INVESTIGATING THE MAXIMUM DETERMINANT FOR AN NXN MATRIX

***I. A. Okello*[1], *C. W. Mwathi*[2] *and B. Kivunge*[3]**
[1,2]*Department of Pure and Applied Mathematics, Jomo Kenyatta University of Agriculture and Technology, Nairobi*
[3]*Kenyatta University, Nairobi*
E-mail: ireneokello@yahoo.com

Abstract

It is known that for any $n \times n$ matrix $A_n = (a_{ij})$; $|a_{ij}| \leq m$, $m \in \mathbb{Z}^+$, $|detA_n| \leq m^n n^{n/2}$, (Garling, 2007). Therefore, $m^n n^{n/2}$ is an upper bound of determinants of all matrices A_n which satisfy the above conditions.

In this research, we determine the maximum determinant of an $n \times n$ matrix A_n, where $A_n = (a_{ij})$, $a_{ij} \in \{0,1,2,3\}$, for n =1, 2, 3, 4 and 5 using the determinant function formula and expansion using minors. For an $n \times n$ $\{0,1,2,3\}$–matrix, the maximum determinants for $n = 1,2,3,4,5$ were found to be 3,9,54,243 and 972 respectively. The number of distinct $\{0,1,2,3\}$–matrices attaining the maximum determinant for n = 1,2,3,4,5 are 1,14,6,24 and 120 respectively. For an $n \times n$ matrix $A_n = (a_{ij})$; $a_{ij} \leq m$, $n, m \in \mathbb{Z}^+$, $|det\ A_n| \leq (n-1)m^n$ with equality if and only if A_n has one and only one zero entry in each row and one and only one zero entry in each column, all the other entries in this matrix are equal to m. The number of distinct such matrices attaining the maximum determinant is $n!$, $n > 2$.

Key words: n × n – matrix, maximum determinant, supremum determinant, minor matrix.

1.0 Introduction

Let A be an $n \times n$ matrix with entries $|a_{ij}| \leq m$. Hadamard's maximum determinant problem asks how large can the absolute value of the determinant of A be? It was shown by Hadamard (1893) that, if all elements of an $n \times n$ matrix of complex numbers have absolute value at most m, then the determinant of the matrix has absolute value at most $m^n n^{n/2}$. For each positive integer n there exist complex $n \times n$ matrices for which this upper bound is attained. For example, the upper bound is attained for $m = 1$ by the matrix $A = \omega^{ij}$ $1 \leq i, j \leq n$, where ω is a primitive n^{th} root of unity or a Vandermonde matrix of the n^{th} root of unity. This matrix is real for $n = 1,2$. However, Hadamard also showed that if the upper bound is attained for a real $n \times n$ matrix, where $n > 2$, then n must be divisible by 4. And equality is attained if $|a_{ij}| = m, \forall\, i, j = 1,2, \ldots, n$ and $A^T A = m^2 n I_n$.

Without loss of generality one may suppose $m = 1$. A real $n \times n$ matrix for which the upper bound $n^{n/2}$ is attained in this case is today called a Hadamard matrix. It is still an open question whether an $n \times n$ Hadamard matrix exists for every positive integer n divisible by 4. However, Hadamard also showed that if the upper bound is attained for a real $n \times n$ matrix, where *n* > *2*, then *n* is divisible by 4.

1.1 Literature Review

For a $\{0,1\}$ −matrix, Hadamard's bound can be improved to $|detA| \leq \frac{(n+1)^{(n+1)/2}}{2^n}$. The largest possible determinant β_n for $n = 1,2, \ldots$ are 1,1,2,3,5,9,32,56,144,320, 1458,3645,9477,.... The numbers of distinct $n \times n$ binary matrices having the largest possible determinant for $n = 1,2,\ldots$ are 1,3,3,60,3600,529299,75600...(Williamson, 1946).

The numbers of distinct $n \times n$ $\{-1,1\}$ −matrices having the largest possible determinant for $n = 1,2, \ldots$ are 1, 4, 96, 384 ... α_n is related to the largest possible $\{0,1\}$ − matrix determinant β_{n-1}, by $\alpha_n = 2^{n-1}\beta_{n-1}$, where α_n is the largest possible determinant for $\{-1,1\}$ −matrices (Brenner and Cummings, 1972).

For an $n \times n$ $\{-1, 0, 1\}$ − matrix, the largest possible determinant is equal to the maximum determinant for $\{-1,1\}$ − matrices. The numbers of $n \times n$ {-1, 0, 1}− matrices having maximum determinants are 1, 4, 240 .... , this was done by Brenner and Cummings (1972).

2.0 Methodology

This paper first computes the maximum determinants of $n \times n$ matrix with entries $a_{ij} = 0, 1, 2, 3$ for $n = 1,2,3,4$ and 5 using the determinant function formula $|det\ A_n| = |\sum_P sgnP\ a_{1j_1}a_{2j_2} \dots a_{nj_n}|$ and expansion using minors. It then studies the properties of matrices with the maximum determinant and generalizes the result.

W.L.O.G suppose $n = 3$ and let A_3 be the 3×3 –matrix ,then

$|detA_3| = |a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}|$, $a_{ij} \in \{0,1,2,3\}, 1 \leq i, j \leq 3$. ……………………………………………………………………………………(3.1)

This can be expressed as

$$|detA_3| = \left| a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \right| = |(a_{11}|A_{3,11}| + a_{12}(-|A_{3,12}|) + a_{13}|A_{3,13}|)| \quad \text{…(3.2)}$$

where $A_{3,1j}$ $1 \leq j \leq 3$ is the minor 2×2 matrix obtained by deleting row 1 and column j from A_3.

For $|detA_3|$ to be maximum, then the determinants of $A_{3,11}, A_{3,12}, A_{3,13}$ should either be maximum positive or maximum negative.

$|A_{3,11}|$ is maximum when $a_{22}a_{33} = 0$ implying that either $a_{22} = 0$ or $a_{33} = 0$ or $a_{22} = a_{33} = 0$ and $a_{23} = a_{32} = 3$. Or $a_{23}a_{32} = 0$ implying that either $a_{23} = 0$ or $a_{32} = 0$ or $a_{23} = a_{32} = 0$ and $a_{22} = a_{33} = 3$.

W.L.O.G suppose $|A_{3,11}|$ is maximum positive then $a_{23}a_{32} = 0$ and $a_{22} = a_{33} = 3$. Suppose $a_{23} = 0$

The determinant of A_3 reduces to,

$|detA_3| = |a_{11}(3.3 - 0.a_{32}) + a_{12}(0.a_{31} - a_{21}.3) + a_{13}(a_{21}a_{32} - 3a_{31})|$……………….(3.3)

$|detA_3| = |9a_{11} + a_{12}(-3a_{21}) + a_{13}(a_{21}a_{32} - 3a_{31})|$…………………………………(3.4)

$|detA_3| = |9a_{11} - 3a_{12}a_{21} + a_{13}a_{21}a_{32} - 3a_{13}a_{31}|$……………………………………(3.5)

The possible values of $a_{ij} = 0,1,2$ and 3. For $|detA_3|$ to be maximum, either case 1 the positive products $9a_{11}$ and $a_{13}a_{21}a_{32}$ are maximum possible while negative products $3a_{12}a_{21}$ and $3a_{13}a_{31}$ are minimum possible or case 2 the positive products $9a_{11}$ and $a_{13}a_{21}a_{32}$ are minimum possible while negative products $3a_{12}a_{21}$ and $3a_{13}a_{31}$ are maximum possible.

If we consider case 1, the products $9a_{11}$ and $a_{13}a_{21}a_{32}$ where $a_{11}, a_{13}, a_{21}, a_{32}$ are in {0,1,2 or 3} are maximum when $a_{11} = a_{13} = a_{21} = a_{32} = 3$.

To get maximum $|det\ A_3|$, $-3a_{12}a_{21} = 0$ and $-3a_{13}a_{31} = 0$ and this is only possible if $a_{12} = a_{31} = 0$ since $a_{13} = a_{21} = 3$.

From the above analysis, $|detA_3| = |9a_{11} - 3a_{12}a_{21} + a_{13}a_{21}a_{32} - 3a_{13}a_{31}|$ is maximum when $a_{11} = 3, a_{12} = 0, a_{13} = 3, a_{21} = 3, a_{22} = 3, a_{23} = 0, a_{31} = 0, a_{32} = 3, a_{33} = 3$, which gives the matrix $\begin{pmatrix} 3 & 0 & 3 \\ 3 & 3 & 0 \\ 0 & 3 & 3 \end{pmatrix}$ with $|detA_3| = 54$. The other matrices attaining the maximum determinant in modulus can be obtained by permuting the columns of the above matrix.

By applying the same method, the maximum determinant for $n = 2,4$ and 5 were computed and the matrices shown in Table I were found to have maximum determinant. Similarly the other matrices attaining the maximum determinant in modulus can be obtained by permuting the columns of the above matrix.

Consider $det\ A_n$,

$|detA_n| = \left|a_{11}|A_{n,11}| + a_{12}(-|A_{n,12}|) + \cdots + a_{1n}(-1^{n+1})|A_{n,1n}|\right|, a_{ij} \leq m, m, n \in \mathbb{Z}^+$ ...(3.6) Let P be the number of positive products in $det\ A_n$ which does not contain any fixed entry, a_{ii} and N be the number of negative products in $det\ A_n$ which does not contain any fixed entry, a_{ii}.

3.0 Proposition

The difference between the number of positive products and the number of negative products of $det\ A_n$ which does not contain any fixed entry, a_{ii}, of A_n in $A_{n,11}$ is $P - N = (n-2)$ while the difference between the number of positive products and the number of negative products of $det\ A_n$ which does not contain any fixed entry, a_{ii}, of A_n in $A_{n,1i}$ is $P - N = -1,\ 2 \leq i \leq n$.

3.1 Proof (by induction)

For $n = 1$ is trivial.

Suppose true for n, then for $n+1$ we have $|detA_{n+1}| = |(a_{11}|A_{n+1,11}| + a_{12}(-|A_{n+1,12}|) + \cdots + a_{1n}(-1^{n+1})|A_{n+1,1n}| + a_{1n+1}(-1^{n+2})|A_{n+1,1n+1}|)|$.....(3.7)

$|A_{n+1,11}|$ is similar to $detA_n = a_{11}|A_{n,11}| + a_{12}(-|A_{n,12}|) + \cdots + a_{1n}(-1^{n+1})|A_{n,1n}|$

with row $1,2,3 \ldots n$ of $detA_n$ corresponding to row $2,3,4 \ldots, n, n+1$ respectively. Therefore, $|A_{n+1\ 11}| = a_{22}|A_{n+1,22}| + a_{23}(-|A_{n+1,23}|) + \cdots + a_{2n}(-1^{n+2})|A_{n+1,2n}| + a_{2\ n+1}(-1^{n+3})|A_{n+1\ 2\ n+1}|$........(3.8) where $A_{n+1,2j}, 2 \leq j \leq n+1$ is the minor matrix of $A_{n+1,11}$. The first expansion term $a_{22}|A_{n+1,22}|$ contains the fixed entry a_{22}, therefore all the products in $|A_{n+1,22}|$ will have a fixed entry a_{22}. In the remaining $n-1$ expansion terms, the difference between the number of positive products and the number of negative products which does not contain any fixed entry a_{ii} is -1. This gives the total difference between the number of positive products and the number of negative products containing no fixed entry a_{ii} in $|A_{n+1,11}|$ to be $-(n-1)$.

$|A_{n+1,1i}|$, $2 \leq i \leq n+1$ is similar to $|A_{n+1,11}|$ where the columns $2,3, \ldots n+1$ corresponds to the columns 1,2,...,n respectively, that is pre-multiplying the permutations of $|A_{n+1,11}|$ by the transpositions $(1i)$, $2 \leq i \leq n+1$, this gives $n-1$ transpositions. Thus the identity permutation of $|A_{n+1,11}|$ becomes even permutation when n is odd and odd permutation when n is even in $|A_{n+1,1i}|$.

Similarly all the positive products become negative and all the negative products become positive after pre-multiplying by the transpositions.

W.L.O.G suppose $i = 2$, $|A_{n+1,12}|$ can be expressed as

$|A_{n+1,12}| = a_{21}(-|A_{n+1,21}|) + a_{23}|A_{n+1,23}| + \cdots + a_{2n}(-1^{n+2})|A_{n+1,2n}| + a_{2n+1}(-1^{n+3})|A_{n+1,2n+1}|$ ...(3.9). Now $|A_{n+1,21}|$ is similar to $|A_{n,11}|$ in $detA_n$ which had the difference between the number of positive products and the number of negative products containing no fixed entry a_{ii} being $P - N = n-2$. Then $-|A_{n+1,21}|$ will have the difference between the number of positive products and the number of negative products being $-(n-2)$. $|A_{n+1,2i}|$, $3 \leq i \leq n+1 = n-1$ expansion terms are similar to $|A_{n,1i}|, 2 \leq i \leq n = n-1$ expansion terms in $detA_n$ of which each had a difference of -1 giving the total difference to be $-(n-1)$. Since the permutations of $detA_n$ are pre-multiplied by the transposition (12), to obtain the corresponding permutations of

$\left|A_{n+1,12}\right|$, all the positive products become negative and vice versa. Thus the total difference $P - N = n - 1$ in $\left|A_{n+1,2i}\right|$, $3 \leq i \leq n + 1$. This gives the total difference in the number to be $P - N = (n - 1) - (n - 2) = 1$ in $\left|A_{n+1,12}\right|$. ……………………………………………(3.10)

Similar arguments for $i = 3,4, \dots, n + 1$ produces the same results. Therefore for each $\left|A_{n+1,1i}\right|, 2 \leq i \leq n + 1$, the difference between the number of positive products and the number of negative products which does not contain any fixed entry a_{ii} is 1.

3.2 Proposition

For $n > 2$, the determinant $|detA_n| \leq (n - 1)m^n$ with equality attained iff A_n has one and only one zero entry in every row and one and only one zero in every column i.e. one of the products in the determinant function has all its entries equal to zero, while the other entries not found in this product are equal to m. The number of matrices attaining the maximum determinant is $n!$.

3.1 Proof (by induction)

$$|detA_n| = |(a_{11}\left|A_{n,11}\right| + a_{12}(-\left|A_{n,12}\right|) + \cdots + a_{1n}(-1^{n+1})\left|A_{n,1n}\right|)|$$

For $n = 3$,

$|detA_3| = |a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}|$……………………………………. (3.11)

$|detA_3|$ is maximum when all the entries of one of its products are equal to zero and the remaining entries are equal to m. This gives a determinant of $2m^3$ with one of the matrix being $\begin{pmatrix} m & 0 & m \\ m & m & 0 \\ 0 & m & m \end{pmatrix}$.

Suppose true for n, then, max $|detA_n| = |(a_{11}\left|A_{n,11}\right| + a_{12}(-\left|A_{n,12}\right|) + \cdots + a_{1n}(-1^{n+1})\left|A_{n,1n}\right|)| = (n - 1)m^n$……………………………………………………………(3.12)

with all the entries of one of its products equals zero and any other entry not found in this product is equal to m.

For $n + 1$ we have $|detA_{n+1}| = |(a_{11}\left|A_{n+1,11}\right| + a_{12}(-\left|A_{n+1,12}\right|) + \cdots + a_{1n}\left|A_{n+1,1n}\right| + a_{1n+1}(-1^{n+2})\left|A_{n+1,1n+1}\right|)|$3.13 for $|detA_{n+1}|$ to be maximum, $a_{1i}, \left|A_{n+1,1i}\right|$ for i odd and $-\left|A_{n+1,1i}\right|$ for i even, $1 \leq i \leq n + 1$, should be maximum positive or maximum negative. Now $A_{n+1,11}$ is similar to A_n with row

1,2,3, ... , n of $detA_n$ corresponding to row 2,3,4, ... , $n, n+1$ respectively and column 1,2,3, ... , n of $detA_n$ corresponding to column 2,3,4, ... , $n, n+1$ respectively. Also $|A_{n+1,11}|$ is similar to $detA_n$ where the terms in the expansion corresponds respectively.

Hence,

$\max|A_{n+1,11}| = \max detA_n = \max|(a_{11}|A_{n,11}| + a_{12}(-|A_{n,12}|) + \cdots + a_{1n}(-1^{n+1})|A_{n,1n}|)|$....(3.14) therefore

$|A_{n+1,11}| = a_{22}|A_{n+1,22}| + a_{23}(-|A_{n+1,23}|) + \cdots + a_{2n}(-1^{n+2})|A_{n+1,2n}| + a_{2n+1}(-1^{n+3})|A_{n+1,2n+1}|$ ..(3.15), is maximum iff all the entries of one of its products are zero and the other entries not found in this product are equal to m. W.L.O.G, suppose $a_{22} = a_{33} = \cdots = a_{n+1n+1} = 0$, then any other product of $|A_{n+1,11}|$ containing a fixed entry a_{ii}, $2 \leq i \leq n+1$, will be reduced to zero and the maximum absolute value of $|A_{n+1,11}|$ is given by the difference in number of positive products and negative products which does not contain any fixed entry a_{ii}. $|A_{n+1,11}|$ has a difference of $(n-1)$ products which does not contain a fixed entry each product containing n entries. The maximum value each product can have is therefore m^n obtained when each entry in these products equal to m. Therefore the maximum value of $|A_{n+1,11}| = \pm(n-1)m^n$.

$|A_{n+1,1i}|$, $2 \leq i \leq n+1$ each have a difference of $P - N = -1$ products which does not contain any fixed entry giving a total of n products each containing n entries and each entry can take a maximum value of m. This gives a maximum $|detA_{n+1,1i}| = \mp m^n$ $2 \leq i \leq n+1$ when all the entries contained in these products are equal to m.

When $|A_{n+1,11}| = (n-1)m^n$, $|A_{n+1,1i}| = -m^n$, $2 \leq i \leq n+1$ as seen earlier. Since the negative values is greater than the positive value, the maximum determinant in absolute value is obtained when the positive term is zero which is only possible when $a_{11} = 0$ and the negative term is maximum possible obtained when $a_{1i} = m$, $2 \leq i \leq n+1$.

When $|A_{n+1,11}| = -(n-1)m^n$, $|A_{n+1,1i}| = m^n$, $2 \leq i \leq n+1$ as seen earlier. Since the positive value is greater than the negative value, the maximum determinant in absolute value is obtained when the negative term is zero which is only possible when $a_{11} = 0$ and the positive term is maximum possible obtained when $a_{1i} = m$, $2 \leq i \leq n+1$.

Therefore for $|detA_{n+1}| = |a_{11}(\pm(n-1)m^n) + a_{1i}(\mp m^n)|,\ 2 \le i \le n+1$, maximum $|detA_{n+1}|$ is obtained when $a_{11} = 0$ and $a_{1i} = m,\ 2 \le i \le n+1$. This gives $|detA_{n+1}| = |0.(\pm(n-1)m^n) + \sum_n m(\mp m^n)| = nm^{n+1}$. Hence true for all values of n.

Note: If two rows/columns in a matrix A_n are interchanged, the determinant is multiplied by -1 thus $|detA_n|$ is the same.

Here, $\begin{pmatrix} 0 & m & \cdots & m \\ m & 0 & & m \\ \vdots & & \ddots & \vdots \\ m & m & \dots & o \end{pmatrix}$ has maximum determinant. The other matrices are purely the permutations of the rows/columns of this matrix. There are $n!$ permutations of this matrix hence there are $n!$ matrices with $|detA_n| = (n-1)m^n$.

Table I

$\boldsymbol{n}$	**No. of matrices with maximum determinant**	**Maximum Determinant**	**Supremum Determinant of all matrices $\boldsymbol{m^n n^{n/2}}$**	**Example of a matrix with maximum determinant**
1	1	3	3	(3)
2	14	9	18	$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$
3	6	54	$81\sqrt{3}$	$\begin{pmatrix} 0 & 3 & 3 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{pmatrix}$
4	24	243	1296	$\begin{pmatrix} 3 & 3 & 3 & 0 \\ 3 & 3 & 0 & 3 \\ 3 & 0 & 3 & 3 \\ 0 & 3 & 3 & 3 \end{pmatrix}$
5	120	972	$6075\sqrt{5}$	$\begin{pmatrix} 0 & 3 & 3 & 3 & 3 \\ 3 & 0 & 3 & 3 & 3 \\ 3 & 3 & 0 & 3 & 3 \\ 3 & 3 & 3 & 0 & 3 \\ 3 & 3 & 3 & 3 & 0 \end{pmatrix}$
n	$n!$	$(n-1)m^n$	$m^n n^{n/2}$	$\begin{pmatrix} 0 & m & \cdots & m \\ m & 0 & & m \\ \vdots & & \ddots & \vdots \\ m & m & \dots & o \end{pmatrix}$

4.0 Conclusion

The main objective was to determine the maximal determinant of a matrix $A_n = (a_{ij}),\quad a_{ij} \in \{0,1,2,\dots,m\}$. The paper has established that for $n > 1$, $|detA_n| \le (n-1)m^n$ with equality attained by matrices with 1 and only 1 zero entry in every row and 1 and only 1 zero entry in every column.

References

Anton H., Roress C. (1994). Elementary Linear Alegbra Application Version 7th edition. Anton Textbooks Inc. New York.

Brenner J. and Cummings L. (1972).*The Hadamard Maximum Determinant Problem.* The American Mathematical Monthly, **79**, pp. 626-630.

Brenner J. (1972). *The Hadamard Maximum Determinant Problem*. The American Mathematical Monthly, **29**.

Garling D. J. H, (2007). *Inequalities: A Journey into Linear Analysis*. Cambridge. USA .pp. 233

Harvey E. R. (2002). Linear Algebra A pure Mathematical Approach, pp 57-60. Birkh*ä*user. Switzerland.

Orrick P W. (2004). *The Maximal (-1,1)-determinant of order 15*. International journal for theoretical and applied statistics, **62(2-3),** pp 195-219.

Orrick P. W., Solomon B. (2007). *Large Determinant Sign Matrices of order* $4k + 1$*.* Discrete Mathematics, **307(2)**, *p*p 226-236.

Orrick P., William S., Bruce D. Roland, and Warrem S. D., (2008), New Lower Bounds for the Maximal Determinant Problem, *Discrete Mathematics*, **309(5)**, pp 533-550. Elsevier.

Wiliamson J. (1946). *Determinants whose elements are zero and 1.* The American Mathematical Monthly, **53**, pp. 427 – 434.