A COMPUTATIONAL ERROR ESTIMATE OF THE TAU METHOD FOR LINEAR INITIAL VALUE PROBLEMS

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Abstract

In Onumanyi and Ortiz (1982), a computationally efficient and general error estimation technique of the tau method for global solution of linear initial value problems (IVP's) defined in the finite interval $[a,b]$ was reported. The result has been extended in different directions and then applied to non-linear problems. In the present paper, we consider a theoretical estimate of the error of tau method for solution of linear initial value problems in the interval [0,1]. The error estimation technique is based on the practical approach of Onumanyi and Ortiz (1982). Numerical examples are given to illustrate the procedures.

Keywords: Error estimate, tau method, Chebyshev polynomials, tau approximant, linear initial value problems, differential system.

1. Introduction

The study of error is of central concern in numerical analysis since most numerical methods only provide approximations to the exact solution of mathematical problems. It is important therefore to estimate or bound the resulting error and any numerical method that fails to provide a suitable procedure for doing this is incomplete.

The first attempt on an error estimation of the tau method was given by Lanczos (1938) where he developed a simple algebraic approach to this problem using the relation of the Chebyshev polynomials to the trigonometric function and which was applied only to the restricted class of 1st order differential equations of the form

$$A(x)y'(x)+B(x)y(x)+C(x)=0$$

where A, B and C are given polynomials. Fox (1962) also reported an error estimation procedure for the tau method which can handle similar problems and of orders higher than one. However, his approach is rather complicated. Hence, the search for other methods which the present work is aimed at addressing. The basic approach of the tau method necessary for a clear understanding of the sequel will be restated in Section 2. Section 3 is on the global error estimation technique of the tau method. Numerical examples will be given in Section 4 and finally our conclusion will be stated in Section 5.

2. Global Tau Method

Accurate approximate polynomial solutions of linear ordinary differential equations with polynomial coefficients can be obtained by the tau method introduced by Lanczos (1938). The method is related to the principle of economization of a differentiable function implicitly defined by a linear differential equation with polynomial coefficients.

To illustrate the method, let us consider the m-th order linear differential equation

$$Ly(x):=\sum_{i=0}^{m}\left(\sum_{j=0}^{N_i}\rho_{ij}x^j\right)y^{(i)}(x)=\sum_{i=0}^{F}f_ix^i, \qquad a\le x\le b \tag{1}$$

with the smooth solution $y(x)$ satisfying a set of multi-point boundary conditions

$$L^*y(x_{ij}) := \sum_{i=0}^{m-1}a_{ij}y^{(i)}(x_{ij})=\beta_j, \qquad j=1,2,\ldots,m \tag{2}$$

where N_i, F are given non-negative integers; a_{ij}, x_{ij}, β_j, f_i, and p_{ij} are given real numbers (x_{ij} are points belonging to the interval $a\le x\le b$ for which the conditions (2) are specified) and $y^{(i)}(x)$ denotes the derivative of order i of $y(x)$.

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The idea of Lanczos is to approximate the solution of the differential equation (1) by the n-th order degree polynomial,

$$y_n(x) = \sum_{i=0}^{n} a_i x^i, \quad n < \infty$$

which is the exact solution of a perturbed equation obtained by adding to the right hand side of (1) a polynomial perturbation term. The polynomial $y_n(x)$ satisfies, then, the differential system

$$Ly_n(x) := \sum_{i=0}^{m}\left(\sum_{j=0}^{N_i} p_{ij}x^j\right)y_n^{(i)}(x) = \sum_{i=0}^{F} f_i x^i + H_n(x) \tag{3}$$

$$L^* y_n(x) := \sum_{i=0}^{m-1} a_{ij} y_n^{(i)}(x_{ij}); j = 1,2,\ldots,n \tag{4}$$

where the perturbation term, $H_n(x)$, is constructed in such a way that the differential system (3) has a polynomial solution of degree n. Generally, $H_n(x)$, following Lanczos, is taken as a linear combination of powers of x multiplied by Chebyshev polynomials. The choice of the Chebyshev polynomial $T_r(x)$ defined by

$$T_r(x) = \cos(r\cos^{-1}[\{2(x-a)/(b-a)\}-1]) = \sum_{j=0}^{r} C_j^{(r)} x^j \tag{5}$$

(with) $C_r^{(r)} = 2^{2r-1}(b-a)^{-r}$ stems from the desire to distribute the error $\max_x |y(x) - y_n(x)|$ evenly throughout the range $[a,b]$. From the point of view of accuracy, the perturbation term of the form

$$H_n(x) = \sum_{i=0}^{m+s-1} T_{m+s-1} T_{n-m+i+1}(x) \tag{6}$$

where $s = \max_{0 \le i \le m}\{N_i - 1\}$ will be considered in the present work. The parameters τ_i, $i = 1,2,\ldots,m+s$ are fixed so that the conditions (4) are satisfied exactly.
To determine the coefficients a_i, $i = 0(1)n$ in $y_n(x)$ involves solving a system of linear algebraic equations $A\tau = b$ obtained by equating corresponding coefficients of like powers of x in (3) and then using conditions (4) $\tau = (a_0,\ldots,a_n,\tau_1,\ldots,\tau_{m+s})^T$

As pointed out by Ortiz (1969), the tau method is of order p, in the sense that if the exact solution of (1) is itself a polynomial of degree less than or equal to p then method will reproduce it.
Definition 1
The differential system (3) will be called the tau problem corresponding to the differential system (1).
Definition 2
The n-th degree polynomial, $y_n(x)$, which satisfies the tau problem (3) will be referred to as the tau approximant of (1) and the tau solution of (3).

3. Global Error Estimation for IVP's

For the tau problem (3), the global error function $e_n(x) = y(x) - y_n(x), \quad a \le x \le b$ satisfies the error differential system

$$Le_n(x) = \sum_{i=0}^{m}\left(\sum_{j=0}^{N_i} p_{ij}x^j\right)e_n^{(i)}(x) = -H_n(x) \tag{7}$$

$$L^* e_n(\alpha) = \sum_{i=0}^{m-1} a_{ij} e_n^{(i)}(\alpha) = 0; \quad j = 1,2,\ldots,m \tag{8}$$

where $\alpha = a$.
By the economization of power series, the reduction of the n-th degree polynomial $\phi_0 + \phi_1 + \cdots + \phi_n x^n$ to one of degree n-1 in the interval $[a,b]$ involves an error polynomial $\phi_n T_n(x)/C_n^{(n)}$ of degree n. Replacing n by n-m+1 in the monomial $T_n(x)/C_n^{(n)}$ and choosing a polynomial $\mu_m = (x-\alpha)^m$, of degree m, then

$$e_n(x) \cong E_{n+1}(x) = \phi_n \mu_m(x) T_{n+1}(x)/C_{n-m+1}^{(n-m+1)} \tag{9}$$

where ϕ_n is now considered an undetermined constant in (9).

The choice $\mu_m(x) = (x-\alpha)^m$
is to ensure that $E_{n+1}(x)$ satisfies all the homogeneous conditions of $e_n(x)$. We economize $E_{n+1}(x)$ by considering the perturbed error equation

$$LE_{n+1}(x) = -H_n(x) + \sum_{i=0}^{m+s-1} \bar{\tau}_{m+s-i} T_{n-m+i+2}(x) \tag{10}$$

where the extra tau parameters $\bar{\tau}_1, \ldots, \bar{\tau}_{m+s}$ are to be determined along with ϕ_n. From the left hand side of (10) we have

$$LE_{n+1}(x) = \frac{\phi_n}{D_1}\left(\lambda_1 x^{n+s+1} + \lambda_2 x^{n+s} + \cdots + \lambda_{m+s+1} x^{n-m+1} + \cdots\right) \tag{11}$$

where $\lambda_v; v =,1,2,\ldots,m+s+1,\ldots;$ are given constants (see equation below) and $D_1 = C_{n-m+1}^{(n-m+1)}$. Now by considering (11) in the form

$$LE_{n+1}(x) = \frac{\phi_n}{D_1}\left[\frac{R_1 T_{n+s+1}(x)}{C_{n+s+1}^{(n+s+1)}} + \frac{R_2 T_{n+s}(x)}{C_{n+s}^{(n+s)}} + \cdots + \frac{R_{m+s+1} T_{n-m+1}(x)}{C_{n-m+1}^{(n-m+1)}} + \cdots\right] \tag{12}$$

we obtain the recursive formula

$$R_v = \lambda_1$$

$$R_v = \lambda_v - \sum_{i=1}^{v-1} \frac{C_{n+s+2-v}^{(n+s+2-v)}}{C_{n+s+2-i}^{(n+s+2-i)}} R_i; v = 2,3,\ldots,m+s+1 \tag{13}$$

where

$$\lambda_v = \sum_{i=1}^{v}\left(\sum_{j=0}^{m} P_{j,s-v+i+j}\right)(j!)\binom{n+2-i}{j} D_i; v = 1,2,\ldots,m+s+1$$

$$D_v = \begin{cases} C_{n-m-v+2}^{(n-m+1)} + \sum_{r=1}^{v-1}(-1)^r \binom{m}{r}\alpha^r C_{n-m-v+r+2}^{(n-m+1)}; & (14) \\ \qquad v = 1,2,\ldots,m & \\ C_{n-m-v+2}^{(n-m+1)} + \sum_{r=1}^{m}(-1)^r \binom{m}{r}\alpha^r C_{n-m-v+r+2}^{(n-m+1)}; & (15) \\ \qquad v = m+1,\ldots,m+s+1 & \end{cases}$$

Finally, comparing the coefficient of $T_{n-m+1}(x)$ in (10) and (12) to determine $\bar{\tau}_1, \ldots, \bar{\tau}_{m+s}$ and ϕ_n leads to

$$\phi_n = \frac{D_1^2 \tau_{m+s}}{R_{m+s+1}}, R_{m+s+1} \neq 0 \tag{16}$$

Substituting (16) into (9) gives the aposteriori estimate

$$\max_x\{|e_n(x)|\} \equiv \max_x\{|E_{n+1}(x)|\}$$
$$\leq \{2^{2n-2m+1}(b-a)^{2m-n-1}|\tau_{m+s}|\}/|R_{m+s+1}|$$
$$R_{m+s+1} \neq 0 \tag{17}$$

The exact error is obtained using 100 equidistant points, as

$$\varepsilon = \max_{a\leq x_i \leq b}|y(x_i) - y_n(x_i)|, \quad i = 0(1)100 \tag{18}$$

The above formulation of the error estimation procedure for tau method gives a better accuracy at the expense of computer time only as demonstrated in the numerical examples considered below. For these examples, the numerical results were obtained using the computer Tau Program which we developed for the tau method and which incorporates the error formula (17) and the exact error (18).

4. Numerical Examples

We consider the following linear initial value problems.

Example 1

We consider the first order differential equation

$$\left.\begin{aligned} &Ly(x) := y'(x) + 2xy(x) = 0, \quad [0,1] \\ &y(0) = 1 \end{aligned}\right\} \qquad (19)$$

For this problem, $m = 1, s = 0, \mu_m = 1, P_{0,0} = 2. P_{1,0} = 1, P_{1,1} = 0$ $f(x) = 0$ and the analytical solution is given by $y(x) = e^{-x^2}$

We assume a tau approximant

$$y_n(x) = \sum_{i=0}^{n} a_i x^i, \qquad n = 6,7,8,9 \qquad (20)$$

Which satisfies the tau problem

$$y'_n(x) + 2xy_n(x) = \tau_1 T_{n+1}(x) \qquad (21)$$

The associated error function $e_n(x)$, then satisfies

$$e'_n(x) + 2xe_n(x) = -\tau_1 T_n(x) \qquad (22)$$

While the error approximant, $E_{n+1}(x)$, defined as

$$E_{n+1}(x) = \frac{\phi_n T_n(x)}{C_n^{(n)}} \qquad (23)$$

Satisfies the perturbed error problem

$$LE_{n+1}(x) = -\tau_1 T_n(x) + \bar{\tau}_1 T_{n+1}(x) \qquad (24)$$

From the left hand side of (24), we have

$$LE_{n+1}(x) = \frac{\phi_n}{D_1}\left(\lambda_1 x^{n+1} + \lambda_2 x^n\right) \qquad (25)$$

Now, equation (25) may be re-written in the form

$$LE_{n+1}(x) = \frac{\phi_n}{D_1}\left[\frac{R_1 T_{n+1}(x)}{C_{n+1}^{(n+1)}} + \frac{R_2 T_n(x)}{C_n^{(n)}}\right] \qquad (26)$$

where $C_n^{(n)} = D_1$.

By comparing the coefficient of $T_n(x)$in (24) and (26) gives

$$\phi_n = \frac{-D_1^2 \tau_1}{R_2} \qquad (27)$$

Substituting (27) into (23) gives the aposteriori estimate

$$E_{n+1}(x) = \frac{-D_1 \tau_1 T_n(x)}{|R_2|} \qquad (28)$$

where τ_1 is obtained from (21) and R_2 is given by

$$R_1 = \lambda_1$$

$$R_2 = \lambda_2 - \frac{C_n^{(n+1)}}{C_{n+1}^{(n+1)}} R_1$$

$$\lambda_1 = \{P_{0,0} + (n+1)P_{1,1}\}D_1 = 2D_1$$

$$\lambda_2 = (n+1)P_{1,0}D_1 + [P_{0,0} + nP_{1,1}]D_2$$

$$= (n+1)D_1 + 2D_2$$

$$D_1 = C_n^{(n)}$$

$$D_2 = C_{n-1}^{(n)}$$

Some numerical results for this example are presented in Table 1 below.

Table 1: Error and Error Estimate for Example 1

n	6	7	8	9
Error Estimate	2.75×10^{-6}	1.80×10^{-7}	3.77×10^{-8}	6.51×10^{-9}
Exact Error	2.27×10^{-6}	1.49×10^{-7}	3.25×10^{-8}	6.03×10^{-9}

Example 2

$$\left.\begin{aligned} &Ly(x) := y''(x) + 3y'(x) = -18x, \quad [0,1] \\ &y(0) = 0, \quad y'(0) = 5 \end{aligned}\right\} \tag{29}$$

In this case, $m = 2, s = 0, \quad f(x) = -18x, \quad P_{0,0} = 0, \quad P_{1,0} = 3. \quad P_{1,1} = 0, P_{2,0} = 1, \quad P_{2,1} = 0, \quad P_{2,2} = 0$ and the analytical solution is given by

$$y(x) = 1 + 2x - 3x^2 - e^{-3x}$$

We seek a tau approximant

$$y_n(x) = \sum_{i=0}^{n} a_i x^i, n = 6,7,8,9 \tag{30}$$

which satisfies the perturbed equation

$$y_n(x) + 3y_n'(x) = \tau_1 T_{n+1}(x) + \tau_2 T_{n-1}(x) \tag{31}$$

The associated error approximant $E_{n+1}(x)$, satisfies

$$LE_{n+1}(x) = -\tau_1 T_{n+1} - \tau_2 T_{n-1} + \tau_1 T_{n+1} + \bar{\tau}_2 T_{n-1} \tag{32}$$

where

$$E_{n+1}(x) = \phi_n T_{n-1} / C_{n-1}^{(n-1)} \tag{33}$$

For this problem, we obtain the error estimate

$$E_{n+1}(x) = \frac{2^{2n-3} \mid \tau_2 \mid}{\mid R_3 \mid} \tag{34}$$

$$R_1 = \lambda_1$$

$$R_2 = \lambda_2 - \frac{C_n^{(n+1)}}{C_{n+1}^{(n+1)}}$$

$$R_1 = \lambda_1$$

$$R_3 = \lambda_3 - \frac{C_{n-1}^{(n+1)}}{C_{n+1}^{(n+1)}} - \frac{C_{n-1}^{(n)}}{C_n^{(n)}}$$

$$\lambda_1 = \{P_{0,0} + (n+1)P_{1,1} + n(n+1)P_{2,2}\}D_1 = 0$$

$$\begin{aligned} \lambda_2 = \{(n+1)P_{1,0} + n(n+1)P_{2,1}\}D_1 + \{P_{0,0} + nP_{1,1} + n(n-1)P_{2,2}\}D_2 \\ + \{P_{0,0} + (n-1)P_{1,1} + n(n-1)(n-2)P_{2,2}\}D_3 \\ = n\{(n+1)D_1 + D_2\} \end{aligned}$$

$$D_1 = C_{n-1}^{(n-1)}$$

$$D_2 = C_{n-2}^{(n-1)}$$

$$D_3 = C_{n-3}^{(n-1)}$$

Table 2 below shows some computed results for Example 2.

Table 2: Error and error estimate for example 2

n	6	7	8	9
Error Estimate	4.60×10^{-6}	7.37×10^{-7}	1.19×10^{-8}	3.86×10^{-9}
Exact Error	4.24×10^{-6}	6.64×10^{-7}	1.09×10^{-8}	3.21×10^{-9}

5. Conclusion

An error estimation of the tau method has been described and applied to smooth polynomial solutions of linear initial value problems. The error estimate is compared with the exact error and we observed that the accuracy of the error estimate obtained improves as n, the degree of the tau solution increases. Also, the estimate is good and gives correctly the order of accuracy of the tau approximant being sought.

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