Gamma and Exponential Autoregressive Moving Average (ARMA) Processes

B. S. Otieno and D.K. Nassiuma

Department of Mathematics, Egerton University, P.O Box 536, Njoro, KENYA

ABSTRACT

Time series data encountered in practice depict properties that deviate from those of gaussian processes. The gamma and exponentially distributed processes which are used as basic models for positive time series fall in the class of non-gaussian processes. In this paper, we develop new and simpler representations of the p^{th} order autoregressive and the q^{th} order moving average processes in gamma and exponential variables. The gamma autoregressive moving average (GARMA(p,q)) model of order p and q and the exponential autoregressive moving average (EARMA(p,q)) model of order p and q are consequently developed. The distributions of developed models, unlike those studied by Lawrance and Lewis (1980), can be determined given either the distribution of the innovation sequence $\{e_t\}$ or that of the process itself. The autocorrelation structure, which is a major identification tool in time series, is discussed for each of the proposed models.

KEY WORDS:

Autocorrelation functions, Exponential distribution, Gamma distribution, non-gaussian processes, positive time series.

1.0 INTRODUCTION

The main goal of the initial studies on processes with non-gaussian distributions was to generate processes with simple properties. These were expected to yield analytically and computationally tractable models for non-independent (markovian) and easily simulated sequences of marginally exponentially distributed random variables.

Lawrance and Lewis (1977) introduced the exponentially moving average of order one process (EMA (1)), while Gaver and Lewis (1980) discussed the exponential autoregressive (EAR(1)) of order one. Jacobs and Lewis (1977) established various properties for the exponential autoregressive moving average of order one (EARMA(1,1)) and Lawrence and Lewis (1980) extended the study to establish conditions for the existence of the exponential autoregressive moving average process (EARMA(p,q)). Sim (1990) proposed a first order Gamma ($1/\lambda$,k) while Tong (1995) pointed out a possible representation of exponential processes as special cases of threshold models.

Processes having gamma distribution form a generalization of the exponential distribution. However, properties of gamma processes are much more complex than the corresponding exponential process. In fact the functional forms of general models with the generating mechanisms having gamma distributions have so far not been studied. However, Gaver and Lewis, (1980) showed that if X_t is a gamma (λ,k) in the linear additive first order autoregressive process of the form $X_t=\phi X_{t-1}+e_t$ for $0 \leq \phi < 1$, then e_t has the moment generating function of an infinitely divisible distribution. This enables the autoregressive, moving average and mixed gamma processes to be constructed.

Lawrence (1982) discusses the innovation distribution of a gamma autoregressive process of order one, denoted as GAR(1) process but he gives no functional form of the model. More recently Lewis, Mackenzie and Hugus (1989) studied a non-linear random coefficient model for gamma processes with the coefficients having beta distributions.

A missing link in these studies is the development of models that have simple properties and whose functional forms facilitate generalization to higher orders. In this study, models of the GARMA (p,q) processes with simple properties are developed. The results are then extended to the development of simpler models for the EARMA(p,q) processes. Linear processes of the autoregressive type that have gamma and exponential distributions are developed in Section 2.0. This is followed by a study of the gamma and exponential moving average processes in Section 3.0 and finally in Section 4.0, the gamma and exponential autoregressive moving average processes are discussed. The autocorrelation structures for the developed models are also discussed. Some distinguishing features between these processes and gaussian processes are also outlined.

2.0 Gamma and Exponential Autoregressive processes

The first-order gamma autoregressive (GAR(1)) process is the simplest in the family of gamma processes and its innovation distribution can be obtained by considering the AR(1) process X_t which has a gamma$(\lambda>0,k)$ distribution. Thus for the AR(1) model of the form $X_t= \phi X_{t-1}+e_t$, where $0<\phi<1$ and $\{e_t\}$ is the innovation sequence, we suppose that the process X_t has a gamma(λ,k), distribution. The distribution of e_t is then derived by the moment generating function technique. That is, if X_t is gamma(λ,k), then its moment generating function is

$$M_{X_t}(s)=\left[\frac{\lambda}{\lambda - s}\right]^k \text{ implying that } M_{e_t}(s)=\left[\phi+(1-\phi)\frac{\lambda}{(\lambda - s)}\right]^k$$

which is the moment generating function of the a particular compound Poisson distribution (Lawrence, 1982) which is also a finite mixture of gamma distributions. When k=1, the moment generating function (m.g.f) of e_t corresponds to that of an exponential $(\lambda > 0)$ random variable. When k=m, m>1, the m.g.f of e_t is obtained as

$$M_{e_t}(s)=\phi^m + m\phi^{m-1}(1-\phi)\left(\frac{\lambda}{\lambda-1}\right)+\ldots+(1-\phi)^m\left(\frac{\lambda}{\lambda-s}\right)^m$$

which is a mixture of a degenerate random variable with mass at zero, a gamma random variable with parameter ($\lambda>0$) and (k-1) gamma random variables with parameters ($\lambda>0$,m), m = 2,3, ... The difference equation for the series X_t with the corresponding probabilities (w.p) takes the form

$$X_t = \begin{cases} \phi X_{t-1} & w.p\ \phi^m \\ \phi X_{t-1} + E_t & w.p\ 1-\phi^m \end{cases}$$

where

$$E_t = \begin{cases} e_t & w.p\ \dfrac{m\phi^{m-1}(1-\phi)}{1-\phi^m} \\ \vdots & \vdots \\ e_t^{m-1}, & w.p\ \dfrac{(1-\phi)^m}{1-\phi^m} \end{cases}$$

which can easily be represented as a threshold models (Tong, 1995).

The autocorrelation function at lag h for the GAR(1) process is obtained as $\rho_h = \phi\rho_{h-1}$ for h = 1, 2, ... Thus $\rho_1 = \phi\rho_0 = \phi$, i.e, $\phi = \rho_1$ which is analogous to the case of the standard AR(1) and for k=1, the autocorrelations are those for the EAR(1) processe.

The AR(2) process is given as $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + e_t$ where $0<\phi_1,\phi_2<1$. Suppose that the process X_t has a gamma($\lambda>0$,k) distribution, then the moment generating function of e_t is obtained as

$$M_{e_t}(s)=\left(\frac{\lambda-\phi_1 s}{\lambda-s}\right)^k\left(\frac{\lambda-\phi_2 s}{\lambda}\right)^k$$

$$=\sum_{x=0}^{m}\binom{m}{x}(1-(\phi_1+\phi_2))^x(\phi_1+\phi_2)^{m-x}\left(\frac{\lambda}{\lambda-s}\right)^x+g(\phi_1,\phi_2,\lambda,s)$$

for m=1,2,...k and $g(\phi_1, \phi_2, \lambda, s)$ is a function of ϕ_1, ϕ_2, λ and s which has a negligible probability. The difference equation for the GAR(2) process X_t takes the form

$$X_t = \begin{cases} \phi_1 X_{t-1} + \phi_2 X_{t-2} & w.p\ (\phi_1 + \phi_2)^m \\ \phi_1 X_{t-1} + \phi_2 X_{t-2} + e_t & w.p\, 2(\phi_1 + \phi_2)(1-(\phi_1 + \phi_2)) \\ . & . \\ . & . \\ . & . \\ \phi_1 X_{t-1} + \phi_2 X_{t-2} + E_t & w.p\ 1-(\phi_1 + \phi_2)^m \end{cases}$$

where

$$E_t = \begin{cases} e_t^{(1)} & w.p\ \dfrac{m(\phi_1 + \phi_2)^{m-1}(1-(\phi_1 + \phi_2))}{1-(\phi_1 + \phi_2)^m} \\ . & . \\ . & . \\ . & . \\ e_t^{x-1}, & w.p\ \dfrac{\binom{m}{x}(\phi_1 + \phi_2)^{m-x}(1-(\phi_1 + \phi_2))^x}{1-(\phi_1 + \phi_2)^m}, x = 3,4,,\ldots m = k. \end{cases}$$

The corresponding autocorrelation function for the case when k=m, m≥2 is obtained as $\rho_h = \phi_1 \rho_{h-1} + \phi_2 \rho_{h-2}$ for h=1,2, ... The autocorrelation functions at lags 1 and 2 are $\rho_1 = \dfrac{\phi_1}{(1 - \phi_2)}$ and $\rho_2 = \phi_1 \rho_1 + \phi_2$ respectively.

In general, for the p[th] order autoregressive gamma (GAR(p)) process, we consider the AR(p) process $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p}$ where $0 < \phi_1, \phi_2, \ldots, \phi_p < 1$ and e_t is the innovation sequence. Suppose X_t is gamma($\lambda>0$, k), then e_t will be a mixture of gamma ($\lambda>0$,k) random variables. The moment generating function of e_t is obtained as

$$M_{e_t}(s) = \left(\frac{\lambda - \phi_1 s}{\lambda - s}\right)^k \left(\frac{\lambda - \phi_2 s}{\lambda}\right)^k \cdots \left(\frac{\lambda - \phi_p s}{\lambda}\right)^k$$

$$= \sum_{x=0}^{m} \binom{m}{x} \left(1 - \sum_{i=1}^{p} \phi i\right)^x \left(\sum_{i=1}^{p} \phi i\right)^{m-x} \left(\frac{\lambda}{\lambda - s}\right)^x + g(\phi_1, \phi_2 \ldots, \phi_p \lambda, s)$$

for m=1, 2, …k and $g(\phi_1, \phi_2, \ldots, \phi_p\ \lambda, s)$ is a function of $\phi_1, \phi_2, \ldots \phi_p$, λ and s which has a negligible probability. This is the moment generating function of a convex mixture of a degenerate random variable with mass at zero, and (k-1) gamma(>0,m) random variables where m=1,2,…k. Thus the difference equation generating the sequence X_t is

$$X_t = \begin{cases} \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p}, & w.p\ a^m \\ \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + e_t, & w.p\ ma^{m-1}b \\ \vdots & \vdots \\ \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + e_t^{(m-1)}, & w.p\ \ b^m \end{cases}$$

where $a = \sum_{i=1}^{p} \phi_i$, $b = 1 - \sum_{i=1}^{p} \phi_i$ and $\{e_t\}$ is a sequence of i.i.d gamma(λ>0,1) random variables, $\{e^{(j)}_t\}$ is a sequence of i.i.d gamma(λ>0, j+1) random variables for j= 1,2,...,(m-2). The autocorrelation function for the GAR(P) when m ≥ 2, at lag h is obtained as $\rho_h = \phi_1 \rho_{h-1} + \ldots + \phi_p \rho_{h-p}$, where h =1, 2, ... , p. It is to be noted these autocorrelation functions have a slower decay but similar to those of the normal distribution. This is due to the distributional similarity of X_t and e_t in the above models which facilitates a simpler derivation of model properties as is the case for the normal distribution. This behaviour contrasts with the autocorrelation of processes studied by Lawrance and Lewis (1980) which decay at a faster rate and make it difficult to distinguish the processes from the gaussian ones.

2.1 Gamma and Exponential Moving Average Gamma Processes

Consider the backward MA(1) process represented by the relation $X_t=\theta e_{t-1}+e_t$ where $0<\theta<1$. Suppose that e_t is distributed as a gamma(λ>0,1) random variable, then

$$M_{X_t}(s) = \frac{\lambda}{\lambda - \theta s} \cdot \frac{\lambda}{\lambda - s} = \left(\theta + (1-\theta)\frac{\lambda}{\lambda - s}\right)\left(\frac{\lambda}{\lambda - \theta s}\right)^2$$
$$= \left(\theta + (1-\theta)\frac{\lambda}{\lambda - s}\right)\left(1 + \frac{\theta s}{\lambda - \theta s}\right)^2$$
$$= \theta + (1-\theta)\frac{\lambda}{\lambda - s} + \frac{2\theta s(\lambda - \theta s) + \theta^2 s^2}{(\lambda - s)(\lambda - \theta s)}.$$

This implies that the random variable X_t is realized as a degenerate random variable with mass at zero, a gamma (λ>0,1) random variable and a random variable ε_t of negligible probability. The differences equation for the first order moving average gamma is obtained as

$$X_t = \begin{cases} \theta e_t\ , & w.p\ \theta \\ \theta e_{t-1} + e_t\ , & w.p\ \ 1-\theta \end{cases}$$

which is corresponds to the EMA(1) proposed by Lawrence and Lewis (1982). For the case when k =2, we define a foreword first-order moving average gamma process by the relation

$$X_t = \begin{cases} \theta e_{t-1}\ , & w.p\ \theta^2 \\ \theta e_{t-1} + e_t\ , & w.p\ \ 2\theta(1-\theta) \\ \theta e_{t-1} + e_t^{(1)}\ , & w.p\ \ (1-\theta)^2 \end{cases} \text{ or } X_t = \begin{cases} \theta e_{t-1}\ , & w.p\ \theta^2 \\ \theta e_{t-1} + \varepsilon_t\ , & w.p\ 1-\theta^2 \end{cases}$$

where

$$\varepsilon_t = \begin{cases} e_t\ , & w.p\ \dfrac{2\theta}{1+\theta} \\ e_t^{(1)}\ , & w.p\ \dfrac{1-\theta}{1+\theta} \end{cases}$$

with $\{e_t\}$ being a sequence of i.i.d gamma(λ>0,1) $\{e^{(1)}{}_t\}$ is a sequence of i.i.d gamma(λ>0,2) and $0 < \theta < 1$. In general, when k = m, we define a first-order moving average by the relation

$$X_t = \begin{cases} \theta e_{t-1}\ , & w.p\ \theta^m \\ \theta e_{t-1} + e_t\ , & w.p\ \ m\theta^{m-1}(1-\theta) \\ \vdots & \vdots \\ \theta e_{t-1} + e_t^{(m-1)}\ , & w.p\ \ (1-\theta)^m \end{cases}$$

where $\{e_t\}$ is a sequence of i.i.d gamma(λ >0,1) random variable and $\{e^{(j)}{}_t\}$ j = 1,2, ...,(m-2), is a sequence of i.i.d gamma (λ>0, j+1) random variables.

The lag one autocorrelation function for GMA(1) defined above are obtained as

$$\rho_1 = \frac{\theta(1-\theta)}{(\theta^2 + (1-\theta)^2)}, \qquad \rho_h = 0, \text{ for h > 1.}$$

Consider now the stationary MA(2) process given by the relation $X_t = \theta_1 e_{t-1} + \theta_2 e_{t-2} + e_t$. Suppose that the innovation sequence is distributed as gamma(,k), then the moment generating function of X_t is obtained as

$$M_{X_t}(s) = \left(\frac{\lambda}{\lambda - \theta_1 s}\right)^k \left(\frac{\lambda}{\lambda - \theta_2 s}\right)^k \left(\frac{\lambda}{\lambda - s}\right)^k$$

When k=1, this simplifies to the form

$$M_{X_t} = \theta_1 + \theta_2 + [(1-\theta_1)(1-\theta_2)\frac{\lambda}{\lambda - s} + g$$

where g has probability $\theta_1\theta_2$. This is the m.g.f of an EMA(1) process. When k=2, the above equation simplifies to

$$M_{X_t} = (\theta_1+\theta_2)^2 + (1-\theta_1)^2(1-\theta_2)^2\left(\frac{\lambda}{\lambda - s}\right)^2 + \ldots$$
$$+ 2(\theta_1+\theta_2)(1-\theta_1)(1-\theta_2)\frac{\lambda}{\lambda - s} + g1$$

where g1 is a function of θ_1, θ_2, λ and s that has negligible probability. The difference equation for the process when k=2 takes the form

$$X_t = \begin{cases} \theta_1 e_{t-1} + \theta_2 e_{t-2}, & w.p\ \theta_1 + \theta_2 \\ \theta_1 e_{t-1} + \theta_2 e_{t-2}, & w.p\ \underset{i=1}{\overset{2}{\pi}}(1-\theta_i) \text{ or} \\ g & w.p\ \theta_1\theta_2. \end{cases}$$

This represents the EMA(2) process where $0<\pi(1-\theta_i)<1$ for i=1,2, implies $\theta_1 <½$, $\theta_2<1$ or $\theta_1 <1$, $\theta_2<1/2$. For this model, if e_t is assumed to be exponential(λ), then X_t is also an exponential random variable. The serial dependency of this model also stops at the second lag and the autocorrelation functions are obtained as

$$\rho_1 = \frac{\theta_1\left(\theta_2 + \prod_{i=1}^{2}(1-\theta_i)\right)}{\theta_1^2+\theta_2^2+\left(\prod_{i=1}^{2}(1-\theta_i)\right)^2} \quad \text{and} \quad \rho_2 = \frac{\theta_2\left(\prod_{i=1}^{2}(1-\theta_i)\right)^2}{\theta_1^2+\theta_2^2+\left(\prod_{i=1}^{2}(1-\theta_i)\right)^2}.$$

Similarly the q^{th} order moving average gamma process is defined by the relation

$$X_t = \begin{cases} \theta_1 e_{t-1} + \ldots + \theta_q e_{t-q}, & w.p\ a^q \\ \theta_1 e_{t-1} + \ldots + \theta_q e_{t-q} + e_t, & w.p\ 2a^{q-1}b \\ \cdot & \\ \cdot & \\ \theta_1 e_{t-1} + \ldots + \theta_q e_{t-q} + e_t^{(q-1)}, & w.p\ b^q \end{cases}$$

where $b=\prod_{i=1}^{q}(1-\theta_i), a=\sum_{i=1}^{q}\theta_i$, $\{e_t\}$ is sequence of i.i.d gamma(λ >0,1) and { $e_t^{(m-1)}$ } is a sequence of i.i.d gamma(λ >0,q) random variables.

2.2 Gamma and Exponential ARMA Processes

The GARMA(1,1) process is a generalization of the GAR(1) and GMA(1) processes discussed in sections two and three above. The probabilistic linear model when k =1 is defined as

$$X_t = \begin{cases} \theta e_{t-1}, & w.p\ \theta \\ \theta e_{t-1} + A_t, & w.p\ (1-\theta) \end{cases}$$

where

$$A_t = \begin{cases} \phi A_{t-1}, & w.p\ \phi \\ \phi A_{t-1} + e_t, & w.p\ (1-\phi) \end{cases}$$

for t = 0, 1, 2,... When k = 2, we define a GARMA(1,1) by the relation

$$X_t = \begin{cases} \theta e_{t-1}, & w.p\ \theta^2 \\ \theta e_{t-1} + A_t, & w.p\ 2\theta(1-\theta) \\ \theta e_{t-1} + A_t^{(1)}, & w.p\ (1-\theta)^2 \end{cases}$$

where

$$A_t = \begin{cases} \phi A_{t-1}, & w.p\ \phi^2 \\ \phi A_{t-1} + e_t, & w.p\ 2\phi(1-\phi) \\ \phi A_{t-1} + e_t^{(1)}, & w.p\ (1-\phi)^2 \end{cases}$$

for t = 0, 1, 2, ... and $e_t^{(1)}$ is a gamma(λ>0,2) random variable.

In general, in the case when k = m, we define a GARMA(1,1) by the relation

$$X_t = \begin{cases} \theta e_{t-1}, & w.p\ \theta^m \\ \theta e_{t-1} + A_t, & w.p\ m\theta^{m-1}(1-\theta) \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \theta e_{t-1} + A_t^{(m-1)}, & w.p\ (1-\theta)^m \end{cases}$$

where

$$A_t = \begin{cases} \phi A_{t-1}, & w.p\ \phi^m \\ \phi A_{t-1} + e_t, & w.p\ m\phi^{m-1}(1-\phi) \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \phi A_{t-1} + e_t^{(m-1)}, & w.p\ (1-\phi^m) \end{cases}$$

for t=0, 1, 2,....and { $A_t^{(m-1)}$ }is a sequence of i.i.d gamma(λ >0,2) random variables.

The EARMA(1,1) process is a generalization of the EAR(1) and EMA(1) processes and has the correlation structure of an autoregressive moving average process of order one and provides an alternative to the poisson process. The probabilistic linear model of the EARMA(1,1) model is defined by

$$X_t = \begin{cases} \theta e_{t-1}, & w.p\ \theta \\ \theta e_{t-1} + A_t, & w.p\ (1-\theta) \end{cases} \quad \text{for } t=0,1,2,\ldots, \quad A_t = \begin{cases} \phi A_{t-1}, & w.p\ \phi \\ \phi A_{t-1} + e_t, & w.p\ (1-\phi) \end{cases}$$

The au correlation function for this model is obtained as

$$\rho_h = \frac{(1-\theta)\phi^h}{[1-\theta+\theta^2(1+\phi)]} \quad \text{for } h=0,1,2,\ldots$$

Using e EAR(2) and the EMA(2) models, a backward EARMA(2,2) process is developed as

$$X_t = \begin{cases} \theta_1 e_{t-1} + \theta_2 e_{t-2}, & w.p\ \theta_1 + \theta_2 \\ \theta_1 e_{t-1} + \theta_2 e_{t-2} + A_t, & w.p\ \pi_{i=1}^{2}(1-\theta_i) \end{cases}$$

where $0<(1-\theta_1),(1-\theta_2)<1$, A_t is an GAR(2) process and $0<\varphi_1,\varphi_2$, for t=0,1,2,......The autocorrelation function at lag h for the above model is obtained as

$$\rho_1 = \frac{(\theta_1\theta_2+(1-\theta_1-\theta_2)(1-\phi_1-\phi_2)[\theta_1\phi_1+\theta_2\phi_2])w+R}{[\theta_1^2+\theta_2^2+(1-\theta_1-\theta_2)(1-\phi_1-\phi_2)(1+\theta_1\phi_1+\theta_2\phi_2)]w+R_1}$$

where $R_1 = (1-\theta_1-\theta_2)[(\phi_1^2+\phi_2^2)\gamma_0^{(1)}+2\phi_1\phi_2\gamma_1^{(1)}]$, ω=var(e_t), $\gamma_h^{(1)}$ are autocovariance functions at lag h , $R=(1-\theta_1-\theta_2)[(\phi_1^2+\phi_2^2)\gamma_o^{(1)}+2\phi_1\phi_2\gamma_1^{(1)}]$ and

$$\rho_h = \frac{(1-\theta_1-\theta_2)(1-\phi_1-\phi_2)[\theta_1\phi_1+\theta_2\phi_2]w+(1-\theta_1-\theta_2)\gamma_h^{(1)}}{(\theta_1^2+\theta_2^2+(1-\theta_1-\theta_2)(1-\phi_1-\phi_2[1+\theta_1\phi_1+\theta_2\phi_2])w+R}$$

for h=2,3,....

The fact that the EMA and EAR models can be expressed in the terms of independent exponential variables makes the two models appealing and tractable.

The GARMA(p,q) process is a generalisation of the GAR(p) and GMA(q) processes and takes the form

$$X_t = \begin{cases} \theta_1 e_{t-1} + \ldots + \theta_q e_{t-q}, & w.p\ a^k \\ \theta_1 e_{t-1} + \ldots + \theta_q e_{t-q} + A_t, & w.p\ 2a^{k-1}b \\ . & \\ . & \\ \theta_1 e_{t-1} + \ldots + \theta_q e_{t-q} + A_t^{(q-1)}, & w.p\ b^k \end{cases} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \quad 1$$

where $b=\prod_{i=1}^{q}(1-\theta_i)$, $a=\sum_{i=1}^{q}\theta_i$ and $A_t^{(k)}$ is a GAR(m) process for m=1,2,..q-1≤p. The case when k=1 in (1) leads to the EARMA(p,q) process with A_t being a GAR(p) process.

3.0 Conclusion

In this paper, the moment generating function technique forms the basis on which the said models are developed starting with the corresponding gaussian models. In fact this approach enables one to determine the distribution of either the innovation sequence or the observed values of such models like the ones given in the literature. The developed AR(p) and MA(q) processes in gamma and exponential variables are generalized into the GARMA(p,q) and EARMA(p,q) models.

A major distinction to the application of the processes developed here and the Gaussian ARMA models is the fact that the serial correlations are all positive. The models developed in this study also show ... ...ve representations of GARMA(p,q) and EARMA(p,q) models lead to simpler assessment of properties for these models. Such models forms a basis for alternative modeling of positively correlated time series.

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