A Note on Interpolation of Stable Processes

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ABSTRACT

Interpolation procedures tailored for gaussian processes may not be applied to infinite variance stable processes. Alternative techniques suitable for a limited set of stable case with index $\alpha \in (1,2]$ were initially studied by Pourahmadi (1984) for harmonizable processes. This was later extended to the ARMA stable process with index $\alpha \in (0,2]$ by Nassiuma (1994). In this paper, the problem of interpolation of stable processes is studied with the aim of developing an algorithm applicable to general linear and nonlinear processes by using the state space formulation. Application of this procedure to the estimation of missing values is discussed.

KEYWORDS

Stable processes, dispersion, filtering, missing observations.

1.0 INTRODUCTION

A wide range of theories have been developed in relation to the state space representation of time series models. Of greatest interest has been the idea of extrapolation, filtering and interpolation. These developments have been exclusively based on the assumption that the underlying process is gaussian. A possible alternative is to consider processes with infinite variance stable errors. Such processes have been found appropriate for data emerging from a wide range of fields (Zolatarev, 1986).

Some missing value procedures for scalar gaussian systems have been studied by Miller and Ferreiro (1983) and Pourahmadi (1989). Techniques for the estimation of missing observations which are based on the state space representation of processes have been studied by Brockwell and Davis (1987] and Abraham and Thavaneswaran (1991) for gaussian processes. These later procedures are applicable to a more general set of linear and nonlinear time series models.

Initial work on interpolation for scalar processes having a stable distribution was carried out by Pourahmadi (1984) whereby he developed an interpolation procedure for harmonizable stable processes with characteristic index $\alpha \in (1,2]$. An approach applicable to symmetric stable processes with index $\alpha \in (0,2]$ was developed by Nassiuma (1994).

This article aims at studying interpolation procedures applicable to a wider class of scalar models with a symmetric stable distribution using the state space representation. Some simplifications in the results are observed and in particular, the covariation function given in Nassiuma (1994) is no longer necessary. The application of the developed procedure to the estimation of missing observations is also discussed.

In Section 2, we start by giving the extrapolation algorithm which was initially studied by Stuck (1978) (although he referred to it as a filtering algorithm). A simple proof of the filtering algorithm is also given. The interpolation procedure is then developed in Section 3 and its application to the estimation of missing observations is discussed in Section 4.

2.0 EXTRAPOLATION AND FILTERING ALGORITHMS

Consider a general state space model of the form

$$X_{t+1} = A_t X_t + B_t u_{t+1} \quad \text{......} \quad 1$$

$$Y_t = C_t X_t + D_t w_t \quad \text{......} \quad 2$$

where A_t, B_t, C_t and D_t are scalar coefficients which are linear or nonlinear functions of the past observations. The sequences $\{u_t\}$ and $\{w_t\}$ are mutually independent random variables defined on the probability space (Ω, F, P) and have scale parameters α_u and α_w respectively. The above two equations (1) and (2) are referred to as the state and the observation equations respectively. The state variable is usually not observed but is evaluated on the basis of the observations. In the case of gaussian systems, α_u and α_w represent variance functions of the respective random variables.

The prediction of scalar valued harmonizable stable processes was studied by Cambanis and Soltani (1983) while stable processes based on the markovian representation were studied by Stuck (1978). The main result obtained by Stuck is given in the following theorem.

Theorem 2.1

Suppose that the set of observations $S_t = (Y_1, Y_2, ..., Y_t)$ are available. Let $\hat{X}_t$ and $\hat{Y}_t$ be projections of X_t and Y_t respectively onto the observation space S_t in the $L^p(\Omega)$ space where $p \in (0,2]$. Using the state and observation equations (1) and (2), the prediction algorithm is obtained as

$$\hat{X}_{t+1} = A_t \hat{X}_t + K_t (Y_t - \hat{Y}_t) \quad \text{......} \quad 3$$

and the dispersion (Brockwell and Davis, 1987) of the prediction error is obtained as

$$\Omega_{t+1} = |(A_t - K_t C_t)|^\alpha \Omega_t + |B_t|^\alpha \alpha_u + |K_t D_t|^\alpha \alpha_w.$$

where K_t is the smoother given as

$$K_t = \frac{A_t}{C_t} \frac{\| C_t|^{\alpha} \Omega_t|^{1/(\alpha-1)}}{\| C_t|^{\alpha} \Omega_t|^{1/(\alpha-1)} + | D_t|^{\alpha/(\alpha-1)} | \alpha_w|^{1/(\alpha-1)}} \quad if\ 0 < \alpha \leq 2$$

$$= A_t / C_t \qquad if\ | c_t / D_t|^{\alpha} \Omega_t > \alpha_w, 0 < \alpha \leq 1$$

$$= 0 \qquad if\ | c_t / D_t|^{\alpha} \Omega_t < \alpha_w, 0 < \alpha \leq 1. \quad (4)$$

Proof: (Stuck 1978).

When the observations ($Y_1, Y_2, ..., Y_{t+1}$) are available and are to be used in evaluating the state variable X_{t+1}, the problem is that of deriving a filtering algorithm. Such an algorithm is given in the following theorem.

Theorem 2.2:

Let the observations $Y_1, Y_2, ..., Y_{t+1}$ be available and suppose that the state and the observation equations are as given in (1) and (2) respectively. The filtered estimate of X_{t+1} is then obtained as

$$\vec{X}_{t+1} = A_t \hat{X}_t + \vec{K}_{t+1}(Y_{t+1} - \hat{Y}_{t+1})$$

where the smoother is given as

$$\vec{K}_{t+1} = \frac{| C_t \Omega_{t+1}|^{1/(\alpha-1)}}{\| C_t|^{\alpha} \Omega_{t+1}|^{1/(\alpha-1)} + \| D_t|^{\alpha} \alpha_w|^{1/(\alpha-1)}} \quad if\ \alpha > 1$$

$$= 1/C_t \qquad if\ | C_t / D_t|^{\alpha} \Omega_{t+1} > \alpha_w, 0 < \alpha \leq 1$$

$$= 0 \qquad if\ | c_t / D_t|^{\alpha} \Omega_{t+1} < \alpha_w, 0 < \alpha \leq 1.$$

Proof:

The filtered value is the projection of X_{t+1} onto the observation space $Y_1, Y_2, ..., Y_{t+1}$ such that

$$\vec{X}_{t+1} = P[X_{t+1} | Y_1, Y_2, ..., Y_{t+1}]$$

$$= P[X_{t+1} | Y_1, Y_2, ..., Y_t] + P[. | Y_{t+1} - \hat{Y}_{t+1}]$$

$$= A_t \hat{X}_t + \vec{K}_{t+1}(Y_{t+1} - \hat{Y}_{t+1}).$$

The error of the filtered estimate is then evaluated as

$$X_{t+1} - \vec{X}_{t+1} = (1 - \vec{K}_{t+1} C_t)(X_{t+1} - \hat{X}_{t+1}) - \vec{K}_{t+1} D_t w_{t+1}$$

with its corresponding dispersion being

$$Disp(X_{t\cdot 1} - \vec{X}_{t\cdot 1}) = \vec{\Omega}_{t\cdot 1} = |1 - \vec{K}_{t\cdot 1} C_t|^{\alpha} \Omega_{t\cdot 1} + |\vec{K}_{t\cdot 1} D_t|^{\alpha} \alpha_w.$$

where Ω_t is as in the extrapolation algorithm. The derivative of the above dispersion with respect to the smoother when equated to zero leads to

$$\vec{\Omega}_{t'\cdot 1} = -\alpha C_t |1 - \vec{K}_{t\cdot 1} C_t|^{(\alpha-1)} \Omega_{t\cdot 1} + |\vec{K}_{t\cdot 1}|^{(\alpha-1)} |D_t|^{\alpha} \alpha_w = 0$$

and solving for $\vec{K}_{t\cdot 1}$ leads to the result as in the theorem.

3.0 INTERPOLATION ALGORITHM

The interpolation problem is such that either one or several observations are randomly missing from a given set and thus one develops ways of estimating the missing observation(s). This implies that there is lack of continuity in the observation space and we define it as $S^*_t = (Y_1, Y_2, ..., Y_{m-1}, Y_{m+1}, ..., Y_t)$. We also define the projection of the state component X_m onto this observation space as being equal to $X_{m|t}$. The interpolation algorithm is then given in the following theorem.

Theorem 3.1:

Let the observation Y_m be missing from a set of t possible observations ($m \leq t$). The estimate of the state component X_m based on minimizing the dispersion of the error of the estimate is then obtained as

$$X_{m|t} = X_{m|t-1} + K^*_t (Y_t - \hat{Y}_t)$$

with the dispersion of the error being obtained recursively as

$$\Omega^*_{m|t} = \Omega_{m|t-1} + |K^*_t C_t|^{\alpha} \Omega_t + |K^*_t D_t|^{\alpha} \alpha_w.$$

where the smoother $K^*_t \neq 0$ 1only if $t = m+1$ and

$$K^*_{m+1} = \frac{|C_{m+1} (A_m - K_m C_m) \Omega_m|^{1/(\alpha-1)}}{||C_{m=1} (A_m - K_m C_m)|^{\alpha} \Omega_m|^{1/(\alpha-1)} + |\alpha_w (|K_m C_{m=1} D_m|^{\alpha} + |D_{m+1}|^{\alpha}) + |B_m C_{m+1}|^{\alpha} \alpha_u|^{1/(\alpha-1)}}$$

if $\alpha > 1$

$$= \frac{1}{C_{m+1}(A_m - K_m C_m)}$$

If $|c_{m=1}(A_m - K_m C_m)|^{\alpha} \Omega_m > \alpha_w (|K_m C_{m+1} D_m|^{\alpha} + |D_{m+1}|^{\alpha}) + |B_m C_{m+1}|^{\alpha} \alpha_u, \; 0 < \alpha \leq 1$

$$= 0$$

if $|C_{m=1}(A_m - K_m C_m)|^{\alpha} \Omega_m < \alpha_w (|K_m C_{m+1} D_m|^{\alpha} + |D_{m+1}|^{\alpha}) + |B_m C_{m+1}|^{\alpha} \alpha_u, \; 0 < \alpha \leq 1$

Proof:

The interpolated state value $X_{m|t}$ is the projection of X_m onto the observation space S_t^*. This implies that we have

$$X_{m|t} = P_{S_t^*}(X_m)$$

$$\simeq A_{m-1} X_{m-1|t-1} + K_t^*(Y_t - \hat{Y}_t)$$

$$= X_{m|t-1} + K_t^*(Y_t - \hat{Y}_t).$$

The error of the interpolated value is then evaluated as

$$X_m - X_{m|t} = [1 - K_{m+1}^* C_{m+1}(A_m - K_m C_m)](X_m - X_{m|t})$$

$$- K_{m+1}^*[K_m C_{m+1} D_m w_m + D_{m+1} w_{m+1} + B_m C_{m+1} u_{m+1}]$$

with its dispersion being

$$Disp(X_m - X_{m|t}) = |1 - K_{m+1}^* C_{m+1}(A_m - K_m C_m)|^\alpha \Omega_m$$

$$+ [|K_{m+1}^* K_m C_{m+1} D_m|^\alpha + |K_{m+1} D_{m+1}|^\alpha] \alpha_w + |K_{m+1} B_m C_{m+1}|^\alpha \alpha_u$$

where K_m and Ω_m are as in the extrapolation algorithm. The solution is then obtained by first of all taking the derivative of the above dispersion with respect to the smoother. This is then equated to zero and solved for the smoother and the result is easily obtained as in the theorem.

4.0 Estimation of Missing Observations

The determination of missing observations is usually necessary for the application of most of the popular computational techniques in time series. This essentially implies that for data that has some observations which are missing, one has to first of all estimate their values. An approach applicable in the analysis of irregularly observed data from a gaussian system without having to evaluate the missing observations was discussed by Nassiuma and Thavaneswaran (1991). This approach was developed with application to ARMA models and it could get quite complicated for more general time series models especially for non-gaussian processes.

In this Section, we consider the application of the interpolation algorithm developed in Section 3 to the estimation of missing observations. For simplicity, we follow the approach of Brockwell and Davis (1987) by using the relation $Z_t = C_t X_t$ in addition to the state and observation equations given in (1) and (2). When the observation Y_m is missing, it is represented by the random variable w_t which implies that under these circumstances, we have $C_m = 0$ and $D_m = 1$. If on the other hand the observation is not missing, we have $Y_t = Z_t = C_t X_t$ which in this case implies that for $t \neq m$, $D_t = 0$ and $C = 1$. It easily follows that $K_m = 0$. Applying these relations to the interpolation algorithm given in the previous Section, we obtain the smoother in a simplified form as

$$K^{*}_{m+1} = \frac{|A_m \Omega_m|^{1/(\alpha-1)}}{|A_m|^{\alpha} \Omega_m|^{1/(\alpha-1)} + |B_m|^{\alpha} \alpha_u|^{1/(\alpha-1)}} \quad if\ \alpha > 1$$

$$= \frac{1}{A_m} \qquad if\ |A_m|^{\alpha} \Omega_m > |B_m|^{\alpha} \alpha_u, 0 < \alpha \leq 1$$

$$= 0 \qquad if\ |A_m|^{\alpha} \Omega_m < +|B_m|^{\alpha} \alpha_u, 0 < \alpha \leq 1.$$

To now obtain an estimate of the missing value, we consider the state variable as representing the missing component and we thus substitute $Z_{m|t}$ for $X_{m|t}$ which leads to the estimate of the missing observation as

$$Z_{m|t} = \hat{Z}_m + K^{*}_{m+1}(Y_{m+1} - \hat{Y}_{m+1}).$$

In the case of a first order autoregressive process with parameter ϕ, in addition to the above conditions, we have $A_m = \phi$ and $B_t = 1$ and the estimated missing observation is of the form

$$Z_{m|t} = \phi Y_{m-1} + \frac{|\phi \Omega_m|^{1/(\alpha-1)}(Y_{m+1} - \hat{Y}_{m+1})}{|\phi|^{\alpha} \Omega_m|^{1/(\alpha-1)} + |\alpha_u|^{1/(\alpha-1)}} \quad if\ \alpha > 1$$

$$= \phi Y_{m-1} + \frac{(Y_{m+1} - \hat{Y}_{m+1})}{\phi} \quad if\ |\phi|^{\alpha} \Omega_m > \alpha_u, 0 < \alpha \leq 1$$

$$= 0 \qquad if\ |\phi|^{\alpha} \Omega_m < \alpha_u, 0 < \alpha \leq 1.$$

When $\alpha = 2$ and $\Omega_m = \alpha_u$, we obtain the gaussian system estimate as

$$Z_{m|t} = \phi Y_{m-1} + \frac{\phi \Omega_m (Y_{m+1} - \hat{Y}_{m+1})}{\phi^2 + 1}$$

$$= \phi Y_{m-1} + \frac{\phi \Omega_m (Y_{m+1} - \hat{Y}_{m+1})}{\phi^2 + 1} \quad if\ |\phi^2 > 1, 0 < \alpha \leq 1$$

$$= 0 \qquad if\ |\phi^2 > 1, 0 < \alpha \leq 1$$

ch is a familiar result. It is however important to note here that the estimate reduces to Y_{m+1}/ϕ in the stationary stable case for $0 < \alpha \leq 1$ otherwise the estimate is zero.

In conclusion, the algorithm developed here facilitate the estimation of missing observations for a e range of scalar valued linear and nonlinear models with stable distributions. The estimation of eral missing observations follows easily from the above results.

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