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EXACT DISTRIBUTION OF ANSARI-BRADLEY TEST STATISTIC: PERMUTATION APPROACH

J.I. ODIASE+, S.M. OGBONMWAN
Department of Mathematics, University of Benin, Benin, Nigeria.

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Abstract

The Ansari-Bradley test is an alternative to the Siegel-Tukey test for the two sample problem for detecting changes in scale between two populations having the same location parameter. The exact sampling distribution of the test statistic for an experiment is compiled by the permutation approach. In the case of independent groups, this test depends on compiling all possible permutations of the values that result from an experiment. An algorithm for the exact permutation distribution of the Ansari-Bradley test statistic is presented and implemented. Tables of exact p-values for the test statistic are created and the probability of a type I error is obtained to be exactly α. The paper reveals that for sample size 15, convergence of the asymptotic distribution to the exact distribution is far from being achieved.

Keywords: Permutation test, Monte Carlotest, p-value, Ansari-Bradley test.

1. Introduction

The p-value is the smallest level of significance at which H_0 would be rejected when a specified test procedure is used on a given data set. The use of the asymptotic test with small sample sizes may yield an incorrect p-value and therefore lead to a false acceptance or rejection of the null hypothesis. Application of the asymptotic test when the sample size is small can lead to a wrong decision, see Mundry and Fischer (1998). Scheffe (1943) asserted that for a general class of problems, the permutation approach is the only possible method of constructing exact tests of significance. The idea of a general method of dealing with the fundamental problem of statistical inference, that is, obtaining exact tests of significance when the underlying probability distribution is unknown is credited to R. A. Fisher, see Wald and Wolfowitz (1944).

Fisher (1936) proposed that randomization should be the basis for experimental design and statistical inference. The premise behind experimental design is that a sample of experimental units, however acquired, is divided randomly into two or more groups. These are then exposed to different treatments. The null hypothesis is that the treatments have no differential effects on the groups with respect to a selected statistic. If there is no requirement that the test statistic should conform to a mathematically definable frequency distribution, then the exact sampling distribution of the test statistic can be compiled by permutation, see Ludbrook and Dudley (1998). Computational advances involving the use of permutations tests are well documented in Good (2000) and Pesarin (2001).

There are two approaches to permutation test; conditional and unconditional approaches. In the unconditional approach, the row (levels of treat-ment) and column (treatments) totals of the tabulated experimental results are allowed to vary along with the permutation of observations or ranks of observations in an experiment while the conditional approach is constrained to have fixed row and column totals as obtained in the actual arrangement of observations of the experiment. According to Agresti (1992), Mehta (1992), Hall and Tajvidi (2002) and Opdyke (2003), the unconditional approach is computationally demanding. Several approaches which are computationally less demanding have been suggested as alternatives. Efron (1979), Efron and Tibshirani (1993), Hall and Tajvidi (2002), Opdyke (2003) presented Monte Carlo approaches. Other approaches like the Bayesian and the likelihood have also been found useful in obtaining approximate exact permutation distribution, see Bayarri and Berger (2004), Spiegelhalter (2004).

The problem with permutation tests has been high computational demands. Computational time for a permutation test is highly prohibitive. R.A. Fisher compiled by hand 32,768 permutations of Charles Darwin's data on the height of cross-fertilized and self-fertilized zea mays plants. It is believed that the enormity of this task is what possibly discouraged Fisher from further research into exact permutation tests, see Bennett (1990) and Ludbrook and Dudley (1998).

Available permutation procedures can sample from the permutation sample space rather than carrying out complete enumeration of all possible distinct

+ *corresponding author* (email:justiceodiase@yahoo.com)

permutations. These available procedures can perform Monte Carlo sampling without replacement within a sample, but none can avoid the possibility of drawing the same sample more than once, thereby reducing the power of the permutation test, see Opdyke (2003). An algorithm for obtaining exact permutation distribution is developed and implemented for the Ansari-Bradley test statistic in this work.

2. Statistical test procedure

Statistical test is based on calculating the test statistic of interest, comparing the calculated test statistic with a critical value and accepting or rejecting the null hypothesis based on the outcome of the comparison. The critical values are usually determined by cutting of the most extreme $100\alpha\%$ of the theoretical frequency distribution of the test statistic, where α is the level of significance, see Siegel and Castellan (1988).

Theoretical frequency distribution of a test statistic for many nonparametric tests is estimated by either using the small sample test statistic or using the large sample asymptotic distribution of the test statistic. For small sample size, the exact probability of obtaining the calculated value of the test statistic or any less likely value has to be determined. The sum of the probabilities of these less likely values is the exact p-value of the test statistic. This procedure to determine the exact probability of a specific value of a test statistic can be obtained through a permutation approach and it is computationally intensive. With large sample size, the frequency distribution of a test statistic is often asymptotically a normal or a chi-square distribution.

The purpose of this paper is to provide a simple but systematic way of obtaining unconditional exact permutation distribution of the Ansari-Bradley test by ensuring that a complete enumeration of all the distinct permutations are generated. The Ansari-Bradley test is an alternative to the Siegel-Tukey test for the two sample problem for detecting changes in scale between two populations having the same location parameter. In order to compute the Ansari-Bradley two-sample scale statistic W, the procedure orders the combined sample (in increasing order) and assigns the score 1 to both the smallest and largest observations in the combined sample, assigns the score 2 to the second smallest and second largest, and so on, that is:

$$\alpha(1)=1,\ \alpha(\mathrm{N})=1,\ \alpha(2)=2,\ \alpha(\mathrm{N}-1)=2,\ldots$$

whereas, the Siegel-Tukey scores are computed as:

$$\alpha(1)=1,\alpha(\mathrm{N})=2,\alpha(\mathrm{N}-1)=3,\alpha(2)=4,\alpha(3)=5,\alpha(\mathrm{N}-2)=6,$$
$$\alpha(\mathrm{N}-3)=7,\alpha(4)=8,\ldots$$

where the score values continue to increase in this pattern towards the middle ranks until all observations have been assigned a score. Ansari-Bradley scores are similar to Siegel-Tukey scores, but Ansari-Bradley assigns the same scores to corresponding extreme ranks.

Fahoome (2002) noted that when $\alpha = 0.05$, the sample size should exceed 15 for the large sample approximation to be adopted for the Ansari-Bradley test. The unconditional exact distribution of the Ansari-Bradley test statistic is generated for m = n ≤15 through the unconditional exact permutation algorithm provided in this paper, where m is the size of the first sample and n is the size of the second sample in a two-sample experiment.

In two independent samples scale tests, the population distributions are usually assumed to have the same location parameter with different spreads. The nonparametric Ansari-Bradley test is a rank test for spread when the population medians are the same or are different but known. The null hypothesis is that the two populations have the same spread, which is tested against the alternative that the variability of the two populations differs.

Combine the two samples and rank the observations as required for the Ansari-Bradley test, keeping track of sample membership. The Ansari-Bradley test statistic, W, is the sum of the ranks of the first sample of size *m*.

$$\mathrm{W} = \sum_{i=1}^{m} R_i \qquad (1)$$

where R_i is the rank of the *ith* observation of the first sample. For large sample sizes, the test statistic becomes

$$\mathrm{W}^* = \frac{W - \dfrac{m(\mathrm{m}+\mathrm{n}+2)}{4}}{\sqrt{\dfrac{\mathrm{mn}(\mathrm{m}+\mathrm{n}+2)(\mathrm{m}+\mathrm{n}-2)}{48(\mathrm{m}+\mathrm{n}-1)}}} \qquad (2)$$

if N = m+n is even and

$$\mathrm{W}^* = \frac{\mathrm{W} - \dfrac{\mathrm{m}(\mathrm{m}+\mathrm{n}+1)^2}{4(\mathrm{m}+\mathrm{n})}}{\sqrt{\dfrac{\mathrm{mn}(\mathrm{m}+\mathrm{n}+1)[3+(\mathrm{m}+\mathrm{n})^2]}{48(\mathrm{m}+\mathrm{n})^2}}} \qquad (3)$$

if N = m+n is odd. W^* is asymptotically normally distributed and hence reject the two-sided null hypothesis if

$$W^* \geq Z_{\alpha/2},$$

see Fahoome (2002).

3. Methodology

The p-value of a test statistic represents the probability of obtaining values of the test statistic that are equal to or more extreme than the observed value of the test statistic. For the continuous case, the p-value is obtained by finding the area under the curve of the theoretical distribution of the test statistic in the direction of the alternative hypothesis, while the discrete case involves adding up the probabilities of events occurring in the direction of the alternative hypothesis that occur at and after the observed value of the test statistic.

In this paper, consideration is given to the unconditional permutation distribution where the row and column totals of the tabulated data of an experiment are allowed to vary with each permutation, see Agresti (1992). In Good (2000), a consideration was given to the tails of permutation distribution in order to arrive at p-values without actually carrying out complete enumeration required for a permutation test. This approach has no precise model for the tail of the distribution from which data are drawn, see Hall and Weissman (1997).

The difficulty in permutation test lies in obtaining all the distinct arrangements of the values obtained in a given experiment. For example, a two-sample experiment with 16 variates in each sample requires

$$\frac{(16+16)!}{(16)!(16)!} = 601{,}080{,}390 \text{ permutations.}$$

A frequency distribution is subsequently arrived at for all the distinct occurrences of the test statistic from which the probability distribution of the test statistic is computed. Permutation test requires few assumptions as a nonparametric procedure. The sufficient condition for a permutation test to be exact and unbiased against shifts in the direction of higher values is the exchangeability of the observations in the combined sample, see Good (2000) and Opdyke (2003).

Suppose two random samples X= $\{x_1, x_2, ..., x_m\}$ and Y=$\{y_1, y_2, ..., y_m\}$ with independent distribution functions F_X and F_Y are simultaneously tested in an experiment with W as the test statistic. Let $H_0 : F_X = F_Y$ against $H_1 : F_X = F_Y$ or $H_1 : F_X < F_Y$ or $H_1 : F_X > F_Y$

Each of the ((m+n)/m) distinct permutations occurs with the probability

$$\frac{m!n!}{(m+n)!}$$

For k distinct values of the test statistic W, the probability distribution of the test statistic $W = w_j, j = 1(1)k$ under the null hypothesis $H_0 : F_X = F_Y$ is given by

$$P(w_j = w_0 | H_0) = f_j \frac{(m!n!)}{(m+n)!},$$

where f_j is the number of occurrences of w_j. For specified values of m,n and the level of significance α, the critical value c corresponds to a level α' closest to α in the cumulative distribution of the test statistic. Ordering all the distinct occurrences of W in ascending order of magnitude, if g is the position of the observed value of W, we have the following significance level for the left tail of the distribution of the test statistic as,

$$\alpha = P(w_g \le c | H_0) = \frac{m!n!}{(m+n)!} \sum_{j=1}^{g} f_j \quad (4)$$

And for the right tail as,

$$\alpha = P(w_g \le c | H_0) = \frac{m!n!}{(m+n)!} \sum_{j=g}^{k} f_j \quad (5)$$

For a two-tailed test, the left and right tails are summed up. Clearly, when the distribution of the test statistic is symmetric,

$$\sum_{j=1}^{g} f_j = \sum_{j=k-g+1}^{k} f_j.$$

In formulating the computer algorithm for unconditional exact permutation distribution, a consideration is given to rank order statistics. First, rank the observed values as required by the Ansari-Bradley test, such that any of the arrangements of the ranks can be used for a full enumeration. For an illustration, take a simple balanced case as the original arrangement of ranks for the observed values of the experiment, that is,

$$\begin{pmatrix} 1 & \frac{N}{2} \\ 2 & \vdots \\ 3 & 3 \\ \vdots & 2 \\ \frac{N}{2} & 1 \end{pmatrix}; \text{where } N = m + n.$$

A discussion of a systematic way of obtaining all the possible permutations of the N variates now follows. Assuming the observed balanced two-sample layout with m = n is represented by,

$$\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_3 \end{pmatrix}, \text{where } x_i \text{ and } y_j$$

are ranks of the two samples in an experiment and constructed as required by the Ansari-Bradley test. All the possible permutations are obtained for $i, j = 1(1)n$ now follow.

The original arrangement of ranks

$$\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_3 \end{pmatrix}, \text{yields} \binom{n}{0}\binom{n}{0} = 1\, permutation.$$

The exchange of one sample rank, that is,

$x_i \leftarrow y_j$, yields $\binom{n}{1}\binom{n}{1}$ *permutations*, see algorithm 1.

The number of ways two sample ranks from one sample can replace two sample ranks from the second sample can be represented as

$$\begin{pmatrix} x_s \\ x_t \end{pmatrix} \leftarrow \begin{pmatrix} y_i \\ y_j \end{pmatrix}; s \neq t, i \neq j, \text{ and yields } \binom{n}{2}\binom{n}{2} \textit{ permutations, see algorithm 2.}$$

Continuing, the number of ways three sample ranks from one sample can replace three sample ranks from the second sample is represented thus:

$$\begin{pmatrix} x_s \\ x_t \\ x_u \end{pmatrix} \leftarrow \begin{pmatrix} y_i \\ y_j \\ y_k \end{pmatrix}; s \neq t \neq u, i \neq j \neq k, \text{ yields } \binom{n}{3}\binom{n}{3} \textit{ permutation.}$$

(see Odiase and Ogbonmwan (2005) for details).

3.1 Presentation of permutation algorithms

Let the sample size of the *jth* sample be *k*. Also, let x_{ij}, for $i = 1(1)k$, $j = 1, 2$, represent the ranks of observations in an experiment as required by the Ansari-Bradley test.

The algorithms presented in this paper can be extended until the desired sample size is attained. The number of permutations grows rapidly as the sample size increases, this translates to increased computer time required to implement the algorithms. Observe that for unequal sample sizes, the number of permutations is

$$\sum_{i=0}^{\min(m,n)} \binom{m}{i}\binom{n}{i} \text{ since } \binom{a}{b} = 0 \text{ for } b > a.$$

The test statistic is computed for each permutation in the complete enumeration of the distinct permutations. The distribution of the statistic is obtained by tabulating the distinct values of the statistic against their probabilities of occurrence in the complete enumeration, bearing in mind that all the permutations are equally likely.

3.2 Tables of critical values

The algorithms were implemented in Intel Visual FORTRAN. The results for the lower and upper critical values W_α for the Ansari-Bradley test statistic are given respectively in Tables 1-2 and Tables 3-4 for $7 \leq m = n \leq 15$, while the exact p-values for $2 \leq m = n \leq 6$ are in the Appendix.

Observe that for a two-sample experiment with less than four variates in each sample, the null hypothesis of no difference between the samples involved cannot be rejected at 5% level of significance because the least p-value for the test statistic is greater than 0.05.

The unconditional permutation algorithm described so far can work for equal and unequal sample sizes but it was implemented for a two-sample problem with equal sample sizes.

The results obtained from the implementation of the permutation algorithms presented in this paper gave rise to the lower and upper critical values (W_α) for the Ansari-Bradley test statistic presented in Tables 1-4.

4. Conclusion

The p-value obtained through the unconditional permutation approach is exact, see Agresti (1992) and Good (2000). Obtaining exact p-values through unconditional permutation has remained difficult because it is computationally intensive. For small sample sizes (m,n < 16), the unconditional permutation distribution of the Ansari-Bradley test statistic is reasonably different from its asymptotic equivalence. In small samples, the use of asymptotic Ansari-Bradley test of significance leads to higher p-values and thus to an increase in the probability of a type II error, that is, false acceptance of the null hypothesis. For a balanced two-sample problem with a sample size of 15, the exact distribution is far from converging to the asymptotic distribution. A sample size much higher than 15 is proposed for the application of the asymptotic version of the test.

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Algorithm 1 Exchange of one sample rank

```
for a ← 1, k do
    swap ← x_a1
    for b ← 1, k do
        x_a1 ← x_b2
        x_b2 ← swap
        Compute W
    end for
end for
```

Algorithm 2 Exchange of two sample ranks

```
for a ← 1,( k-1) do
    swap1 ← x_a1
    for b ← 1, k do
        swap2 ← x_b1
        for c ← (b + 1), k do
            t ← (c + 1)
            for d ← t, k do
                x_a1 ← x_c2
                x_c2 ← swap1
                x_b1 ← x_d2
                x_d2 ← swap2
                Compute W
            end for
        end for
    end for
end for
```

Algorithm 3 Exchange of three sample ranks

```
for a ← 1,( k-2) do
    swap1 ← x_a1
    for b ← (a +1), (k-1) do
        swap2 ← x_b1
        for c ← (b +1), k do
            swap3 ← x_c1
            for d ← 1, k do
                t ← (d +1)
                for e ← t, k do
                    t1 ← (e + 1)
                    for f ← t1, k do
                        x_a1 ← x_d2
                        x_d2 ← swap1
                        x_b1 ← x_e2
                        x_e2 ← swap2
                        x_c1 ← x_f2
                        x_f2 ← swap3
                        Compute W
                    end for
                end for
            end for
        end for
    end for
```

Table 1: Lower critical values W_α for the Ansari-Bradley test statistic (α = 0.001, 0.0025, 0.005, 0.01) $\alpha' = P(W \leq W_\alpha)$, $7 \leq m = n \leq 15$

α		0.001		0.0025		0.005		0.01	
m	n	W_α	α'	W_α	α'	W_α	α'	W_α	α'
7	7	16	0.0006 (0.0144)	17	0.0017 (0.0226)	18	0.0052 (0.0343)	19	0.0122 (0.0506)
8	8	22	0.0011 (0.0182)	23	0.0026 (0.0261)	24	0.0059 (0.0365)	19	0.0115 (0.0501)
9	9	28	0.0009 (0.0165)	29	0.0019 (0.0224)	30	0.0037 (0.0299)	32	0.0118 (0.0514)
10	10	35	0.0008 (0.0159)	37	0.0029 (0.0267)	38	0.0049 (0.0341)	39	0.0081 (0.0430)
11	11	43	0.0009 (0.0161)	45	0.0025 (0.0252)	46	0.0040 (0.0312)	48	0.0094 (0.0468)
12	12	52	0.0010 (0.0167)	54	0.0024 (0.0247)	56	0.0054 (0.0359)	58	0.0109 (0.0508)
13	13	62	0.0011 (0.0175)	64	0.0025 (0.0249)	66	0.0050 (0.0346)	68	0.0094 (0.0473)
14	14	72	0.0009 (0.0158)	75	0.0026 (0.0254)	77	0.0048 (0.0342)	80	0.0112 (0.0518)
15	15	84	0.0011 (0.0172)	87	0.0028 (0.0263)	89	0.0049 (0.0343)	92	0.0103 (0.0500)

The asymptotic p-values of W are in parentheses

Table 2: Lower critical values W_α for the Ansari-Bradley test statistic (α = 0.025, 0.05, 0.1) $\alpha' = P(W \leq W_\alpha)$, $7 \leq m = n \leq 15$

α		0.025		0.05		0.1	
m	n	W_α	α'	W_α	α'	W_α	α'
7	7	20	0.0256 (0.0726)	21	0.0466 (0.1012)	22	0.0804 (0.1373)
8	8	26	0.0211 (0.0676)	28	0.0572 (0.1160)	29	0.0867 (0.1478)
9	9	34	0.0305 (0.0838)	35	0.0460 (0.1048)	37	0.0938 (0.1578)
10	10	42	0.0282 (0.0815)	44	0.0560 (0.1189)	46	0.1007 (0.1671)
11	11	51	0.0274 (0.0812)	53	0.0501 (0.1129)	56	0.1076 (0.1758)
12	12	61	0.0274 (0.0820)	63	0.0467 (0.1097)	66	0.0932 (0.1630)
13	13	72	0.0281 (0.0836)	74	0.0450 (0.1083)	78	0.1012 (0.1724)
14	14	83	0.0232 (0.0760)	87	0.0539 (0.1206)	90	0.0924 (0.1644)
15	15	96	0.0249 (0.0792)	100	0.0529 (0.1200)	104	0.1004 (0.1736)

The asymptotic p-values of W are in parentheses

Table 3: Upper critical values W_α for the Ansari-Bradley test statistic (α = 0.001, 0.0025, 0.005, 0.01)

$\alpha' = P(W \leq W_\alpha), 7 \leq m = n \leq 15$

α		0.001		0.0025		0.005		0.01	
m	n	W_α	α'	W_α	α'	W_α	α'	W_α	α'
7	7	40	0.0006 (0.0144)	39	0.0017 (0.0226)	38	0.0052 (0.0343)	37	0.0122 (0.0506)
8	8	50	0.0011 (0.0182)	49	0.0026 (0.0261)	48	0.0059 (0.0365)	47	0.0115 (0.0501)
9	9	62	0.0009 (0.0165)	61	0.0019 (0.0224)	60	0.0037 (0.0299)	58	0.0118 (0.0514)
10	10	75	0.0008 (0.0159)	73	0.0029 (0.0267)	72	0.0049 (0.0341)	71	0.0081 (0.0430)
11	11	89	0.0009 (0.0161)	87	0.0025 (0.0252)	86	0.0040 (0.0312)	84	0.0094 (0.0468)
12	12	104	0.0010 (0.0167)	102	0.0024 (0.0247)	100	0.0054 (0.0359)	98	0.0109 (0.0508)
13	13	120	0.0011 (0.0175)	118	0.0025 (0.0249)	116	0.0050 (0.0346)	114	0.0094 (0.0473)
14	14	138	0.0009 (0.0158)	135	0.0026 (0.0254)	133	0.0048 (0.0342)	130	0.0112 (0.0518)
15	15	156	0.0011 (0.0172)	153	0.0028 (0.0263)	151	0.0049 (0.0343)	148	0.0103 (0.0500)

The asymptotic p-values of W are in parentheses

Table 4: Upper critical values W_α for the Ansari-Bradley test statistic (α = 0.025, 0.05, 0.1)

$\alpha' = P(W \leq W_\alpha), 7 \leq m = n \leq 15$

α		0.025		0.05		0.1	
m	n	W_α	α'	W_α	α'	W_α	α'
7	7	36	0.0256 (0.0726)	35	0.0466 (0.1012)	34	0.0804 (0.1373)
8	8	46	0.0211 (0.0676)	44	0.0572 (0.1160)	43	0.0867 (0.1478)
9	9	56	0.0305 (0.0838)	55	0.0460 (0.1048)	53	0.0938 (0.1578)
10	10	68	0.0282 (0.0815)	66	0.0560 (0.1189)	64	0.1007 (0.1671)
11	11	81	0.0274 (0.0812)	79	0.0501 (0.1129)	76	0.1076 (0.1758)
12	12	95	0.0274 (0.0820)	93	0.0467 (0.1097)	90	0.0932 (0.1630)
13	13	110	0.0281 (0.0836)	118	0.0450 (0.1083)	114	0.1012 (0.1724)
14	14	127	0.0232 (0.0760)	123	0.0539 (0.1206)	120	0.0924 (0.1644)
15	15	144	0.0249 (0.0792)	140	0.0529 (0.1200)	136	0.1004 (0.1736)

The asymptotic p-values of W are in parentheses

Appendix

Unconditional Permutation (α) and Asymptotic (α') p-values for Ansari-Bradley test

W	2x2		2x3		2x4		2x5		2x6	
	α	α'	α	α'	α	α'	α	α'	α	α'
2	0.1667	0.1103								
3	0.8333	0.5000								
4	1.0000	0.8897	0.1000	0.0984						
5			0.3000	0.2593						
6			0.7000	0.5000	0.0143	0.0471				
7			0.9000	0.7407	0.0714	0.1047				
8			1.0000	0.9016	0.2000	0.2014				
9					0.3714	0.3379	0.0079	0.0359		
10					0.6286	0.5000	0.0238	0.0668		
11					0.8000	0.6621	0.0714	0.1151		
12					0.9286	0.7986	0.1508	0.1841	0.0011	0.0197
13					0.9857	0.8953	0.2698	0.2743	0.0054	0.0336
14					1.0000	0.9529	0.4127	0.3821	0.0152	0.0546
15							0.5873	0.5000	0.0368	0.0848
16							0.7302	0.6179	0.0736	0.1262
17							0.8492	0.7257	0.1342	0.1800
18							0.9286	0.8159	0.2154	0.2462
19							0.9762	0.8849	0.3193	0.3236
20							0.9921	0.9332	0.4351	0.4095
21							1.0000	0.9641	0.5649	0.5000
22									0.6807	0.5905
23									0.7846	0.6764
24									0.8658	0.7538
25									0.9264	0.8200
26									0.9632	0.8738
27									0.9848	0.9152
28									0.9946	0.9454
29									0.9989	0.9664
30									1.0000	0.9803