THE EXPONENTIATED GENERALIZED PARETO DISTRIBUTION

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Abstract

Recently Gupta *et al.* (1998) introduced the exponentiated exponential distribution as a generalization of the standard exponential distribution. In this paper, we introduce a three-parameter generalized Pareto distribution, the exponentiated generalized Pareto distribution (EGP). We present a comprehensive treatment of the mathematical properties of this new distribution.

Keywords: Three-parameter generalized Pareto distribution, moments, hazard rate function.

1. Introduction

Gupta *et al.* (1998) introduced the exponentiated exponential (EE) distribution as a generalization of the standard exponential distribution. The EE distribution is given by

$$F(x) = [1 - \exp(-\lambda x)]^{\alpha} \qquad (1.1)$$

for $x>0, \lambda>0$ and $\alpha > 0$, which is simply the α-th power of the cdf of the standard exponential distribution. The mathematical properties of this EE distribution have been studied in detail by Gupta and Kundu (2001) and Nadarajah and Kotz (2003). We introduce, in this paper, a three-parameter generalized Pareto distribution in the same way (1.1) generalizes the standard exponetial distribution, and study its properties. We know that the cdf of the generalized Pareto distribution is given by

$$F(x) = 1 - \left(1 - \frac{k}{\alpha}x\right)^{\frac{1}{k}}; k \neq 0$$

$$= 1 - \exp\left(-\frac{x}{\alpha}\right); k = 0 \qquad (1.2)$$

for $0 \leq x \leq \infty$, *for* $k \leq 0$ and $0 \leq x \leq \frac{\alpha}{k}$ for $k>0$. The parameters of the distribution are α, the scale parameter, and k the shape parameter.

Moments of the distribution are readily obtained by noting that

$$E\left(1 - \frac{k}{\alpha}x\right)^r = \frac{1}{1+rk}, 1 + rk > 0 \qquad (1.3)$$

The r-th moment of X exists if $k > -\frac{1}{r}$ and thus all moments exist for $0 \leq x \leq \frac{\alpha}{k}$. The mean and variance of X are

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$$\mu = \frac{\alpha}{1+k}, \sigma^2 = \frac{\alpha^2}{(1+k)^2(1+2k)} \tag{1.4}$$

respectively.

2. The Exponentiated Generalized Pareto (EGP) Distribution

In this section we define the new EGP distribution by

$$F(x) = \left[1-(1-\frac{k}{\alpha}x)^{\frac{1}{k}}\right]^{\beta} \text{ for } \beta>0, \quad k\neq 0$$

$$= \left[1-\exp\left(-\frac{x}{\alpha}\right)\right]^{\beta}; \quad \beta>0, \quad k=0 \tag{2.1}$$

We refer to the distribution (2.1) as the exponentiated generalized Pareto (EGP) distribution. The corresponding pdf is given by

$$f(x) = \frac{\beta}{\alpha}\left[1-\left(1-\frac{k}{\alpha}\right)^{\frac{1}{k}}\right]^{\beta-1}\left(1-\frac{k}{\alpha}x\right)^{\frac{1}{k}-1}; \quad k\neq 0$$

$$= \frac{\beta}{\alpha}\left[1-\exp\left(-\frac{x}{\alpha}\right)\right]^{\beta-1}\exp\left(-\frac{x}{\alpha}\right); \quad k=0 \tag{2.2}$$

The generalized Pareto is a special case of (2.2) for $\beta=1$

Using the series representation (Nadarajah and Kotz (2003)),

$$(1+z)^a = \sum_{j=0}^{\infty}\frac{\Gamma(a+1)\ z^j}{\Gamma(a-j+1)j!} \tag{2.3}$$

(2.2) can be expressed in the mixture form

$$f(x) = \frac{\Gamma(\beta+1)}{\alpha}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\Gamma(\beta-j)}\left(1-\frac{k}{\alpha}x\right)^{\frac{j-k+1}{k}}; k\neq 0$$

$$= \frac{\Gamma(\beta+1)}{\alpha}\sum_{j=0}^{\infty}\frac{(-1)^j}{j!\Gamma(\beta-j)}e^{-(j+1)\frac{x}{\beta}}; k=0 \tag{2.4}$$

The exponentiated generalized Pareto distribution has an attractive physical interpretation. Suppose that the lifetimes of n components in a series are independently and identically distributed according to (2.2), then the lifetime of the system is also EGP. An additional motivation comes from many applications of the generalized Pareto distribution in modelling the extremes and in quality control, Gumbel (1958). Adeyemi (2002) and Adeyemi and Ojo (2003) also discussed recurrence relations for moments of order statistics and applied the properties of order statistics from the generalized Pareto distribution to least squares estimation of location-scale parameters and presented examples in both singly and doubly censored lifetime data.

3. The Moments

If X has the pdf (2.2) then the r-th moment can be written as

$$E(X^r) = \int_0^{\frac{\alpha}{k}} x^r f(x)dx$$

On setting $y=1-\frac{k}{\alpha}x$ we have

$$E(X^r)=\beta \int_0^1 y^{rk}(1-y)^{\beta-1}dy = \beta B(rk+1,\beta) \quad (3.1)$$

where B(. .) is the complete beta function.
The moments can be obtained recursively from the recurrence formula

$$\sum_{j=0}^{r}\binom{r}{j}\left(-\frac{k}{\alpha}\right)^j \mu^j = \beta B(rk+1,\beta); k \neq 0 \quad (3.2a)$$

$$\mu^r = \beta\alpha^r \sum_{j=0}^{\beta-1}\binom{\beta-1}{j}\frac{(-1)^j}{(j+1)^r}\Gamma(r+1); k=0 \quad (3.2b)$$

From the relation (3.2), we obtain the first four moments, respectively, as follows

$$\mu = \frac{\alpha}{k}[1-\beta B(k+1,\beta)] \quad (3.3a)$$

$$\mu^2 = \frac{\alpha}{k}[1-\beta B(2k+1,\beta)] \quad (3.4a)$$

$$\mu^3 = \frac{\alpha}{k}[1-\beta B(3k+1,\beta)] \quad (3.5a)$$

$$\mu^4 = \frac{\alpha}{k}[41-\beta B(k+1,\beta)] \quad (3.6a)$$

and

$$\mu = \alpha\beta \sum_{j=0}^{\beta-1}\binom{\beta-1}{j}\frac{(-1)^j}{(j+1)^2} \quad (3.3b)$$

$$\mu^2 = 2\alpha^2\beta \sum_{j=0}^{\beta-1}\binom{\beta-1}{j}\frac{(-1)^j}{(j+1)^3} \quad (3.4b)$$

$$\mu^3 = 6\alpha^3\beta \sum_{j=0}^{\beta-1}\binom{\beta-1}{j}\frac{(-1)^j}{(j+1)^4} \quad (3.5b)$$

$$\mu^4 = 24\alpha^4\beta\sum_{j=0}^{\beta-1}\binom{\beta-1}{j}\frac{(-1)^j}{(j+1)^5} \tag{3.6b}$$

The coefficients of skewness and kurtosis can be calculated using [(3.3) - (3.6)] for all $\beta > 0, 0 \le k \le \frac{\alpha}{k}, \alpha > 0$ for instance

$$\gamma_1 = \left(\frac{\alpha}{k}\right)^{\frac{2}{3}} \frac{[1-\beta B(3k+1,\beta)]}{[1-\beta B(2k+1,\beta)]^2} \tag{3.7}$$

and

$$\gamma_2 = \frac{k[1-\beta B(4k+1,\beta)]}{\alpha[1-\beta B(2k+1,\beta)]^2} \tag{3.8}$$

The moments generating function can be obtained as

$$M_X(t) = \frac{\beta}{\alpha}\int_0^{\frac{\alpha}{k}} e^{t(1-\frac{k}{\alpha}x)}\left[1-(1-\frac{k}{\alpha}x)^{\frac{1}{k}}\right]^{\beta-1}(1-\frac{k}{\alpha}x)^{\frac{1}{k}-1}dx \tag{3.9}$$

$$= \frac{\beta}{\alpha}\int_2^1 e^{ty}(1-y^{\frac{1}{k}})^{\beta-1}y^{\frac{1}{k}-1}dy \tag{3.10}$$

having changed variable. Though (3.10) is not in closed form, however, the moments can be obtained by fixing k and β.

4. The Shape

The first derivative of $\ln f(x)$ for the EGP distribution is given by

$$\frac{d\ln f(x)}{dx} = \frac{(\beta-1)(1-\frac{k}{\alpha}x)^{\frac{1}{k}} - \left[1-(1-\frac{k}{\alpha}x)^{\frac{1}{k}}\right](1-k)}{\alpha[1-(1-\frac{k}{\alpha}x)^{\frac{1}{k}}](1-\frac{k}{\alpha}x)^{\frac{1}{k}}},\ k \ne 0 \tag{4.1a}$$

and

$$\frac{d\ln f(x)}{dx} = \frac{(\beta-1)e^{\left(-\frac{x}{\alpha}\right)}}{\alpha(1-e^{(-\frac{x}{\alpha})})} - \frac{1}{\alpha};\ k=0 \tag{4.1b}$$

Standard calculations based on this derivative show that *f(x)* exhibits a single mode at $x=x_0$ with $f(0)=0=f\left(\frac{\alpha}{k}\right)$ where x_0 is the solution of $\frac{d\ln f(x)}{dx}=0$. Refer to the Appendix for illustrations of some of the possible values of x_0 for some selected values β, α and k.

5. The Hazard Rate Function

The hazard rate function defined by $h(x)=\frac{f(x)}{\bar{F}(x)}$ is an important quantity characterizing life phenomena where $\bar{F}(x)=Pr\{X>x\}$. For the EGP distribution, *h(x)* takes the form

$$h(x)=\frac{\beta[1-(1-\frac{k}{\alpha}x)^{\frac{1}{k}}]^{\beta-1}(1-\frac{k}{\alpha}x)^{\frac{1}{k}-1}}{\alpha-\alpha\left[1-(1-\frac{k}{\alpha}x)^{\frac{1}{k}}\right]^{\beta}};k\neq 0 \qquad (5.1a)$$

$$h(x)=\frac{\beta}{\alpha}[1-e^{(-\frac{x}{\alpha})}]^{\beta-2}e^{(-\frac{x}{\alpha})};k=0 \qquad (5.1b)$$

The first derivative of *ln h(x)* with respect to *x* is

$$\frac{d\ln h(x)}{dx}=(1-Q)^{\beta}[Q^{k+1}-Q^{k}+\left(\frac{2\beta-1}{\alpha}\right)Q]+Q\left(\frac{\beta-k}{\alpha}\right),k\neq 0 \qquad (5.2a)$$

where $Q=(1-\frac{k}{\alpha})^{\frac{1}{k}}$ (5.2a) can be solved for fixed β, α and k. Elementary calculations show that *h(x)* is multimodal for example when $\beta=1=k=\alpha$, *h(x)* exhibits two modes at 0 and 1. Possible shapes of *h(x)*, for some selected values of β, α and k, can be obtained using (5.2).

Also

$$\frac{d\ln h(x)}{dx}=\frac{(\beta-2)e^{\left(-\frac{x}{\alpha}\right)}}{\alpha(1-e^{(-\frac{x}{\alpha})})}-\frac{1}{\alpha};k=0 \qquad (5.2b)$$

Equation (5.2b) is easily solved to give $x=-\alpha\ln\left(\frac{1}{\beta-1}\right)$

6. Estimation of Parameters

Estimation of the three parameters are considered by the method of Maximum Likelihood. The loglikelihood for a random sample $x_1, x_2, ..., x_n$ from (2.2) is

$$\ln L(\alpha,\beta,k) = n\ln\beta - n\ln\alpha + n(\beta-1)\Sigma\ln[1-(1-\frac{k}{\alpha}x_i)^{\frac{1}{k}}] + n(\tfrac{1}{k}-1)\Sigma\ln\left(1-\frac{k}{\alpha}x_i\right) \quad (6.1)$$

The first order derivative of (6.1) with respect to each of the parameters give

$$\frac{\partial \ln L}{\partial\alpha} = \frac{n(\beta-1)\Sigma x_i(1-\frac{k}{\alpha}x_i)^{\frac{1}{k}-1}}{\alpha^2\Sigma[1-(1-\frac{k}{\alpha}x_i)^{\frac{1}{k}}]} - \frac{nk(\frac{1}{k}-1)\Sigma x_i}{\alpha^2\Sigma(1-\frac{k}{\alpha}x_i)} - \frac{n}{\alpha} \quad (6.2)$$

$$\frac{\partial \ln L}{\partial\beta} = \frac{n}{\beta} + n\Sigma\ln[1-(1-\frac{k}{\alpha}x_i)^{\frac{1}{k}}] \quad (6.3)$$

$$\frac{\partial \ln L}{\partial k} = \frac{n(\beta-1)}{1-\Sigma(1-\frac{k}{\alpha}x_i)^{\frac{1}{k}}}[\frac{1}{k}\Sigma\left(\frac{x_i}{\alpha-kx_i}\right) + \frac{1}{k^2}\Sigma\ln(1-\frac{k}{\alpha}x_i)]$$

$$-\frac{n}{k^2}\Sigma\ln(1-\frac{k}{\alpha}x_i) + \frac{n(1-\frac{1}{k})}{\alpha\Sigma(1-\frac{k}{\alpha}x_i)} \quad (6.4)$$

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REFERENCES

Adeyemi, S., 2002. Some recurrence relations for single and product moments of order statistics from the generalized Pareto distribution. *Journal of Statistical Research*. 36(2), 168-177.

Adeyemi, S. and Ojo, M.O., 2003. Higher moments of order statistics from the generalized Pareto distribution. *Journal of Statistical Research* (to appear).

Adeyemi, S. and Ojo, M.O., 2003. A generalization of the Gumbel distribution. *Kragujevac Journal of Mathematics*, 25, 19-29.

Gumbel, E.J., 1958. Statistics of Extremes. Colombia Press.

Gupta,R.C., Gupta,P.L. and Gupta, R.D., 1998. Modelling failure time data by Lehman alternatives. *Comm. Statist.-Theor, Meth.*, 27, 887-904.

Gupta, R.D. and Kundu, D., 2001. Exponentiated exponential family: an alternative to gamma and Weibull distributions. *Biometrical Journal*, 43, 117-130.

Nadarajah, S. and Kotz, S., 2003a. The exponentiated Fretchet distribution. *InterStat.* no. 1, pp. 1-7.

Nadarajah, S. and Kotz, S., 2003b. On the exponentiated exponential distribution (to appear in Statistica).

APPENDIX

The mode of the exponentiated generalized Pareto distribution for some selected values of k, α and β.

k	β	α	mode
0.05	2	1.5	2.44
0.1	2.1	2	3.21
0.15	2.2	2.5	3.95
0.2	2.3	3	4.65
0.25	2.4	3.5	5.33
0.3	2.5	4	5.97
0.35	2.6	4.5	6.57
0.4	2.7	5	7.13
0.45	2.8	5.5	7.65
0.5	2.9	6	8.13
0.55	3	6.5	8.57
0.6	3.1	7	8.97
0.65	3.2	7.5	9.34
0.7	3.3	8	9.67
0.75	3.4	8.5	9.97
0.8	3.5	9	10.23
0.85	3.6	9.5	10.47
0.9	3.7	10	10.67
0.95	3.8	10.5	10.85
1	3.9	11	11