NECESSARY CONDITIONS FOR EXISTENCE OF A FRIEND OF 38

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Abstract

This paper sets out to, systematically, use properties of the abundancy index function to prove that a friend m of 38 must be an odd non-square multiple of 19^2 which is not divisible by 3, and that every prime factor q of m such that $4|(q+1)$ has an even exponent in the prime factorization of m. In addition, if the power of 19 in m is 2, then $127|m$ in which case the power of 127 must be even, larger than 2 and not equal to 8, and if the power of 19 is 6, both 701 and 70841 would be compulsory prime factors of m, where the power of 701 cannot equal 1 or 3. The paper also establishes that it is not possible to have 8 as the power of 19 in the prime factorization of m.

Key words: Abundancy index, sum of divisors, friend, prime factor, power

1.0 Introduction

Let a be a positive integer, and $\sigma(a)$ denote the sum of the positive factors of a. If $a = \prod_{i=1}^{k} p_i^{\alpha_i}$, where $p_1, p_2, \ldots, p_k$ are distinct primes, k is a positive integer, and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are nonnegative integers, then

$$\sigma(a) = \sigma\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right) = \prod_{i=1}^{k}\left(\sum_{j=0}^{\alpha_i} p_i^j\right) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1}-1}{p_i-1} \quad \ldots\ldots(1)$$

(Ward, 2008; Ryan, 2009; Ryan, 2006), where σ is a multiplicative function (i.e. if x and y are relatively prime then $\sigma(xy) = \sigma(x)\sigma(y)$), (Rosen, 1993; Ryan, 2006; Dris, 2008; Ryan, 2009). The abundancy ratio/abundancy index of a, denoted $I(a)$, is defined by $I(a) = \frac{\sigma(a)}{a}$, and in this case, $I(a)$ is given by

$$I(a) = I\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right) = \prod_{i=1}^{k} I(p_i^{\alpha_i}) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1}-1}{p_i^{\alpha_i}(p_i-1)} \quad \ldots\ldots(2)$$

(Ryan, 2006; Ryan, 2009)

1.1 Friendly Numbers and Solitary Numbers

Two distinct positive integers m and n are friends if $I(m) = I(n)$ and the pair $\{m, n\}$ is called a friendly pair. Numbers that are not friendly are called solitary. If m and n are friends and k a positive integer coprime to both m and n, then mk and nk are friends (Ward, 2008). If $m|n$, then m and n cannot be friends (Ward, 2008). All primes, prime powers and all positive integers n, $(n, \sigma(n)) = 1$ are solitary (Dris, 2008). There are also numbers such as $n = 18$, 45, 48, and 52 which are solitary but for which $(n, \sigma(n)) \neq 1$ (Dris, 2008). However, there exist numbers such as 10, 14, 22, 26, 34 and 38, whose categorization as either friendly or solitary is undetermined (Weinstein, 2011). In fact, numbers of the form $n = 2p$ for primes $p > 3$ have so far not been classified as either friendly or solitary. According to Ward (2008), if n is a friend of 10 then it must be a square with at least 6 distinct prime factors, the smallest being 5, such that at least one of the prime factors must be congruent to 1 modulo 3 and appear with an exponent congruent to 2 modulo 6 in the prime power factorization of n. This paper is intended to explore some necessary conditions for existence of a friend of 38.

2.0 Method

2.1 Some Basic Properties of the Abundancy Index

Let m and n be positive integers.

$I(n) \geq 1$ with equality only if $n = 1$. (Ward, 2008)
If $m|n$, then $I(m) \leq I(n)$ with equality only if $m = n$. (Weiner, 2000; Ward 2008)

I is multiplicative, i.e. if $(m,n) = 1$, then $I(mn) = I(m)I(n)$. (Ryan, 2006; Ryan, 2009; Ward, 2008)
$I(p^a)$ is an increasing function of a when p is fixed but it is a decreasing function of p when a is fixed (Laatsch, 1986).
If the distinct prime factors of n are p_1, p_2,..., p_k, then $I(n) < \prod_{i=1}^{k} \frac{p_i}{p_i - 1}$. This follows from definition of $I(n)$, and the fact that for $p > 1$, $\frac{p^{e+1}-1}{p^{e+1}-p^e} = \frac{p-\frac{1}{p^e}}{p-1} \to \frac{p}{p-1}$ as $e \to \infty$ (Ward, 2008).
In this paper, the definition and properties of the abundancy index function, together with very basic number theoretic properties, shall be used to determine some necessary conditions that a friend of 38 must satisfy.

3.0 Results

3.1 On Potential Friends of 38

Lemma

If p is an odd prime and r an odd positive integer, then $\sigma(p^r)$ is even.
If q is an odd prime and s is an even positive integer, then $\sigma(q^s)$ is odd.
Proof: a) $\sigma(p^r) = 1 + p + p^2 + \cdots + p^r$ is a sum of even number of odd terms which is even.
b) $\sigma(q^s) = 1 + q + q^2 + \cdots + q^s$ is a sum of odd number of odd terms which is odd. □

3.1 Theorem Any friend of 38 is an odd non-square multiple of 19.

Proof: Suppose m is a friend of 38. By definition, $I(m) = \frac{\sigma(m)}{m} = I(38) = \frac{30}{19}$ and hence,

$$30m = 19\sigma(m) \quad (3)$$

From divisibility properties, $19|m$. Now, $2 \nmid m$, otherwise we would have $38|m$ cancelling possibility of friendship, from an observation in section 1.1. Since m is odd and from Equation (3), $\sigma(m)$ is even, and hence m must be a non-square. This is seen by applying the equation $\sigma\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right) = \prod_{i=1}^{k}\left(\sum_{j=0}^{\alpha_i} p_i^j\right)$ whereby if all the p_i are odd and the product is even, at least one α_i must be odd, by Lemma 3.1. So, m is of the form $m = 19^x y$ for x, $y \in \mathbb{N}$, $(19, y) = 1$, where either x is odd or y is not a square. □

3.2 Theorem Let m be a friend of 38. If q is a prime factor of m such that $4|(q + 1)$, then the power of q in the prime power factorization of m must be even.

Proof: Let k be the power of q in m. Suppose k is odd. Then, $k+1$ is even and by definition,

$\sigma(q^k) = 1 + q + \ldots + q^{k-1} + q^k$, which is the sum of even number of terms. Pairing consecutive summands,

$$\sigma(q^k) = (1+q) + (q^2+q^3) + \cdots + (q^{k-1}+q^k)$$
$$= (1+q) + q^2(1+q) + \cdots + q^{k-1}(1+q)$$
$$= (1+q)(1+q^2+q^4+\cdots+q^{k-1}).$$

So, $(q+1)|\sigma(q^k)$. Now, since $\sigma(q^k)|\,\sigma(m)$ and σ is multiplicative, then $(q+1)|\,\sigma(m)$. From Equation (3), $(q+1)|30m$ implying $4|30m$ and hence $2|m$, a contradiction since m must be odd. This means that k cannot be odd and must therefore be even. □

Since 19 is a compulsory prime factor of m, then from Theorem 3.2, its power must be even and since m is odd and non-square, then $m = 19^{2a}b$, a, $b \in \mathbb{N}$, $(2, b) = (19, b) = 1$, where b is not a square.

3.3 Theorem Suppose m is a friend of 38. Then, m is not divisible by 3.

Proof: Suppose $3|m$. By property (v) in section 2.1, $I(3^4 19^e) < \frac{121}{81}\frac{19}{18} = \frac{2299}{1458} < \frac{30}{19}$ $\forall e \in \mathbb{N}$. Hence, $I(3^4 19^2) < \frac{30}{19}$. However, $I(3^5 19^2) = \frac{364}{243}\frac{381}{361} = \frac{46228}{29241} > \frac{30}{19}$. So, for all $y \geq 2$, $I(3^x 19^y) < \frac{30}{19}$ $\forall\, x \in \{1,2,3,4\}$, but $I(3^x 19^y) > \frac{30}{19}$ $\forall\, x > 4$. Recall that the power of 19 is even and greater than or equal to 2. So, if $3|m$, the power of 3 in m is either 1, 2, 3 or 4. By Theorem 3.2, the only possible power of 3 is either 2 or 4. Let $m = 3^x 19^y z$ for x, y, $z \in \mathbb{N}$, $(3, z) = (19, z) = 1$. Then, $x \in \{2,4\}$ and y is an even positive integer. If $x = 2$, $m = 3^2 19^y z$, and from Equation (3), $30m = 19\sigma(3^2)\sigma(19^y z) = 19.13.\sigma(19^y z)$ implying that $13|m$. However, $I(3^2.13.19^2) = \frac{13}{9}\frac{14}{13}\frac{381}{361} = \frac{1778}{1083} > \frac{30}{19}$, meaning that it is not possible to have $13|m$. So, $x \neq 2$. If $x = 4$, then $m = 3^4 19^y z$ and from Equation (3), $30m = 19\sigma(3^4)\sigma(19^y z) = 19.121.\sigma(19^y z) = 19.11^2.\sigma(19^y z)$ implying that $11^2|m$. However, $I(3^4.11^2.19^2) = \frac{121}{81}\frac{133}{121}\frac{381}{361} = \frac{889}{513} > \frac{30}{19}$, meaning that it is not possible to have $11^2|m$. So, $x \neq 4$. This has just shown that neither 2 nor 4 can be a power of 3. But 2 and 4 are the only possible powers of 3, as earlier seen. So, 3 can never be a factor of m. □

So far it has been established that if m is a friend of 38, then $m = 19^{2a}b$ for some a, $b \in \mathbb{N}$, where $(2, b) = (3, b) = (19, b) = 1$ and b is non-square. With this definition of m, we have the following additional results:

3.4 Theorem Suppose that a friend m of 38 exists so that it is of the form $m = 19^{2a}b$, for some $a,\ b \in \mathbb{N}$, with $(2, b) = (3, b) = (19, b) = 1$ and b a non-square. Then, the following statements are true:

If $a = 1$, then $b = 127^{2c}d$, c, $d \in \mathbb{N}$, $(19, d) = (127, d) = 1$, d is not a square, and $c \notin \{1, 4\}$.
If $a = 3$, then $b = 701^r 70841^s t$, $r, s, t \in \mathbb{N}$, $(19, t) = (701, t) = (70841, t) = 1$, t is not a square, and $r \notin \{1, 3\}$.
The value of a cannot be equal to 4.

Proof:
Suppose $a = 1$ so that $m = 19^2 b$. From Equation (3),

$$30m = 19\sigma(19^2)\sigma(b) = 19 \times 381 \times \sigma(b) = 19 \times (3 \times 127) \times \sigma(b),$$

implying $127|m$. By Theorem 3.2, the power of 127 in m must be even. So, $m = 19^2 127^{2c} d$ where $c, d \in \mathbb{N}$ and $(19, d) = (127, d) = 1$, and since b is a non-square, so is d. If $c = 1$, then $m = 19^2 127^2 d$ and from Equation (3),

$$30m = 19\sigma(19^2)\sigma(127^2)\sigma(d) = 19 \times 381 \times 16257 \times \sigma(d)$$

$$= 19 \times (3 \times 127) \times (3 \times 5419) \times \sigma(d)$$ which would imply that $3^2|30m$ and hence $3|m$, which does not hold, by Theorem 3.3. So, $c = 1$ is an impossibility. On the other hand, if $c = 4$, $m = 19^2 127^8 d$ and from Equation (3),

$$30m = 19\sigma(19^2)\sigma(127^8)\sigma(d) = 19 \times 381 \times 68212339274677761 \times \sigma(d)$$

$$= 19 \times (3 \times 127) \times (3 \times 22737446424892587) \times \sigma(d),$$

so that $3^2|30m$ and hence $3|m$, contradicting Theorem 3.3. So, $c = 4$ is another impossibility.

If $a = 3$, then $m = 19^6 b$. From Equation (3),

$$30m = 19\sigma(19^6)\sigma(b) = 19 \times 49659541 \times \sigma(b) = 19 \times (701 \times 70841) \times \sigma(b),$$

implying that both 701 and 70841 divide m. So, $m = 19^6 701^r 70841^s t$ where r, s, $t \in \mathbb{N}$ and $(19, t) = (701, t) = (70841, t) = 1$. If $r = 1$, $m = 19^6 \times 701 \times 70841^s \times t$, and hence

$$30m = 19\sigma(19^6)\sigma(701)\sigma(70841^s t)$$

$$= 19 \times 49659541 \times 702 \times \sigma(70841^s t)$$

$$= 19 \times (701 \times 70841) \times (2 \times 3^3 \times 13) \times \sigma(70841^s t).$$

From divisibility properties, $3^3|30m$ and hence $3|m$, a contradiction. Therefore, it is not possible to have $r = 1$. If $r = 3$, $m = 19^6 701^3 70841^s t$ so that

$$
\begin{aligned}
30m &= 19 \times \sigma(19^6)\sigma(701^3)\sigma(70841^s t) \\
&= 19 \times 49659541 \times 344964204 \times \sigma(70841^s t) \\
&= 19 \times (701 \times 70841) \times (4 \times 86241051) \times \sigma(70841^s t),
\end{aligned}
$$

implying $4|30m$ and hence $2|m$, a contradiction. Therefore, $r = 3$ is another impossibility.

Now, suppose $a = 4$ so that $m = 19^8 b$. From Equation (3),

$$
\begin{aligned}
30m = 19\sigma(19^8)\sigma(b) &= 19 \times 17927094321 \times \sigma(b) \\
&= 19 \times (3^2 \times 127 \times 523 \times 29989) \times \sigma(b),
\end{aligned}
$$

implying that $3^2|30m$ and hence $3|m$, a contradiction. So, it is not possible to have $a = 4$. □

4.0 Conclusion

The results above prove that a friend m of 38 is an odd non-square multiple of 19^2 which is not divisible by 3. It has also been shown that every prime factor q of m such that $4|(q + 1)$ must have an even exponent in the prime factorization of m. If the power of 19 in m is 2, then 127 emerges as a compulsory prime factor of m, in which case the power of 127 must be even, larger than 2 and not equal to 8. This in turn means that it is not possible to have 2 as the power of all prime factors of m that have even powers in the prime factorization of m. It has also been established that it is not possible to have 8 as the power of 19 in the prime factorization of m.

These findings are just some of the necessary, but not sufficient, conditions for a friend of 38 to exist. It, therefore, remains a challenge to identify a friend of 38 or even prove the non-existence of one. If this is done, the concept used in this paper might be helpful in determining the classification, as either friendly or solitary, of other numbers whose category is undetermined so far.

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