ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS OF A GENERALIZED LIPSCHITZ ORDINARY DIFFERENTIAL EQUATIONS

T.A. ADEMOLA+ and M.O. OGUNDIRAN
Department of Mathematics and Statistics, Bowen University, Iwo, Osun State, Nigeria.

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Abstract
In this paper, the existence and uniqueness theorem for the solutions of the initial value problem (1.1), (1.2) and (3.12) shall be proven, where the function f, in (1.1) and (3.12) satisfy generalized Lipschitz continuous.

Key words: Generalized Lipschitz continuous, fixed point, Gronwall inequalities, existence and uniqueness of solutions, contraction mapping principle.

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1. Introduction

Let R^n denote the real Euclidean n-dimensional space and R denote the real interval $-\infty < t < \infty$
We shall consider here the system of real differential equation of the form

$$\dot{x}(t) = f(t, x(t)) \qquad (1.1)$$

where $x \in R^n, t \in R$ and $f : R^{n+1} \to R^n$.

The dots here, as elsewhere, stand for differentiation with respect to independent variable t. Assume some smoothness condition on f, say f is carathéodory in t and x. The initial value problems consist of finding a solution $x(t, t_0, x_0)$ for given $t_0 \varepsilon R$, $x_0 \varepsilon R^n$, which reduce to x_0 at $t = t_0$ that is

$$x(t, t_0, x_0) = x_0 \qquad (1.2)$$

In [3], [5], [6], [10] and [11] the existence and uniqueness theorem for the solutions (1.1) satisfying (1.2) with the function f(t, x) in (1.1) satisfying Lipschitz condition

$$\| f(t, x(t)) - f(t, y(t)) \| \le L|x(t) - y(t)| \qquad (1.3)$$

for some constant $L > 0$, has been considered.
In [10; Chapter 5, 133-140], Ladas and Lakshmikantham, discussed (local) existence and uniqueness of solutions of (1.1) satisfying (1.2) in Banach space with function f in the right hand sides of (1.1) satisfying (1.3). Hochstadt in [6; Chapter 6, 204-206], also established conditions for the existence of a unique solution of (1.1) satisfying (1.2), where the function f in the right hand side of (1.1) satisfies (1.3).
In [5; Chapter 1, 7-45], Cronin considered two versions of the basic existence theorem for the solutions of (1.1) satisfying (1.2). In both versions, the concept of a constant Lipschitz condition (1.3) was used. Himmelberg and Van vleck [9] discussed an existence theorem for (1.1) and (1.2) under Lipschitz carathéodory condition.
Let $(F, \|.\|)$ be a Fréchet space,
$f : [0, t] \times F \to F$ continuous and $x_0 \varepsilon F$. Then the initial value problem (1.1) and (1.2), where f in (1.1) satisfies (1.2) for all (t, x), (t, y) ε $[0, t]\times F$, and L is a row finite matrix with nonnegative entries was considered in [7] and [8]. Here, the author established criteria for an existence and uniqueness for the initial value problem (1.1) and (1.2) in Fréchet spaces using Lipschitz conditions formulated with a generalized distance and row finite matrices.

+ *corresponding author* (email: ademola672000@yahoo.com)

In [1] Agarwal and Bahuguna, considered the following nonlocal nonlinear functional differential equation in a real reflexive Banach space x,

$$u'(t) + Au(t) = f(t, u(t), u(b_1(t)), u(b_2(t)), \cdots, u(b_m(t))), t \in (0, T]$$
$$h(u) = \varphi_0 \text{ on } [-\tau, 0] \qquad (1.4)$$

where $0 < T < \infty, \phi_0 \in C_0 := C([-\tau, 0]; X)$,
the nonlinear operator A is singled-valued and m-accretive defined from the domain $D(A) \subset X$ into X, the nonlinear map f is defined from $[0, T] \times X^{m+1}$ into X and map h is defined from

$$C_T := C([-\tau, T]; X) \text{ into } C_T.$$

Here, method of lines was applied to establish the existence and uniqueness of a strong solution. The f in (1.4) is assumed to satisfies a local Lipschitz-like condition

$$\|f(t, u_1, u_2, \cdots, u_{m+1}), f(s, v_1, v_2, \cdots, v_{m+1})\| \le L_f \, [|t - s| + \sum_{i=1}^{m+1} \|u_i - v_i\|] \qquad (1.5)$$

$$\text{for all } (u_1, u_2, \cdots, u_{m+1}), (v_1, v_2, \cdots, v_{m+1}) \in B_r(X^{m+1}, x_0, x_0, \cdots, x_0,)$$

and t,s ε [0,T] where $L : R^+ \to R^+$ is a non-decreasing function. Dhage, Saluunkhe, Agarwal and Zhang in [4], considered the first order functional differential equation

$$\left(\frac{x(t)}{f(t, x(t))}\right)' = g(t, x_t) \text{ a.e } t \in I \qquad (1.6)$$
$$x(t) = \varphi(t), t \in I_0,$$

where $f : I \times R \to R - \{0\}$ is continuous, $g : I \times C \to R$

and the function $x_t(\theta) : I_0 \to C$ is defined by $x_t(\theta) := x(t + \theta) \forall \theta \in I_0$.

In this paper, we shall be concerned with existence and uniqueness of solutions for the initial value problem (1.1) (1.2) and (3.12) with f in (1.1) and (3.12) satisfying generalized Lipschitz continuity, that is L in (1.3) being a real-valued function. In which case (1.3), is a particular case. The motivation for the present work has come from the work mentioned above.
In some other applications generalized Gronwall inequality has been proved and will be extensively used here for the prove of uniqueness of solutions of equation (1.1) and (3.12).

2. Preliminaries

In this section we shall consider some definitions and notions as will be used in subsequent section.
Definition 2.1:
Let f be a function

$$f : R^{n+1} \to R^n \text{ be carathéodory in a domain } D \subset R^{n+1}.$$

f is said to be generalized Lipschitz continuous (with respect to x) if for every two points $(t, x), (t, y) \in D$ there exists an integrable function $K, K : I \subset R \to R$

such that K(t) is bounded for all $t \in I$, and

$$\| f(t, x) - f(t, y) \| \le K(t) \| x - y \|.$$

Remark 2.2:
The known Lipschitz continuity is equivalent to the one above for $K(t) \equiv L$ for some constant $L > 0$.

Definition 2.3:
Let (X, d) be a metric space. A function $f : X \to X$ is called a contraction if there exists a real positive number, $0 \le \alpha \le 1$ such that $d(f(x), f(y)) \le \alpha d(x, y)$ for all $x, y \in X$.

Theorem 2.4:
Every contraction mapping on a complete metric space has a unique fixed point. *Proof.* [6].
Theorem 2.5:
Let X be a complete metric space and $A: X \to X$. Suppose that for some $n > 1$, $A^{(n)}$ has a unique fixed point, x_0. Then x_0 is a fixed point of A.
Proof:
Suppose that $A^{(n)}(x_0) = x_0$. We know that

$$A\left[A^{(n)}(x)\right] = A^{(n)}[A(x)] \qquad (2.1)$$

for all $x \in X$.

Since this is true for all $x \in X$ then this is true for x_0 that is

$$\begin{aligned} A(x_0) &= A\left[A^{(n)}(x_0)\right] \\ &= A^{(n)}\left[A(x_0)\right] \text{ from } (2.1) \\ &= A^{(n+1)}(x_0) \\ &= x_0. \end{aligned}$$

3. Main Results

In this section we shall state and prove the existence and uniqueness theorems for initial value problem (1.1) and (3.12) such that f satisfies generalized Lipschitz condition.
Theorem 3.1:
Suppose $f: R^{n+1} \to R^n$ be defined and continuous in a certain domain $D \subset R^{n+1}$ and that f be as in Definition 2.1.

Then forevery point $(t_0, x_0) \in D$ there exists a unique solution $x = \varphi(t)$ of (1.1) defined in some interval containing t_0, and which satisfies $\phi(t_0) = x_0$.

Proof:
Let f be a carathéodory function of the variable (t, x). Then the initial value problem (1.1) satisfying (1.2) has a solution $x(t) = x(t, t_0, x_0)$ on $I \subset R$ ($t_0 \in I$, I an open interval).

It is also known in [4] that x is a continuous solution of the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))d\tau \qquad (3.1)$$

and conversely. Thus the existence problem for (1.1) satisfying (1.2) is equivalent to that for (3.1). Now let u(t) be such that f(t, u(t)) is defined and continuous on I such that $t_0 \varepsilon$ I. Then

$$x_0 + \int_{t_0}^{t} f(\tau, x(\tau))d\tau \qquad (3.2)$$

defines a continuously differentiable function F(t), say, on I such that $F(t_0) = x_0$. Thus (3.2) gives rise to an operator A which maps the function u(t) onto the function F(t). That is

$$\begin{aligned} \left[A(u)\right](t) &= x_0 + \int_{t_0}^{t} f(\tau, x(\tau))d\tau \\ &= F(t) \qquad (3.3) \end{aligned}$$

for all $t \in I$. Thus if x is a solution of (3.1) so that

$$x_0 + \int_{t_0}^{t} f(\tau\, x(\tau))d\tau = x(t).$$

Then $[A(u)](t) = x(t)$ and so the solutions of (3.1) are fixed points of the operator A. Since $(t_0, x_0) \in D$, D is an open connected subset of R^{n+1} there exist positive numbers $\mu > 0$, $r > 0$ such that

$$S = \{(t, x) : |t - t_0| \leq, \|x - x_0\| \leq r\} \qquad (3.4)$$

lies in D (i.e. $S \subset D$).

Since S is compact and f is continuous on S, there exists M > 0 such that

$$\|f(t, x)\| \leq M \qquad (3.5)$$

for all $(t, x) \in S$.

Let $u = u(t)$ be an arbitrary function for which $A(u)$ is defined. Then

$$\| A(u)(t_2) - A(u)(t_1) \| \leq \left\| \int_{t_1}^{t_2} f(\tau, u(\tau))d\tau \right\| \leq M|t_2 - t_1| \qquad (3.6)$$

so that A(u) is continuous whenever it is defined. This suggests that we consider as domain for A a function space consisting of all continuous vector functions defined on some suitable interval of real line containing t_0. Note that in view of (3.4) it is suffices to prove the existence of a unique function

$x = x(t) \in S$, which is defined on some interval I, $t_0 \in I$

and contained in $|t-t_0| \leq T$, $T \leq \mu$, such that $x(t)$ satisfies (3.1).

Let $q > 0$ be an arbitrary number and

$Cq = \{\phi : \phi : R \to R^n$, which are defined and continuous on $|t-t_0| \leq q\}$

such that,

$$\|\phi(t) - x_0\| \leq r \qquad (3.7)$$

and $q \leq T$. For any f, $g \in C_q$ define the metric ρ by

$$\rho(f, g) = \sup_{|t-t_0| \leq q} \|f(t) - g(t)\|.$$

C_q is a complete metric space. For any $\varphi \in C_q$, $A\varphi$ is defined by

$$A\varphi = x_0 + \int_{t_0}^{t} f(\tau, \varphi(\tau))d\tau. \qquad (3.8)$$

We now show that subject to suitable restriction on q, that

(i) if $\varphi \in C_q$ then $A\varphi \in C_q$

(ii) if $\varphi_1, \varphi_2 \in C_q$, then there exists, $0 \leq \alpha < 1$ such that

$$\rho(A\varphi_1, A\varphi_2) \leq \alpha\rho(\varphi_1, \varphi_2).$$

Let $\varphi \in C_q$, then by (3.8)

$$\| A\varphi - x_0 \| \leq \left\| \int_{t_0}^{t} \|f(\tau, \varphi(\tau))\| d\tau \right\| \leq qM$$

Since $|t - t_0| \leq q$. Therefore $\| A\varphi - x_0 \| \leq r$ if

$$q < \frac{r}{M} \qquad (3.9)$$

Thus $A : C_q \to C_q$. If (3.8) is satisfied and $\varphi_1, \varphi_2 \in C_q$, then by (3.6)

$$\|A\phi_2 - A\phi_1\| \leq \left\| \int_{t_0}^{t} \|f(\tau, \phi_2(\tau)) - f(\tau, \phi_1(\tau))\| d\tau \right\|$$

$$\leq \left\| \int_{t_0}^{t} |K(\tau)| \| \phi_2(\tau) - \phi_1(\tau)\| d\tau \right\|$$

$$\leq \left\| \int_{t_0}^{t} |K(\tau)| (\| \phi_2(\tau) - \phi_1(\tau)\|) d\tau \right\|$$

(Cauchy Schwartz inequality)

$$\leq K_0 \left\| \int_{t_0}^{t} (\| \varphi_2(\tau) - \varphi_1(\tau)\|) d\tau \right\|$$

(for some $K_0 > 0$ a bound of $K(t)$)

$$\leq qK_0 \sup_{|t-t_0| \leq q} \| \varphi_2(t) - \varphi_1(t) \|$$

Thus provided

$$q \le \frac{\alpha}{K_0}, 0 \le \alpha < 1 \qquad (3.10)$$

$$\rho(A\varphi_2, A\varphi_1) \le \alpha\rho(\varphi_2, \varphi_1).$$

Let $q < \tau$ be fixed such that (3.9) and (3.10) hold, that is

$$q \le \min\left(T, \frac{r}{M}, \frac{\alpha}{K_0}\right).$$

Then A: $C_q \to C_q$ and A is a contraction on C_q, since C_q is complete there exists $x \varepsilon C_q$ such that Ax = x. Finally, we need to establish the uniqueness of this solution. Suppose the solution is not unique, then there exists another solution $z(t) = z(t, t_0, x_0)$ for (1.1) satisfying

$z(t_0) = x_0$ on $|t - t_0| \le q$. By (3.1) we have

$$x(t) = x(t_0) + \int_{t_0}^{t} f(\tau, x(\tau))d\tau$$

$$z(t) = z(t_0) + \int_{t_0}^{t} f(\tau, z(\tau))d\tau$$

By definition (2.1), we have that

$$\|z(t)\text{-}x(t)\| \le \|z(t_0)\text{-}x(t_0)\| + \int_{t_0}^{t} \|f(\tau, z(\tau))\text{-}f(\tau,x(\tau))\| d\tau$$

$$\le \|z(t_0)\text{-}x(t_0)\| + \int_{t_0}^{t} |K(\tau)| \|z(\tau)\text{-}x(\tau)\| d\tau$$

$$\le \|z(t_0)\text{-}x(t_0)\| + K_0 \int_{t_0}^{t} \|z(\tau)\text{-}x(\tau)\| d\tau$$

for some $K_0 > 0$, where K_0 is a bound of K(t). Now set

$$V(t) = \|z(t)\text{-}x(t)\| \qquad (3.11)$$

then the last inequality becomes,

$$V(t) \le V(t_0) + K_0 \int_{t_0}^{t} V(s)ds.$$

By generalized Gronwall's inequality, we have that

$$V(t) \le V(t_0) \, exp\left[K_0 |t\text{-}t_0|\right].$$

But $V(t_0) = \| z(t_0) - x(t_0)\| = 0$. Therefore, $V(t) \le 0$. By (3.11) $V(t) \ne 0$, hence

$$V(t) = 0$$

$$z(t) = x(t).$$

Thus the solution $x(t) = x(t, t_0, x_0)$ of (1.1) satisfying (1.2) exists and is unique.

Next, we consider n-vectors equations

$$\dot{x}(t) = f(t, x(t), \mu)$$

$$x(t_0, t_0, x_0, \mu_0) = x_0 \qquad (3.12)$$

where $\mu = col(\mu_1, \mu_2, \cdots, \mu_i, \cdots \mu_l)$, $x = col(x_1, x_2, \cdots, x_i, \cdots x_n)$,

$t \in R$, $x_0 \in R^n$, and $f: \bar{\Omega} \to R^n$, $\bar{\Omega} \subset R^{l+n+l}$

Theorem 3.2:

Let f be a carathéodory function defined on $\bar{\Omega}$ of the space variables $(t, x, \mu) \subset R^{l+n+l}$ suppose that for arbitrary points (t, x, μ), $(t, y, \mu) \in \bar{\Omega}$,

f satisfies

$$\|f(t, x, \mu) - f(t, y, \mu)\| \le K(t)\|x - y\|$$

where K(t) is the function defined in Definition 2.1. Then there exist positive constants r, ρ such that for $|\mu - \mu_0| < \rho$ the unique solution $x = \varphi(t, \mu)$ of the initial value problem (3.12) is defined for $|t - t_0| < r$ and is carathéodory in t, x, μ.

Proof:

Similar to the proof of Theorem 3.1 hence, it is omitted. This completes the proof of Theorem 3.2.

Example 1:

Given that F is a multi-valued map, defined on a subset

$$D \subset R^{n+1} \text{ that is } F: D \subset R^{n+1} \to R^n$$

$$\text{such that } F(t,x) \subset R^n \text{ for each } (t,x) \in D.$$

Suppose F is absolutely continuous, there exists a continuous selection f of F such that f satisfies the generalized Lipschitz condition (Definition 2.1). Then the differential inclusion

$$\dot{x}(t) \in F(t,x)$$
$$x(t_0) = x_0 \qquad (3.13)$$

has a solution in the domain D.

Proof:

$$F(t,x) = \{f(t,x) : (t,x) \in D\} \text{ for every continuous selection f of F}$$

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau, x(\tau))d\tau \quad (\text{since F is absolutely continuous})$$

which is a form of Theorem 3.1, since f satisfies generalized Lipschitz continuous condition. Hence there exists a solution to differential inclusion (3.13).

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