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FIXED POINT OF A CLASS OF TOTAL ASYMPTOTICALLY ϑ-QUASINONEXPANSIVE MAPPINGS

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ABSTRACT

In this paper, a new class of total asymptotically quasinonexpansive nonself-mappings is introduced and strong convergence of Ishikawa type iteration is established for this class of mappings. The result is presented under the framework of Banach spaces with uniform convexity.

Keywords: Total asymptotically ϑ-quasinonexpansive mappings, Strong convergence, Common fixed point, Banach spaces.

1. INTRODUCTION AND PRELIMINARIES

Let X be nonempty set with a self-map $T{:}X\to X$, a fixed point of T is said to exist at a point $a\in X$ if $Ta=a$. The set $\{a\in X{:}Ta=a\}$ of all the fixed points of T in X is commonly represented by F_T. Suppose E is a normed linear space, then $T{:}E\to E$ is a contractive map if there is a constant $\vartheta\in[0,1)$ so that $||Ta - Tb||\leq\vartheta||a - b||$ for all $a,b\in E$. A mapping T is nonexpansive if $\vartheta = 1$ so that $||Ta - Tb||\leq||a - b||$ for all $a,b\in E$. T is called quasi-nonexpansive if there exists a point $p\in F_T$ so that $||Ta - Tp||\leq||a - p||$ for all $a\in E$. T is asymptotically nonexpansive if there is a sequence $\{\vartheta_n\}\in[1,\infty)$ for all $n\in\mathbb{N}$ with $\lim_{n\to\infty} k_n = 1$ so that $||T^na - T^nb||\leq\vartheta_n||a - b||$ for all $a,b\in E$. The mapping is asymptotically quasinonexpansive if $F_T\neq\varnothing$ and there is $\{\vartheta_n\}\in[1,\infty)$ with $\lim_{n\to\infty}\vartheta_n = 1$ so that $\forall\, n\in\mathbb{N}$, the subsequent inequality follows:

$$||T^na - p||\leq\vartheta_n||a - p||,\ \forall a\in E, p\in F_T.$$

The study of asymptotically nonexpansive, quasi-nonexpansive maps concerning the existence of fixed points has become appealing to several researchers working in nonlinear analysis (Olatinwo, 2008; Ariza-Ruiz, 2012; Ajibade *et al.*, 2022). This study, is often carried out both in self and nonself mappings in different ambient spaces such as metric space, complete normed vector space and uniformly convex Banach spaces. For instance, the conception of asymptotically nonexpansive mappings in Banach spaces was presented by Goebel and Kirk (1972) where the existence of fixed points for asymptotic nonexpansive maps was presented under the framework of Banach spaces with uniform convexity. Recently, Gunduz *et al.* (2017) established the convergence of an iterative process developed by Ishikawa to a class of nonself maps that are totally asymptotically nonexpansive. For more results in the fixed point of nonself total asymptotically nonexpansive mappings (Khan and Hussain, 2008; Thianwan, 2009; Khan *et al.*, 2015).

1.1 Some Definitions

Suppose $X\subset V$. If $\exists$ a continuous map $Q{:}V\to X$ such that $Qa = a$, $\forall\ a\in X$, then X is said to be a retract of V. A map $Q{:}V\to V$ is said to be retraction if $Q^2 = Q$. Given that $Q{:}V\to X$ is a nonexpansive retraction of V onto X then it follows that,

Definition 1.1 (Alter *et al.*, 2006) *A nonself mapping $Q{:}X\to V$ is asymptotically nonexpansive given that the sequence $\{\gamma_n\}\subset[1,\infty\}$ with $\lim_{n\to\infty}\gamma_n = 1$, we get*

$$||T(QT)^{n-1}a - T(QT)^{n-1}b||\leq\gamma_n||a - b||$$ *for all* $a,b\in X$ *and* $n\in\mathbb{N}$.

Definition 1.2 Suppose $X\subset V$ and let $Q{:}V\to X$ be a quasinonexpansive retraction of V onto X. Suppose the set of all fixed points $F_T\neq\varnothing$ and let ϑ:

$R^+ \rightarrow R^+$ with $\vartheta(r) \leq r$ for all $r > 0$. A nonself mapping $T{:}X \rightarrow V$ is called asymptotically ϑ-quasinonexpansive if there is a sequence $\{\varrho_n\} \in [0,\infty)$ with $\lim_{n\to\infty} \varrho_n = 1$ so that

$$||T(QT)^{n-1}a - T(QT)^{n-1}p|| \leq \varrho_n \vartheta(||a-p||),$$

$\forall a \in X$, and

The idea of total asymptotically nonexpansive maps was initiated by Alber *et al.* [8] where they demonstrated some strong and weak convergence results for the kind of maps defined as follows;
Let X be a closed subset of a normed linear real space V, then $Q{:}X \rightarrow X$ is a total asymptotically nonexpansive map, suppose $\exists$ a real non-negative sequences $\{\gamma_n\}$, $\{\varrho_n\}$ with $\gamma_n, \varrho_n \longrightarrow 0$ as $n \rightarrow \infty$ and a precisely expanding function ϑ: $R^+ \rightarrow R^+$ with $\vartheta(0) = 0$:

$$\| T^n a - T^n b \| \leq \| a - b \| + \gamma_n(\| a - b \|) + \varrho_n, n \in \mathbb{N} \quad 1.$$

for all $a, b \in X$.

Definition 1.3 Let X be a nonempty subset of V and let $Q{:}V \rightarrow X$ be a quasinonexpansive retraction of V onto X with all of the fixed points' set $F_T \neq \varnothing$. A nonself mapping $T{:}X \rightarrow V$ is total asymptotically quasinonexpansive if there are sequences $\{\gamma_n\}, \{\varrho_n\}$ with $\gamma_n, \varrho_n \longrightarrow 0$ $n \longrightarrow \infty$ and a precisely expanding function $\vartheta{:}\, \mathbb{R} \longrightarrow \mathbb{R}$ with $\vartheta(r) \leq r \;\; \forall \; r > 0$ so that

$$||T(QT)^{n-1}a - T(QT)^{n-1}p|| \leq ||a-p|| + \gamma_n \vartheta(||a-p||) + \varrho_n, \quad 2.$$

for all $a \in X$, and $p \in F_T$, $n \in \mathbb{N}$.

Remark 1.1 (Khan *et al.*, 2015) *Given that the mapping* $T{:}X \rightarrow V$ *be asymptotically quasi-nonexpansive and* $Q{:}V \rightarrow X$ *is a quasi-nonexpansive retraction, then* $QT{:}X \rightarrow X$ *is asymptotically quasi-nonexpansive mapping.*

$$||(QT)^n a - (QT)^n p|| = ||QT(QT)^{n-1}a - QT(QT)^{n-1}p||$$
$$\leq ||a-p|| + \gamma_n \vartheta(||a-p||) + \varrho_n,$$
$$\leq ||T(QT)^{n-1}a - T(QT)^{n-1}p||$$

for all $a \in X$, and $p \in F_T$, $n \in \mathbb{N}$.

Definition 1.4 If there is a $\delta(\varepsilon) > 0$ that correlates to each ε, $0 < \varepsilon \leq 2$, then the Banach space E with a uniform convexity fulfils

$||u|| = ||v|| = 1, ||u-v|| \geq \varepsilon$, it *implies that*

$$\left|\left|\frac{u+v}{2}\right|\right| \geq 1 - \delta(\varepsilon) \quad 3.$$

Definition 1.5 If any sequence in a normalized space $(V, ||.||)$ is such that $b_n \rightharpoonup b_0$, it is said to fulfill the Opial condition if for all $c \in V, c \neq b_0$,

$$\limsup_{n\to\infty} ||b_n - b_0|| < \limsup_{n\to\infty} ||b_n - c||, \quad 4.$$

which can be proved from the extraction of those appropriate subsequences, that the lower limits can be replaced with upper limit in the definition above.

Recently, iterative procedures and estimation of fixed points of nonlinear maps have been studied widely by many authors. See (Chidume *et al.*, 2003, Rashwan and Altwqi, 2011, Khan *et al.*, 2015, Gunduz *et al.*, 2017) among others.

1.2 Some Iterative Procedures

Rashwan and Altwqi (2011) Let X be a closed, nonempty subset of convexity of a Banach space V with uniform convexity which is a nonexpansive retract of V. Let $T_\tau{:}X \rightarrow V$ where $\tau = 1,2,3$ consist of three non-self asymptotically non-expansive maps having sequences $\varphi_n, \eta_n \subset \{0,\infty)$ so that $\sum_{n=0}^{\infty} \varphi_n < \infty$, $\sum_{n=0}^{\infty} \eta_n < \infty$. Thus, given that $a_1 \in X$,

$$\begin{cases} a_{n+1} = Q((1-\alpha_n)b_n + \alpha_n T_1(QT_1)^{n-1}b_n), \\ b_n = Q((1-\beta_n)c_n + \beta_n T_2(QT_2)^{n-1}c_n), \\ c_n = Q((1-\lambda_n)a_n + \lambda_n T_3(QT_3)^{n-1}a_n), \quad n \in \mathbb{N}. \end{cases} \quad 5.$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ are sequences in [0,1]. Within the framework of continuously convex Banach space, they were able to determine the unique fixed points of three nonself asymptotically nonexpansive maps.

In the iterative procedure (5) above, if $\lambda_n = 0$, according to Thianwan [3], it reduces to

$$\begin{cases} a_{n+1} = Q((1-\alpha_n)b_n + \alpha_n T_1(QT_1)^{n-1}b_n), \\ b_n = Q((1-\beta_n)c_n + \beta_n T_2(QT_2)^{n-1}a_n), n \in \mathbb{N}. \end{cases} \quad 6.$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1]. Furthermore, other authors have also reduces (5) above by putting $\lambda_n = 0$, $\beta_n = 0$ to obtain

$a_{n+1} = Q((1-\alpha_n)b_n + \alpha_n T(QT)^{n-1}a_n), n \in \mathbb{N},$

where $\{\alpha_n\}$ is a sequences in [0,1]. Assume that X is a closed convex nonempty subset of a real, uniformly convex Banach space V that is a nonexpansive retract of V. Let $T_\tau: X \to V$ where τ = 1,2,3 consist of three non-self asymptotically non-expansive maps having sequences $\varphi_n, \eta_n \subset \{0, \infty)$ such that $\sum_{n=0}^{\infty} \varphi_n < \infty$,

$\sum_{n=0}^{\infty} \eta_n < \infty$. Thus, given that $a_1 \in X$,

$$\begin{cases} a_{n+1} = (1-\alpha_n)b_n + \alpha_n (QT_1)^n b_n, \\ b_n = (1-\beta_n)c_n + \beta_n (QT_2)^n c_n, \\ c_n = (1-\lambda_n)a_n + \lambda_n (QT_3)^n a_n, \quad n \in \mathbb{N}. \end{cases} \quad 7.$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ are sequences in [0,1]. They approximated a common fixed points at a faster rate than (5).

Recently, Gunduz *et al.* (2017) introduced a new iterative procedure for computing common fixed points of two nonself total asymptotically nonexpansive mappings and produced a tool for convergence of the iterative procedure in Banach spaces. Let U be a nonempty closed convex subset of a real normed linear space V with a retraction Q. Let S_1, S_2: $U \to V$ be two nonself asymptotically nonexpansive mappings with respect to Q.

$$\begin{cases} a_1 \in U, \\ a_{n+1} = (1-\alpha_n)(QS_1)^n a_n + \alpha_n (QS_2)^n b_n, \\ b_n = (1-\beta_n)a_n + \beta_n (QS_1)^n a_n, \quad n \in \mathbb{N}. \end{cases} \quad 8.$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1].

For the sake of this paper, a new class of nonself total asymptotically ϑ-quasinonexpansive mapping is introduced and an Ishikawa type iteration is considered for strong convergence for two nonself total asymptotically ϑ-quasinonexpansive mappings. The result is established within the context of Banach spaces V with uniform convexity.

The above definitions and the following lemma are useful in proving our main theorems.

For the next section, we always assume $F = F_T \cap F_S = \{a \in Ta = Sa = a\} \neq \emptyset$ as a common fixed point set of T, S.

Lemma 1.1 [5] Let $\alpha_n \geq 0, \sigma_n \geq 0$ be such that

$$\alpha_{n+1} \leq (1+\alpha_n)\alpha_n + \sigma_n. \quad 9.$$

If, a. $\sum_{n=1}^{\infty} \alpha_n < \infty$; b. $\sum_{n=1}^{\infty} \sigma_n < \infty$; and

c. $\lim_{n\to\infty} \inf a_n = 0$, then

$\lim_{n\to\infty} a_n = 0$.

2. MAIN RESULTS

Definition 2.1 Let X be a nonempty subset of V and let $Q: V \to X$ be the quasinonexpansive retraction of V onto X. Suppose the set of all fixed points $F_T \neq \emptyset$ and a function $\vartheta: R^+ \to R^+$ with $\vartheta(r) \leq r$ for all $r > 0$. A nonself map $T_\tau: X \to V$ is totally asymptotically ϑ-quasinonexpansive if the sequences $\{\gamma_n\}$, $\{\lambda_n\}$ exists where $\gamma_n, \lambda_n \to 0$ as $n \to \infty$ such that

$$||(QT)^n a - (QT)^n p|| \leq (1+\gamma_n)\vartheta(||a-p||) + \lambda_n \quad 10.$$

for all $a \in X$, and $p \in F_T$, $n \in \mathbb{N}$.

Remark 2.1 A nonself total asymptotically ϑ-quasinonexpansive mapping becomes

(i) asymptotically ϑ-quasinonexpansive mappings if $\lambda_n = 0$, for all $n \geq 1$.

(ii) ϑ-quasinonexpansive mapping if $\gamma_n = 0$, and $\lambda_n = 0$ for all n

(iii) quasi-nonexpansive mapping if ϑ® $\leq r$ for all $r > 0$, $\gamma_n = 0$ and $\lambda_n = 0$ for all $n \in \mathbb{N}$

Let $T, S: X \to V$ be two quasi-nonexpansive, nonself asymptotically maps with respect to a quasi-nonexpansive retraction Q.

We shall employ the iterative scheme (8) in obtaining our result in this section.

$$\begin{cases} a_1 \in X, \\ a_{n+1} = (1-\alpha_n)(QT)^n a_n + \alpha_n (QS)^n b_n, \\ b_n = (1-\beta_n)a_n + \beta_n (QT)^n a_n, \quad n \in \mathbb{N}. \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1].

Lemma 2.1 Let X be a closed nonempty subset of V that is also a quasi-nonexpansive retract with retraction Q and let V be a Banach space with uniform convexity. Let $T, S:X\to V$ be two nonself total asymptotically ϑ-quasinonexpansive maps with sequences $\{\gamma_n\}$, $\{\lambda_n\}$ both in (0,1) such that $\gamma_n, \lambda_n \to 0$ as n $\to\infty$. Consider a function $\vartheta: R^+\to R^+$, with $\vartheta(r)\leq r \ \forall r>0, F_T\neq\emptyset$. $\{a_n\}$ is defined by (8). Then with the bounded sequence $\{a_n\}$ and $p\in F_T$, the $\lim_{n\to\infty}||a_n-p||$ exists.

Proof

Since $p \in F_T$, it follows from (10) and (8) that

$$\begin{aligned}||b_n-p|| &= ||(1-\beta_n)a_n+\beta_n(QT)^n a_n-p||\\ &= ||(1-\beta_n)(a_n-p)+\beta_n[(QT)^n x_n-p]||\\ &\leq (1-\beta_n)||a_n-p||+\beta_n||(QT)^n a_n-p||\\ &\leq (1-\beta_n)||a_n-p||+\beta_n[(1+\gamma_n)\vartheta(||a_n-p||)+\lambda_n]\\ &\leq (1-\beta_n)||a_n-p||+(\beta_n+\beta_n\gamma_n)||a_n-p||+\beta_n\lambda_n\\ &= ||a_n-p||+\beta_n\gamma_n||a_n-p||+\beta_n\lambda_n\end{aligned}$$

$$||b_n-p|| \leq (1+D_n)||a_n-p|| + C_n \qquad 11.$$

Where $\beta_n\gamma_n = D_n$ and $\beta_n\lambda_n = C_n$

Also, from equations (10) and (8) we have

$$||a_{n+1}-p|| = ||(1-\alpha_n)(QT)^n a_n+\alpha_n(QS)^n b_n-p||$$
$$= ||(1-\alpha_n)(QT)^n(a_n-p)+\alpha_n[(QS)^n b_n-p]||$$
$$\leq (1-\alpha_n)||(QT)^n a_n-p||+\alpha_n||(QS)^n b_n-p||$$
$$\leq (1-\alpha_n)[(1+\gamma_n)\vartheta(||a_n-p||)+\lambda_n]+\alpha_n[(1+\gamma_n)$$
$$\vartheta(||b_n-p||)+\lambda_n]$$

$$||a_{n+1}-p|| \leq (1-\alpha_n)[(1+\gamma_n)(||a_n-p||)+\lambda_n]$$
$$+\alpha_n[(1+\gamma_n)\cdot$$
$$((1+D_n)||a_n-p||+C_n)+\lambda_n]$$
$$= ||a_n-p||+\gamma_n||a_n-p||+\lambda_n+D_n\alpha_n||a_n-p||$$
$$+D_n\alpha_n\gamma_n||a_n-p||$$
$$+C_n\alpha_n+C_n\alpha_n\gamma_n$$

$$||a_{n+1}-p|| \leq (1+U_n)||a_n-p||+W_n \qquad 12.$$

Where $(1+\gamma_n+D_n\alpha_n+D_n\alpha_n\gamma_n)=U_n$ and $\lambda_n+C_n\alpha_n+C_n\alpha_n\gamma_n=W_n$

Since $\sum_{n=1}^{\infty} U_n<\infty$ and $\Sigma_{n=1}^{\infty} W_n<\infty$, by Lemma (1.1), we have $\lim_{n\to\infty}||a_n-p||$ exists.

We now proceed to prove a new result with the aid of the above lemma.

Theorem 2.1 Let X be a quasi-nonexpansive retraction of Q and a closed nonempty subset of a Banach space V with uniform convexity which define $T, S:X\to V$ as two continuous ϑ-quasinonexpansive mappings that are nonself total asymptotically (10) so that $\Sigma_{n=1}^{\infty}\gamma_n<\infty, \Sigma_{n=1}^{\infty}\lambda_n<\infty$.

Consider that $\vartheta: R^+\to R^+$, with $\vartheta(r)\leq r \ \forall r>0$, then the sequence $\{a_n\}$ defined by (8) strongly converges to a common fixed point of T and S if and only if $\liminf_{n\to\infty}||a_n-F||=0$, where $||a_n-F||=\inf_{p\in F}||a_n-p||,\ n\in\mathbb{N}$.

Proof

Assuming that $\liminf_{p\in F}||a_n - F|| = 0$, with the sequence $\{a_n\}$ converging to a common fixed point of T and S. We now show that the sequence $\{a_n\}$ is a Cauchy sequence in V. Suppose the $\lim_{n\to\infty}||a_n-p||$ exists. Then, from

Lemma 2.1, we have

$$||a_{n+1}-p|| \leq (1+U_n)||a_n-p||+W_n$$
$$\leq (1+U_n)(||a_n-p||+W_n)$$

then by iterating,

$$||a_{n+3}-p|| \leq (1+U_{n+2})(||a_{n+2}-p||+W_{n+2})$$
$$\leq (1+U_{n+1})(1+U_n)[(||a_n-p||+W_n+W_{n+1}]$$
$$\leq (1+U_{n+1})[(1+U_n)(||a_n-p||+W_n)+W_{n+1}]$$
$$||a_{n+2}-p|| \leq (1+U_{n+1})(||a_{n+1}-p||+W_{n+1})$$
$$\leq (1+U_{n+2})[(1+U_{n+1})(1+U_n)(||a_n-p||$$
$$+W_n+W_{n+1})+W_{n+2}]$$
$$\leq (1+U_{n+2})(1+U_{n+1})(1+U_n)[||a_n-p||$$
$$+W_n+W_{n+1}+W_{n+2}]$$
$$||a_{n+4}-p|| \leq (1+U_{n+3})(||a_{n+3}-p||+W_{n+3})$$
$$\leq (1+U_{n+3})[(1+U_{n+2})(1+U_{n+1})(1+U_n)(||x_n-p||$$
$$+W_n+W_{n+1}+W_{n+2})+W_{n+3}]$$
$$||a_{n+4}-p|| \leq (1+U_{n+3})(1+U_{n+2})(1+U_{n+1})(1+U_n)\cdot$$
$$[||a_n-p||+W_n+W_{n+1}+W_{n+2}+W_{n+3}]$$

For any $m\in\mathbb{N}$ we have

$$||a_{n+m}-p|| \leq U_{n+m-1}(||a_{n+m-1}-p||+W_{n+m-1})$$
$$\leq (1+U_{n+m-1})\dots[(1+U_{n+3})(1+U_{n+2})(1+U_{n+1})$$
$$(1+U_n)(||a_n-p||+W_n+\dots+W_{n+3})+\dots+W_{n+m-1}]$$
$$W_n+W_{n+1}+W_{n+2}+\dots+W_{n+m-1}]$$
$$\leq (1+U_{n+m-1})\dots(1+U_{n+3})(1+U_{n+2})(1+U_{n+1})$$
$$(1+U_n)[||a_n-p||+$$

with the fact that $1+t\leq \exp(t)$, we have

$||a_{n+m} - p|| \leq \exp(U_{n+m-1} + \ldots + U_{n+3} + U_{n+2}$

$+ U_{n+1} + U_n)$

$.(||a_n - p|| + W_n + W_{n+1} + W_{n+2} + W_{n+3} + \ldots + W_{n+m-1})$

$\leq \exp(\sum_{i=n}^{n+m-1} U_i)(||a_n - p|| + \sum_{i=n}^{n+m-1} W_i)$

$\leq \exp(\sum_{i=n}^{\infty} U_i)(||a_n - p|| + \sum_{i=n}^{\infty} W_i)$.

Therefore, for $m \in \mathbb{N}$ and $p \in F$

$|a_{n+m} - a_n|| \leq ||a_{n+m} - p|| + ||a_n - p||$

$\leq [1 + (\exp(\sum_{i=n}^{\infty} U_i))]||a_n - p|| +$

$(\exp(\sum_{i=n}^{\infty} U_i))(\sum_{i=n}^{\infty} W_i)$

$||a_{n+m} - a_n|| \leq [1 + \exp(\sum_{i=n}^{\infty} U_i)](||a_n - p||$

$+ (\sum_{i=n}^{\infty} W_i))$

$||a_{n+m} - a_n|| \leq H(||a_n - p|| + \sum_{i=n}^{\infty} W_i)$,

$Where \;\; H = 1 + \exp(\sum_{i=n}^{\infty} U_i)$

$||a_{n+m} - a_n|| \leq H||a_n - p|| + H(\sum_{i=n}^{\infty} W_i)$

For $H > 0$ and $p \in F$

$$||a_{n+m} - a_n|| \leq Hd(a_n, F) + H\left(\sum_{i=n}^{\infty} W_i\right).$$

For $\in > 0$ and $\lim_{n\to\infty} d(a_n, F) = 0$ and $\sum_{i=n}^{\infty} W_i < \infty$

there exist an integer $n_0 > 0$ such that $n > n_0$,

$d(a_n, F) < \frac{\epsilon}{2H}$ and $\sum_{i=n}^{\infty} W_i \;<\; \frac{\epsilon}{2H}$. Then

$||a_{n+m} - a_n|| \leq H\left(\frac{\epsilon}{2H} + \frac{\epsilon}{2H}\right) = \epsilon$

$||a_{n+m} - a_n|| \leq \epsilon$

which means that $\{a_n\}$ is a Cauchy sequence in V. Thus, the completeness of space V guarantees the existence of $\{a_n\}$. Therefore, $\lim_{n\to\infty}\{a_n\}$ exists.
Let $\lim_{n\to\infty}\{a_n\} = p^*$, already T and S are continuous mappings.
Now, we show that p^* is a common fixed point of T and S. Suppose $p^* \in F$ since F is a closed subset of V, we can see that $||p^* - F|| > 0$.
But for all $p \in F$, we have

$||p^* - p|| \leq ||p^* - a_n|| + ||a_n - p||$

$\leq ||p^* - a_n|| + ||a_n - F||$

which gives

$d(p^*, F) \leq ||p^* - a_n|| + ||a_n - F||$

since $\{a_n\}p^* \to 0$ as n $\to\infty$ which contradicts $||p^* - F|| > 0$. Hence, p^* is a common fixed points of T and S as required.

Remark 2.2
(i) Our result is an improvement on the work of Khan *et al.* (2015) and Gunduz *et al.* (2017).
(ii) In conclusion, our results provided useful information on the establishment of convergence of Ishikawa iterative procedure to a common fixed points of the class of ϑ-quasinonexpansive. Finally, going by the research results so far, a new generalised map was introduced, the convergence of newly modified S-iterative procedure to a common fixed point of nonself total asymptotically ϑ-quasinonexpansive maps in Banach spaces that are uniformly convex were investigated.

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