \neq 0$.

Theorem 2.4

In an affinely connected G(2-F_n), if the associated recurrence vector K_m is independent of the line elements along with condition $G^i_{jkh} = 0$, then the projective curvature tensor field satisfies the relation

$$W^i_{jkh(\ell)(m)} = W^i_{jkh(\ell)} K_m + W^i_{jkh} a_{\ell m}$$

Proof

The covariant differentiation of (1.1) with respect to the indices ℓ and m yields.

$$W^i_{jkh(\ell)(m)} = H^i_{jkh(\ell)(m)} + \frac{\delta^i_j}{n+1}\left[H_{kh(\ell)(m)} - H_{hk(\ell)(m)}\right]$$

$$+\frac{\dot{x}^i}{n+1}\left[\dot{\partial}_j\left(H_{kh(\ell)(m)}\right)-\dot{\partial}_j\left(H_{hk(\ell)(m)}\right)\right]$$

$$+\frac{\delta^i_k}{n^2-1}\left[nH_{jh(\ell)(m)}+H_{hj(\ell)(m)}+\dot{x}^r\dot{\partial}_j\left(H_{hr(\ell)(m)}\right)\right] \qquad \text{.........................(2.13)}$$

$$-\frac{\delta^i_h}{n^2-1}\left[nH_{jk(\ell)(m)}+H_{kj(\ell)(m)}+\dot{x}^r\dot{\partial}_j\left(H_{kr(\ell)(m)}\right)\right]$$

by virtue of $G^i_{jkh}=0$ and the equation (1.10). In view of the equations (1.14), (1.16) and the Cor. 2.1 the equation (2.13) becomes

$$W^i_{jkh(\ell)(m)}=K_m\left\{H^i_{jkh(\ell)}+\frac{\delta^i_j}{n+1}\left[H_{kh(\ell)}-H_{hk(\ell)}\right]\right.$$

$$+\frac{\dot{x}^i}{n+1}\left[\dot{\partial}_j\left(H_{kh(\ell)}\right)-\dot{\partial}_j\left(H_{hk(\ell)}\right)\right]$$

$$+\frac{\delta^i_k}{n^2-1}\left[nH_{jh(\ell)}+H_{hj(\ell)}+\dot{x}^r\dot{\partial}_j\left(H_{hr(\ell)}\right)\right]$$

$$\left.-\frac{\delta^i_h}{n^2-1}\left[nH_{jk(\ell)}+H_{kj(\ell)}+\dot{x}^r\dot{\partial}_j\left(H_{kr(\ell)}\right)\right]\right\}$$

$$+a_{\ell m}\left[H^i_{jkh}+\frac{\delta^i_j}{n+1}\left(H_{kh}-H_{hk}\right)+\frac{\dot{x}^i}{n+1}\left(\dot{\partial}_jH_{kh}-\dot{\partial}_jH_{hk}\right)\right. \qquad \text{........(2.14)}$$

$$\left.+\frac{\delta^i_k}{n^2-1}\left(nH_{jh}+H_{hj}+\dot{x}^r\dot{\partial}_jH_{hr}\right)-\frac{\delta^i_h}{n^2-1}\left(nH_{jk}+H_{kj}+\dot{\partial}_jH_{kr}\right)\right]$$

Simplifying (2.14) by the equation (1.1), we get

$$W^i_{jkh(\ell)(m)}=W^i_{jkh(\ell)}K_m+W^i_{jkh}a_{\ell m} \qquad \text{...................(2.15)}$$

which proves the theorem.

Definition 2.2

The projective curvature tensor field W^i_{jkh} satisfying the condition (2.15) is called generalised birecurrent projective curvature tensor field.

Remark 2.2

Theorem 2.1 holds good in G(2-F_n) but Theorem 2.4 is true in affinely connected G(2-F_n) when the associated recurrence vector field K_m is independent of the line elements. Conversely if the project curvature tensor field W^i_{jkh} satisfies (2.15), then from (1.4) the projective tensor field W^i_{jk} and the projective deviation tensor field W^i_k satisfy (2.2) and (2.3), respectively.

Theorem 2.5

In G(2-F_n), if W^i_k is G-2-recurrent projective deviation tensor field which is neither zero nor a recurrent tensor field, then the associated recurrence vector and tensor fields are unique.

Proof

The equation (2.3) shows that the projective deviation tensor field W^i_k is G-2 - recurrent. Without loss of generality we can assume that it also satisfies the relation

$$W^i_{k(\ell)(m)} = W^i_{k(\ell)}K'_m + W^i_k a'_{\ell m} \quad \text{.................(2.16)}$$

Subtracting (2.16) from (2.3), we obtain

$$W^i_{k(\ell)}B_m + W^i_k A_{\ell m} = 0, \quad \text{.................(2.17)}$$

where $B_m = K_m - K'_m$ and $A_{\ell m} = a_{\ell m} - a'_{\ell m}$.

Here we discuss the following cases:

Case I

Let $K_m = K'_m$ and $a_{\ell m} \neq a'_{\ell m}$. Then $B_m \neq 0$ and $A_{\ell m} \neq 0$ which yields $W^i_k = 0$ from equation (2.17).

Case II

Let $K_m \neq K'_m$ and $a_{\ell m} = a'_{\ell m}$. Then $B_m \neq 0$ but $A_{\ell m} = 0$ which gives $W^i_{k(\ell)} = 0$ from (2.17) and hence $W^i_k = 0$.

Case III

Let $K_m \neq K'_m$ and $a_{\ell m} \neq a'_{\ell m}$. In this case both B_m and $A_{\ell m}$ are non zero. Let B'^m be any contravariant vector field such that $B'^m B_m \neq 0$, then the inner product with B'^m in (2.17) yields

$$W^i_{k(\ell)} = v_\ell W^i_k , \quad \text{..................(2.18)}$$

where $v_\ell = - A_{\ell m}B'^m / B_m B'^m$

In (2.18), if $v_\ell = 0$, then $W^i_{k(\ell)} = 0$ and hence $W^i_k = 0$

But if $v_\ell \neq 0$, then W^i_k is a recurrent projective deviation tensor field.

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