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SOME EXTENDED PARETO TYPE I DISTRIBUTIONS

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ABSTRACT

Probability distributions are essential in data modeling. Introduction of parameter(s) into existing probability distributions is a method of extending or generalizing distributions to produce more flexible distributions and for better fit to data. The Pareto type 1 distribution (PT1) is a right skewed continuous distribution originally used in description of wealth and income but also used for modeling other right skewed data. To add flexibility, Pareto type 1 distribution was extended by introducing parameter(s) into its probability distribution to accommodate more types of data. Some functions of the extended Pareto type 1 distributions were derived using five parameter induction methods. Flexibility of extended distributions was demonstrated through comparisons of density and hazard function shapes of some of the extended distributions with those of the PT1. Further study on properties of non-existing extended Pareto Type I distributions and real-life applications are recommended.

Keywords: Alpha Power Transformed Model, Extended Pareto Type 1 Distribution, Marshall-Olkin Model, Pareto Type 1 Distribution, Proportional Hazard Model, Proportional Reversed Hazard Model.

INTRODUCTION

The PT1 is a right skewed distribution named after an economist and socialist, Vilfredo Pareto. It was originally used in description of wealth and income and also in the analysis of other skewed data. PT1 is skewed with a long and heavy tail on the right and has a decreasing hazard rate function therefore not suitable for the analysis of other types of data with different characteristics. This necessitates its extension or generalization to provide greater flexibility for the analysis of a wider variety of data.

There are different approaches to extending or generalizing distributions which include the use of generators to generate generalized or extended distributions from any base distribution. These extended distributions embed the base distributions and belong to different families of distributions, such as exponentiated family, Burr X-G family, Beta-G family, Kumaraswamy-G family, Beta extended Weibull family, odd gamma-G family, odd Burr-G family and so on. Given any base distribution, the functions and properties of these families of distributions are used as generators to derive those of new generalized distributions which are extensions of the base distribution. Parameter induction is one of the methods used in generating families of distributions.

Lehmann alternatives often referred to as exponentiated family have been used in different studies to extend distributions for additional flexibility by addition of an extra parameter. Gupta and Kundu (2009) gave a different interpretation to Lehmann alternatives 1 and II presenting properties of proportional reversed hazard models and proportional hazard models which are same as those of any generalized distribution obtainable from Lehmann alternatives 1 and II respectively. The Power Transformed Model (PTM) was also discussed by Gupta and Kundu (2009) as a parameter induction method. Another parameter induction method is that introduced by Marshall and Olkin (1997). Gupta and Kundu (2009) identified the generalized distributions derived from Marshall and Olkin parameter induction method as proportional odds models. The alpha power transformed method was introduced by Mahdavi and Kundu (2017). Application of one or more parameter induction methods produce families of generalized or extended distributions having new parameter(s).

Bourguignon *et al.* (2012) introduced the Kumaraswamy Pareto distribution based on the Kumaraswamy generalized (Kw-G) family of distributions studied by Cordeiro and de Castro (2011). Urama *et al.* (2021) generalized the Kumaraswamy Pareto distribution to obtain the transmuted Kumaraswamy Pareto distribution using quadratic rank transmutation map. Similarly, Merovci and Puka (2014) used the quadratic rank transmutation map method to derive the transmuted Pareto distribution, a generalization of the Pareto distribution. Chhetri *et al.* (2017) proposed the beta transmuted Pareto, a member of the beta-generated (beta-G) family of distributions developed by Eugene *et al.* (2002). The beta transmuted Pareto is an extension of transmuted Pareto distribution.

Another member of the beta-G family is the beta-Pareto distribution by Akinsete *et al.* (2008). In the beta-G family is yet the beta exponentiated Pareto distribution (Zea *et al.*, 2012), an extention of the exponentiated Pareto distribution by Gupta *et al.* (1998). The exponentiated Pareto distribution belongs to the exponentiated family of distributions and is a generalization of the Pareto distribution having its CDF raised to the power of the new parameter. The exponentiated Pareto was also generalized by Mead (2014) using the generalized beta generated distribution developed by Alexander *et al.* (2012). The new distribution was called the generalized beta exponentiated Pareto type 1 (GBEP) [McDonald exponentiated Pareto].

Ihtisham *et al.* (2019) introduced the alpha power Pareto density, a generalization of the Pareto distribution with an additional parameter by applying the alpha power transformation family method by Mahdavi and Kundu (2017). The Marshall-Olkin Alpha Power Pareto was studied by Almetwally and Haj Ahmed (2020) which is a member of the Marshall-Olkin alpha power family of distributions with two additional parameters introduced by Nassar *et al.* (2019). Ghitany (2005) introduced the Marshall-Olkin extended Pareto distribution which extends the Pareto type 1 distribution using Marshall-Olkin parameter induction method. Other extensions of the PT1 include Modified weighted Pareto distribution (Sahmram, 2020), Burr X Pareto distribution (Korkmaz *et al.*, 2018) based on Burr X generator of distributions by Yousef *et al.* (2016), Gamma- Pareto distribution (Alzaatreh *et al.*, 2012), exponentiated gamma-Pareto distribution (Alzaghal A., 2020), new Weibull-Pareto distribution (Tahir *et al.*, 2016 and also Nasiru and Luguterah, 2015), Exponentiated new Weibull-Pareto distribution (Al-Omari *et al.*, 2019), transmuted new Weibull-Pareto distribution (Tahir *et al.*, 2018), Weibull-Pareto distribution (Alzaatreh *et al.*, 2013), Exponentiated Weibull-Pareto distribution (Afify *et al.*, 2016), amongst others.

This study extends PT1 by introducing additional parameters (not more than two) to improve flexibility. Some of the extended PT1s were used to demonstrate flexibility of extended distributions. Some of the extended PT1s have not been studied and are proposed for further study and real-life applications.

MATERIALS AND METHODS

Derivation of Extended Pareto Type 1 Distributions

Let the Probability Density Function (PDF), Cumulative Distribution Function (CDF) Survival Function (SF), Hazard Function (HF), and Reversed Hazard Function (RHF) of a base random variable, X, be denoted respectively by $f(x)$, $F(x)$, $\bar{F}(x)$, $h(x)$, and $r(x)$.

To derive extended Pareto Type 1 distribution, PT1 was used as base distribution. A Pareto Type 1 distributed random variable, X, has the following functions;

$$f(x) = \frac{ab^a}{x^{a+1}}, \quad (1.1)$$

$$F(x) = 1 - \left(\frac{b}{x}\right)^a \quad (1.1a)$$

$$\bar{F}(x) = \left(\frac{b}{x}\right)^a \quad (1.1b)$$

$$h(x) = \frac{a}{x} \quad (1.1c)$$

$$r(x) = \frac{ab^a}{x^{a+1}\left(1 - \left(\frac{b}{x}\right)^a\right)} = \frac{ab^a}{x(x^a - b^a)} \quad (1.1d)$$

$x \geq b$, where $b > 0$ is the minimum possible value of x and $a > 0$ is the shape parameter.

The approach employed for extension of PT1 is parameter induction. Functions of extended PT1s introducing additional parameter(s) were derived from exponentiated, Marshall-Olkin, power transformation and alpha power transformation methods of generalization and extended distributions were referred to as proportional hazard models, proportional reversed hazard models, power transformed models, Marshall-Olkin models and alpha power transformed models.

Given the Pareto distribution as base distribution, let Y be a random variable whose distribution is obtainable from the distribution of the base random variable, X;

A Proportional Hazard Model (PHM) for modeling Y has the following respective PDF, CDF, SF, and HF;

$$f_Y(x) = \theta f(x)(\bar{F}(x))^{\theta-1} \quad (1.2)$$

$$F_Y(x) = 1 - (\bar{F}(x))^{\theta} \quad (1.2a)$$

$$\bar{F}_Y(x) = [\bar{F}(x)]^{\theta} \quad (1.2b)$$

$$h_Y(x) = \theta h(x) \quad (1.2c)$$

$\theta>0$ is an additional shape parameter. The PHM is an extension or modification of the base distribution (Gupta and Kundu, 2009).

A Proportional Reversed Hazard Model (PRHM) for Y has the following respective PDF, CDF, SF, and RHF;

$$f_Y(x) = \theta f(x)F(x)^{\theta-1} \quad (1.3)$$

$$F_Y(x) = F(x)^{\theta} \quad (1.3a)$$

$$\bar{F}_Y(x) = 1 - F(x)^{\theta} \quad (1.3b)$$

$$r_Y(x) = \theta r(x) \quad (1.3c)$$

$\theta>0$ is an additional shape parameter. The PRHM is an extension or modification of the base distribution (Gupta and Kundu, 2009).

The Power Transformed Model (PTM) was also discussed by Gupta and Kundu (2009). Given X to be a non-negative random variable (base random variable), considering a new random variable Y with an additional parameter ($\theta > 0$) such that $Y=X^{1/\theta}$, the PDF, CDF, and SF of the PTM obtainable from the distribution of X are as follows;

$$f_Y(x) = \theta x^{\theta-1} f(x^{\theta}) \quad (1.4)$$

$$F_Y(x) = F(x^{\theta}) \quad (1.4a)$$

$$\bar{F}_Y(x) = 1 - F(x^{\theta}) \quad (1.4b)$$

The PDF, CDF, and SF of the Marshall-Olkin Model (M-OM) for modeling Y, ($y \in \Re$) proposed by Marshall and Olkin (1997) with an extra parameter ($\theta > 0$) are as follows;

$$f_Y(x) = \frac{\theta f(x)}{[1-(1-\theta)(1-F(x))]^2} = \frac{\theta f(x)}{[\theta+(1-\theta)F(x)]^2} \quad (1.5)$$

$$F_Y(x) = \frac{F(x)}{1-(1-\theta)(1-F(x))} = \frac{F(x)}{\theta+(1-\theta)F(x)} \quad (1.5a)$$

$$\bar{F}_Y(x) = \frac{\theta\bar{F}(x)}{1-(1-\theta)(1-F(x))} = \frac{\theta\bar{F}(x)}{\theta+(1-\theta)F(x)} \quad (1.5b)$$

The α-Power transformations of the CDF and PDF of X ($x \in \Re$) are defined as follows:

$$f_Y(x) = \begin{cases} \frac{\log\theta}{\theta-1} f(x)\theta^{F(x)}, & \theta \neq 1 \\ f(x), & \theta = 1 \end{cases} \quad (1.6)$$

$$F_Y(x) = \begin{cases} \frac{\theta^{F(x)}-1}{\theta-1}, & \theta \neq 1 \\ F(x), & \theta = 1 \end{cases} \quad (1.6a)$$

$$\bar{F}_Y(x) = \begin{cases} 1-\frac{\theta^{F(x)}-1}{\theta-1} = \frac{\theta-\theta^{F(x)}}{\theta-1}, & \theta \neq 1 \\ \bar{F}(x), & \theta = 1 \end{cases} \quad (1.6b)$$

1.6, 1.6a, and 1.6b are the PDF, CDF, and SF of an Alpha Power Transformed Model (APTM) with an additional parameter ($\theta > 0$). The Alpha Power Transformed Family of generalized distributions was introduced by Mahdavi and Kundu (2017).

Function of these models were used as generators. The functions of Extended PT1s with an additional parameter (θ) were obtained by substituting functions of Pareto distribution in equations 1.1 through to 1.1d into functions of each model as given in equations 1.2 through 1.6b. Extended PT1s with an additional parameter were used as base distributions and similarly extended to introduce another parameter, λ. The extended PT1s with an additional parameter ($\theta > 0$) reduce to PT1 when $\theta = 1$. The extended PT1s having two additional parameters ($\theta > 0, \lambda > 0$) reduce to the extended PT1 with one extra parameter ($\theta > 0$) used as base distribution in the second generalization when the newest parameter assumes the value of 1 ($\lambda = 1$) but reduce to PT1 when both added parameters assume the value of

1 (θ = 1, λ = 1).

Validity of PDFs of Extended Pareto Distributions

For *b=1*, the PDF of continuous random variable, Y, is valid if it satisfies the following two properties:

a. $f(y) \geq o$ for any $y \in \Re$ (2.1)

b. $\int_1^\infty f(y)\,dx = 1$ (2.2)

The validity of PDFs derived for extended PT1s given the functions of Pareto distribution in equations 1.1 through 1.1d and those of each model used for generalization in equations 1.2 through 1.6b was demonstrated using the PDFs of the first four extended PT1s.

Flexibility of Extended Pareto Type 1 Distributions

The density and hazard functions of Pareto distribution and those of the first four extended PT1s were plotted and compared to demonstrate flexibility of extended PT1s. PT1 is skewed with a long and heavy tail on the right and has a decreasing hazard rate. For each of the four distributions, the parameter, *b*, was held constant at a value of 1 while varying other parameter values to ascertain the effect of the new parameters on the density and hazard shapes. Existence of new shapes established flexibility.

RESULTS

Derivation of Extended Pareto Type 1 Distributions

The PT1 was used as base distribution and extending it using functions in equations 1.2 through 1.6b as generators produce extended PT1s with an additional shape parameter (θ > 0). Subsequently extending any of the extended PT1s with an extra shape parameter (θ > 0) yields in turn another extended PT1 with another shape parameter (λ > 0).

The PDF, CDF, SF, HF, and RHF of extended PT1s were obtained for the first seven extended distributions while only the PDF and CDF were given for the rest. The SF, HF, and RHF of remaining distributions can be similarly obtained using the details below;

$h(y) = \frac{f(y)}{\bar{F}(y)}, r(y) = \frac{f(y)}{F(y)}, and\ \bar{F}(y) = 1 - F(y)\cdot$

Extended PT1 1 (EPT1-1)

EPT1-1 is a PHM with an additional shape parameter (θ > 0) obtained as an extension of PT1 (base distribution) using the functions in 1.2 through 1. 2c as generators. The PDF, CDF, SF, HF, and RHF of this extended distribution are then respectively derived as follows;

Using 1.1 and 1.1b in 1.2,

$$f(y) = \frac{\theta a b^a}{y^{a+1}}\left(\left(\frac{b}{y}\right)^a\right)^{\theta-1} = \frac{a\theta b^{a\theta}}{y^{a\theta+1}} \quad (3.1)$$

Using 1.1b in 1.2a,

$$F(y) = 1 - \left(\frac{b}{y}\right)^{a\theta} \quad (3.1a)$$

Using 1.1b in 1.2b,

$$\bar{F}(y) = \left(\frac{b}{y}\right)^{a\theta} \quad (3.1b)$$

Using 1.1b in 1.2c,

$$h(y) = \frac{a\theta}{y} \quad (3.1c)$$

$$r(y) = \frac{f(y)}{F(y)} = \frac{\frac{\theta a b^a}{y^{a+1}}\left(\left(\frac{b}{y}\right)^a\right)^{\theta-1}}{1-\left(\frac{b}{y}\right)^{a\theta}} = \frac{a\theta b^{a\theta}}{y^{a\theta+1}\left(1-\left(\frac{b}{y}\right)^{a\theta}\right)} = \frac{a\theta b^{a\theta}}{y(y^{a\theta} - b^{a\theta})} \quad (3.1d)$$

Extended PT1 2 (EPT1-2)

EPT1-2 with two additional shape parameters (θ > 0, λ > 0) is obtained as an extension of EPT1-1 using the functions in 1.3 through 1.3c as generators but now with additional parameter, λ > 0, and EPT1-1 as base distribution. It is therefore referred to in this study as a PRHM with respect to EPT1-1. The PDF, CDF, SF, HF, and RHF of this extended distribution are derived respectively as follows;

Using 3.1 and 3.1a in 1.3,

$$f(y) = \frac{\lambda\theta a b^a}{y^{a+1}}\left(\left(\frac{b}{y}\right)^a\right)^{\theta-1}\left(1-\left(\frac{b}{y}\right)^{a\theta}\right)^{\lambda-1} \quad (3.2)$$

Using 3.1a in 1.3a,

$$F_Y(x) = \left(1-\left(\frac{b}{y}\right)^{a\theta}\right)^{\lambda} \quad (3.2a)$$

Using 3.1a in 1.3b

$$\bar{F}(y) = 1-\left(1-\left(\frac{b}{y}\right)^{a\theta}\right)^{\lambda} \quad (3.2b)$$

$$h(y) = \frac{f(y)}{\bar{F}(y)} = \frac{\lambda\theta a b^a\left(\left(\frac{b}{y}\right)^a\right)^{\theta-1}\left(1-\left(\frac{b}{y}\right)^{a\theta}\right)^{\lambda-1}}{y^{a+1}\left(1-\left(1-\left(\frac{b}{y}\right)^{a\theta}\right)^{\lambda}\right)} \quad (3.2c)$$

Using 3.1d in 1.3c,

$$r(y) = \frac{\lambda\theta ab^a\left(\left(\frac{b}{y}\right)^a\right)^{\theta-1}}{y^{a+1}\left(1-\left(\frac{b}{y}\right)^{a\theta}\right)} = \frac{\lambda a\theta b^{a\theta}}{y^{a\theta+1}\left(1-\left(\frac{b}{y}\right)^{a\theta}\right)} = \frac{\lambda a\theta b^{a\theta}}{y(y^{a\theta}-b^{a\theta})} \quad (3.2c)$$

Extended PT1 3 (EPT1-3)

EPT1-3 with two additional shape parameters ($\theta > 0$, $\lambda > 0$) is obtained as an extension of EPT1-1 using the functions in 1.5 through 1.5b as generators but now with additional parameter, $\lambda > 0$, and EPT1-1 as base distribution. It is therefore referred to in this study as M-OM with respect to EPT1-1. The PDF, CDF, SF, HF, and RHF of this extended distribution are derived respectively as follows;

Using 3.1 and 3.1a in 1.5,

$$f(y) = \frac{\lambda\theta ab^a((b/y)^a)^{\theta-1}}{y^{a+1}[1-(1-\lambda)(1-(1-(b/y)^{a\theta}))]^2} = \frac{\lambda\theta ab^a((b/y)^a)^{\theta-1}}{y^{a+1}[1-(1-\lambda)((b/y)^{a\theta})]^2} \quad (3.3)$$

Using 3.1a in 1.5a,

$$F(y) = \frac{(1-(b/y)^{a\theta})}{1-(1-\lambda)\left(1-(1-(b/y)^{a\theta})\right)} = \frac{(1-(b/y)^{a\theta})}{1-(1-\lambda)((b/y)^{a\theta})} \quad (3.3a)$$

Using 3.1a and 3.1b in 1.5b,

$$\bar{F}(y) = \frac{\lambda(b/y)^{a\theta}}{1-(1-\lambda)\left(1-(1-(b/y)^{a\theta})\right)} = \frac{\lambda(b/y)^{a\theta}}{1-(1-\lambda)((b/y)^{a\theta})} \quad (3.3b)$$

$$h(y) = \frac{f(y)}{\bar{F}(y)} = \frac{\lambda\theta ab^a((b/y)^a)^{\theta-1}[1-(1-\lambda)((b/y)^{a\theta})]}{y^{a+1}[1-(1-\lambda)((b/y)^{a\theta})]^2(\lambda(b/y)^{a\theta})} = \frac{\theta ab^a}{y^{a+1}((b/y)^a)[1-(1-\lambda)((b/y)^{a\theta})]} \quad (3.3c)$$

$$r(y) = \frac{f(y)}{F(y)} = \frac{\lambda\theta ab^a((b/y)^a)^{\theta-1}[1-(1-\lambda)((b/y)^{a\theta})]}{y^{a+1}[1-(1-\lambda)((b/y)^{a\theta})]^2(1-(b/y)^{a\theta})} = \frac{\lambda\theta ab^a((b/y)^a)^{\theta-1}}{y^{a+1}[1-(1-\lambda)((b/y)^{a\theta})](1-(b/y)^{a\theta})} \quad (3.3d)$$

The functions of other extended PT1s are obtained similarly and given in the following subsections.

Extended PT1 4 (EPT1-4)

EPT1-4 with two additional shape parameters ($\theta > 0$, $\lambda > 0$) is obtained as an extension of EPT1-1 using the functions in 1.4 through 1.4b as generators but now with additional parameter, $\lambda > 0$, and EPT1-1 as base distribution. It is therefore referred to in this study as a PTM with respect to EPT1-1. The PDF, CDF, SF, HF, and RHF of this extended distribution are respectively given below.

$$f(y) = \theta\lambda y^{\lambda-1}\frac{ab^a}{y^{\lambda(a+1)}}\left(\left(\frac{b}{y^\lambda}\right)^a\right)^{\theta-1} \quad (3.4)$$

$$F(y) = 1-\left(\frac{b}{y^\lambda}\right)^{a\theta} \quad (3.4a)$$

$$\bar{F}(y) = \left(\frac{b}{y^\lambda}\right)^{a\theta} \quad (3.4b)$$

$$h(y) = \frac{\lambda\theta y^{\lambda-1}\frac{ab^a}{y^{\lambda(a+1)}}\left(\left(\frac{b}{y^\lambda}\right)^a\right)^{\theta-1}}{\left(\frac{b}{y^\lambda}\right)^{a\theta}} = \theta\lambda y^{\lambda-1}\frac{ab^a}{y^{\lambda(a+1)}}\left(\left(\frac{b}{y^\lambda}\right)^a\right)^{-1} \quad (3.4c)$$

$$r(y) = \frac{\lambda\theta y^{\lambda-1}\frac{ab^a}{y^{\lambda(a+1)}}\left(\left(\frac{b}{y^\lambda}\right)^a\right)^{\theta-1}}{1-\left(\frac{b}{y^\lambda}\right)^{a\theta}} = \theta\lambda y^{\lambda-1}\frac{ab^a}{y^{\lambda(a+1)}}\left(\left(\frac{b}{y^\lambda}\right)^a\right)^{\theta-1}\left(1-\left(\frac{b}{y^\lambda}\right)^{a\theta}\right)^{-1} \quad (3.4d)$$

Extended PT1 5 (EPT1-5)

EPT1-5 with two additional shape parameters ($\theta > 0$, $\lambda > 0$) is obtained as an extension of EPT1-1 using the functions in 1.6 through 1.6b as generators but now with additional parameter, $\lambda > 0$, and EPT1-1 as base distribution. It is therefore referred to in this study as an APTM with respect to EPT1-1. The PDF, CDF, and SF of this extended distribution are respectively given below.

$$f(y) = \begin{cases} \left(\frac{\log\lambda}{\lambda-1}\right)\frac{\theta ab^a}{y^{a+1}}\left(\left(\frac{b}{y}\right)^a\right)^{\theta-1}\lambda^{(1-(b/y)^{a\theta})}, \theta > 0, & \lambda > 0, \lambda \neq 1 \\ \frac{\theta ab^a}{y^{a+1}}\left(\left(\frac{b}{y}\right)^a\right)^{\theta-1}, \theta > 0, \lambda = 1 \\ \frac{ab^a}{y^{a+1}}, \theta = 1, \lambda = 1 \end{cases} \quad (3.5)$$

$$F(y) = \begin{cases} \frac{\lambda^{1-(b/y)^{a\theta}}-1}{\lambda-1}, \ \theta > 0, \lambda > 0, \lambda = 1 \\ 1-\left(\frac{b}{y}\right)^{a\theta}, \theta > 0, \lambda = 1 \\ 1-\left(\frac{b}{y}\right)^a, \theta = 1, \lambda = 1 \end{cases} \quad (3.5a)$$

$$\bar{F}(y) = \begin{cases} 1-\left(\frac{\lambda^{1-(b/y)^{a\theta}}-1}{\lambda-1}\right) = \frac{\lambda-\lambda^{1-(b/y)^{a\theta}}}{\lambda-1}, & \theta > 0, \lambda > 0, \lambda = 1 \\ \left(\frac{b}{y}\right)^{a\theta}, \theta > 0, \lambda = 1 \\ \left(\frac{b}{y}\right)^a, & \theta = 1, \lambda = 1 \end{cases} \quad (3.5b)$$

$$h(y) = \begin{cases} \frac{\theta ab^a\log(\lambda)((b/y)^a)^{\theta-1}\lambda^{(1-(b/y)^{a\theta})}}{y^{a+1}\left(\lambda-\lambda^{(1-(b/y)^{a\theta})}\right)}, \theta > 0, & \lambda > 0, \lambda \neq 1 \\ \frac{\theta a}{y}, \theta > 0, \lambda = 1 \\ \frac{a}{y}, \theta = 1, \lambda = 1 \end{cases} \quad (3.5c)$$

$$r(y) = \begin{cases} \frac{\theta ab^a\log(\lambda)((b/y)^a)^{\theta-1}\lambda^{(1-(b/y)^{a\theta})}}{y^{a+1}\left(\lambda^{(1-(b/y)^{a\theta})}-1\right)}, \theta > 0, & \lambda > 0, \lambda \neq 1 \\ \frac{\theta ab^{a\theta}}{y(y^{a\theta}-b^{a\theta})}, \theta > 0, \lambda = 1 \\ \frac{ab^a}{y(y^a-b^a)}, & , \theta = 1, \lambda = 1 \end{cases} \quad (3.5d)$$

Extended PT1 6 (EPT1-6)

EPT1-6 is a PRHM with an additional shape parameter (θ > 0) obtained as an extension of PT1 (base distribution) using the functions in 1.3 through 1. 3c as generators. The PDF, CDF, and SF, HF, and RHF of this extended distribution are respectively given below.

$$f(y) = \frac{\theta a b^a}{y^{a+1}}[1-(b/y)^a]^{\theta-1} \quad (3.6)$$

$$F(y) = [1-(b/y)^a]^{\theta} \quad (3.6a)$$

$$\bar{F}(y) = 1-[1-(b/y)^a]^{\theta} \quad (3.6b)$$

$$h(y) = \frac{\theta a b^a}{y^{a+1}}[1-(b/y)^a]^{\theta-1}\left[1-[1-(b/y)^a]^{\theta}\right]^{-1} \quad (3.6c)$$

$$r(y) = \frac{\theta a b^a}{y^{a+1}\left(1-\left(\frac{b}{y}\right)^a\right)} = \frac{\theta a b^a}{y(y^a-b^a)} \quad (3.6d)$$

Extended PT1 7 (EPT1-7)

EPT1-7 with two additional shape parameters (θ > 0, λ > 0) is obtained as an extension of EPT1-6 using the functions in 1.2 through 1.2c as generators but now with additional parameter, λ > 0, and EPT1-6 as base distribution. It is therefore referred to in this study as a PHM with respect to EPT1-6. The PDF, CDF, SF, and HF of this extended distribution are respectively given below.

$$f(y) = \frac{\lambda\theta a b^a}{y^{a+1}}[1-(b/y)^a]^{\theta-1}\left(1-[1-(b/y)^a]^{\theta}\right)^{\lambda-1} \quad (3.7)$$

$$F(y) = 1-\left(1-[1-(b/y)^a]^{\theta}\right)^{\lambda} \quad (3.7a)$$

$$\bar{F}(y) = \left(1-[1-(b/y)^a]^{\theta}\right)^{\lambda} \quad (3.7b)$$

$$h(y) = \frac{\lambda\theta a b^a}{y^{a+1}}[1-(b/y)^a]^{\theta-1}\left[1-[1-(b/y)^a]^{\theta}\right]^{-1} \quad (3.7c)$$

$$r(y) = \frac{\lambda\theta a b^a[1-(b/y)^a]^{\theta-1}\left[1-[1-(b/y)^a]^{\theta}\right]^{\lambda-1}}{y^{a+1}(1-(1-[1-(b/y)^a]^{\theta})^{\lambda})} \quad (3.7d)$$

Extended PT1 8 (EPT1-8)

EPT1-8 with two additional shape parameters (θ > 0, λ > 0) is obtained as an extension of EPT1-6 using the functions in 1.5 through 1.5b as generators but now with additional parameter, λ > 0, and EPT1-6 as base distribution. It is therefore referred to in this study as M-OM with respect to EPT1-6. The PDF and CDF of this extended distribution are respectively given below.

$$f(y) = \frac{\lambda\theta a b^a[1-(b/y)^a]^{\theta-1}}{y^{a+1}[1-(1-\lambda)(1-[1-(b/y)^a]^{\theta})]^2} \quad (3.8)$$

$$F(y) = \frac{[1-(b/y)^a]^{\theta}}{1-(1-\lambda)(1-[1-(b/y)^a]^{\theta})} \quad (3.8a)$$

Extended PT1 9 (EPT1-9)

EPT1-9 with two additional shape parameters (θ > 0, λ > 0) is obtained as an extension of EPT1-6 using the functions in 1.4 through 1.4b as generators but now with additional parameter, λ > 0, and EPT1-6 as base distribution. It is therefore referred to in this study as a PTM with respect to EPT1-6. The PDF and CDF of this extended distribution are respectively given below.

$$f(y) = \lambda y^{\lambda-1}\frac{\theta a b^a}{y^{\lambda(a+1)}}\left[1-\left(\frac{b}{y^{\lambda}}\right)^a\right]^{\theta-1} \quad (3.9)$$

$$F_Y(x) = \left[1-\left(\frac{b}{y^{\lambda}}\right)^a\right]^{\theta} \quad (3.9a)$$

Extended PT1 10 (EPT1-10)

EPT1-10 with two additional shape parameters (θ > 0, λ > 0) is obtained as an extension of EPT1-6 using the functions in 1.6 through 1.6b as generators but now with additional parameter, λ > 0, and EPT1-6 as base distribution. It is therefore referred to in this study as an APTM with respect to EPT1-6. The PDF and CDF of this extended distribution are respectively given below.

$$f(y) = \begin{cases} \left(\frac{\log\lambda}{\lambda-1}\right)\frac{\theta a b^a}{y^{a+1}}\left[1-\left(\frac{b}{y}\right)^a\right]^{\theta-1}\lambda^{\left[1-\left(\frac{b}{y}\right)^a\right]^{\theta}}, \theta>0, \quad \lambda>0, \lambda\neq 1 \\ \frac{\theta a b^a}{y^{a+1}}\left[1-\left(\frac{b}{y}\right)^a\right]^{\theta-1}, \theta>0, \lambda=1 \\ \frac{a b^a}{y^{a+1}}, \theta=1, \lambda=1 \end{cases} \quad (3.10)$$

$$F(y) = \begin{cases} \frac{\lambda^{[1-(b/y)^a]^{\theta}}-1}{\lambda-1}, \quad \theta>0, \lambda>0, \lambda=1 \\ \left[1-\left(\frac{b}{y}\right)^a\right]^{\theta}, \theta>0, \lambda=1 \\ 1-\left(\frac{b}{y}\right)^a, \theta=1, \lambda=1 \end{cases} \quad (3.10a)$$

Extended PT1 11 (EPT1-11)

EPT1-11 is M-OM with an additional shape parameter (θ > 0) obtained as an extension of PT1 (base distribution) using the functions in 1.5 through 1. 5b as generators. The PDF, CDF, and SF of this extended distribution are respectively given below.

$$f(y) = \frac{\theta a b^a}{y^{a+1}\left[1-(1-\theta)\left(1-(1-(b/y)^a)\right)\right]^2} = \frac{\theta a b^a}{y^{a+1}[1-(1-\theta)(b/y)^a]^2} \quad (3.11)$$

$$F(y) = \frac{1-(b/y)^a}{1-(1-\theta)\left(1-(1-(b/y)^a)\right)} = \frac{1-(b/y)^a}{1-(1-\theta)(b/y)^a} \quad (3.11a)$$

$$\bar{F}(y) = \frac{\theta(b/y)^a}{1-(1-\theta)\big(1-(1-(b/y)^a)\big)} = \frac{\theta(b/y)^a}{1-(1-\theta)(b/y)^a} \tag{3.11b}$$

Extended PT1 12 (EPT1-12)

EPT1-12 with two additional shape parameters (θ > 0, λ > 0) is obtained as an extension of EPT1-11 using the functions in 1.3 through 1.3c as generators but now with additional parameter, λ > 0, and EPT1-11 as base distribution. It is therefore referred to in this study as a PRHM with respect to EPT1-11. The PDF and CDF of this extended distribution are respectively given below.

$$f(y) = \frac{\lambda\theta ab^a}{y^{a+1}[1-(1-\theta)(b/y)^a]^2}\left(\frac{1-(b/y)^a}{1-(1-\theta)(b/y)^a}\right)^{\lambda-1} = \frac{\lambda\theta ab^a(1-(b/y)^a)^{\lambda-1}}{y^{a+1}[1-(1-\theta)(b/y)^a]^{\lambda+1}} \tag{3.12}$$

$$F(y) = \left(\frac{1-(b/y)^a}{1-(1-\theta)(b/y)^a}\right)^{\lambda} \tag{3.12a}$$

Extended PT1 13 (EPT1-13)

EPT1-13 with two additional shape parameters (θ > 0, λ > 0) is obtained as an extension of EPT1-11 using the functions in 1.2 through 1.2c as generators but now with additional parameter, λ > 0, and EPT1-11 as base distribution. It is therefore referred to in this study as a PHM with respect to EPT1-11. The PDF and CDF of this extended distribution are respectively given below.

$$f(y) = \frac{\lambda\theta ab^a}{y^{a+1}[1-(1-\theta)(b/y)^a]^2}\left(\frac{\theta(b/y)^a}{1-(1-\theta)(b/y)^a}\right)^{\lambda-1} = \frac{\lambda\theta^{\lambda} ab^a((b/y)^a)^{\lambda-1}}{y^{a+1}[1-(1-\theta)(b/y)^a]^{\lambda+1}} \tag{3.13}$$

$$F(y) = 1-\left(\frac{\theta(b/y)^a}{1-(1-\theta)(b/y)^a}\right)^{\lambda} \tag{3.13a}$$

Extended PT1 14 (EPT1-14)

EPT1-14 with two additional shape parameters (θ > 0, λ > 0) is obtained as an extension of EPT1-11 using the functions in 1.6 through 1.6b as generators but now with additional parameter, λ > 0, and EPT1-11 as base distribution. It is therefore referred to in this study as a APTM with respect to EPT1-11. The PDF and CDF of this extended distribution are respectively given below.

$$f(y) = \begin{cases} \dfrac{\log(\lambda)\theta ab^a}{y^{a+1}(\lambda-1)\left[1-(1-\theta)\left(\frac{b}{y}\right)^a\right]^2}\lambda^{\frac{1-\left(\frac{b}{y}\right)^a}{1-(1-\theta)\left(\frac{b}{y}\right)^a}}, \theta > 0,\ \lambda > 0, \lambda \neq 1 \\ \dfrac{\theta ab^a}{y^{a+1}\left[1-(1-\theta)\left(\frac{b}{y}\right)^a\right]^2}, \theta > 0, \lambda = 1 \\ \dfrac{ab^a}{y^{a+1}}, \theta = 1, \lambda = 1 \end{cases} \tag{3.14}$$

$$F(y) = \begin{cases} \dfrac{\lambda^{\frac{1-(b/y)^a}{1-(1-\theta)(b/y)^a}}-1}{\lambda-1}, \quad \theta > 0, \lambda > 0, \lambda = 1 \\ \dfrac{1-(b/y)^a}{1-(1-\theta)(b/y)^a}, \theta > 0, \lambda = 1 \\ 1-\left(\dfrac{b}{y}\right)^a, \theta = 1, \lambda = 1 \end{cases} \tag{3.14a}$$

Extended PT1 15 (EPT1-15)

EPT1-15 with two additional shape parameters (θ > 0, λ > 0) is obtained as an extension of EPT1-11 using the functions in 1.4 through 1.4b as generators but now with additional parameter, λ > 0, and EPT1-11 as base distribution. It is therefore referred to in this study as a PTM with respect to EPT1-11. The PDF and CDF of this extended distribution are respectively given below.

$$f_Y(x) = \frac{\lambda y^{\lambda-1}\theta ab^a}{y^{\lambda(a+1)}[1-(1-\theta)(b/y^{\lambda})]^2} \tag{3.15}$$

$$F(y) = \frac{1-(b/y^{\lambda})}{1-(1-\theta)(b/y^{\lambda})} \tag{3.15a}$$

Extended PT1 16 (EPT1-16)

EPT1-16 is an APTM with an additional shape parameter (θ > 0) obtained as an extension of PT1 (base distribution) using the functions in 1.6 through 1. 6b as generators. The PDF, CDF, and SF of this extended distribution are respectively given below.

$$f(y) = \begin{cases} \dfrac{ab^a\log(\theta)}{y^{a+1}(\theta-1)}\theta^{1-(b/y)^a}, \theta > 0, \quad \theta \neq 1 \\ \dfrac{ab^a}{y^{a+1}}, \quad \theta = 1 \end{cases} \tag{3.16}$$

$$F(y) = \begin{cases} \dfrac{\theta^{1-(b/y)^a}-1}{\theta-1}, \theta > 0, \theta \neq 1 \\ 1-\left(\dfrac{b}{y}\right)^a, \quad \theta = 1 \end{cases} \tag{3.16a}$$

$$\bar{F}(y) = \begin{cases} 1-\dfrac{\theta^{1-\left(\frac{b}{y}\right)^a}-1}{\theta-1} = \dfrac{\theta-\theta^{1-\left(\frac{b}{y}\right)^a}}{\theta-1}, \theta > 0, \theta \neq 1 \\ \left(\dfrac{b}{y}\right)^a, \quad \theta = 1 \end{cases} \tag{3.16b}$$

Extended PT1 17 (EPT1-17)

EPT1-17 with two additional shape parameters (θ > 0, λ > 0) is obtained as an extension of EPT1-16 using the functions in 1.3 through 1.3c as generators but now with additional parameter, λ > 0, and EPT1-16 as base distribution. It is therefore referred to in this study as a PRHM with respect to EPT1-16. The PDF and CDF of this extended distribution are respectively given below.

$$f(y) = \begin{cases} \dfrac{\lambda ab^a \log(\theta)}{y^{a+1}(\theta-1)} \theta^{1-(b/y)^a} \left(\dfrac{\theta^{1-(b/y)^a}-1}{\theta-1}\right)^{\lambda-1}, \theta > 0, & \theta \neq 1 \\ \dfrac{\lambda ab^a}{y^{a+1}} \left(1-\left(\dfrac{b}{y}\right)^a\right)^{\lambda-1}, & \theta = 1 \end{cases} \tag{3.17}$$

$$F(y) = \begin{cases} \left(\dfrac{\theta^{1-(b/y)^a}-1}{\theta-1}\right)^{\lambda}, \theta > 0, & \theta \neq 1 \\ \left(1-\left(\dfrac{b}{y}\right)^a\right)^{\lambda}, & \theta = 1 \end{cases} \tag{3.17a}$$

Extended PT1 18 (EPT1-18)

EPT1-18 with two additional shape parameters (θ > 0, λ > 0) is obtained as an extension of EPT1-16 using the functions in 1.2 through 1.2c as generators but now with additional parameter, λ > 0, and EPT1-16 as base distribution. It is therefore referred to in this study as a PHM with respect to EPT1-16. The PDF and CDF of this extended distribution are respectively given below.

$$f(y) = \begin{cases} \dfrac{\lambda ab^a \log(\theta)}{y^{a+1}(\theta-1)} \theta^{1-(b/y)^a} \left(\dfrac{\theta-\theta^{1-(b/y)^a}}{\theta-1}\right)^{\lambda-1}, \theta > 0, & \theta \neq 1 \\ \dfrac{\lambda ab^a}{y^{a+1}} \left(\dfrac{b}{y}\right)^{a(\lambda-1)}, & \theta = 1 \end{cases} \tag{3.18}$$

$$F(y) = \begin{cases} 1-\left(\dfrac{\theta-\theta^{1-(b/y)^a}}{\theta-1}\right)^{\lambda}, \theta > 0, & \theta \neq 1 \\ 1-\left(\dfrac{b}{y}\right)^{a\lambda}, & \theta = 1 \end{cases} \tag{3.18a}$$

Extended PT1 19 (EPT1-19)

EPT1-19 with two additional shape parameters (θ > 0, λ > 0) is obtained as an extension of EPT1-16 using the functions in 1.5 through 1.5b as generators but now with additional parameter, λ > 0, and EPT1-16 as base distribution. It is therefore referred to in this study as M-OM with respect to EPT1-16. The PDF and CDF of this extended distribution are respectively given below.

$$f(y) = \begin{cases} \dfrac{\lambda ab^a \log(\theta)\theta^{1-(b/y)^a}}{y^{a+1}(\theta-1)\left[1-(1-\lambda)\left(1-\left(\dfrac{\theta^{1-(b/y)^a}-1}{\theta-1}\right)\right)\right]^2}, \theta > 0, & \theta \neq 1 \\ \dfrac{\lambda ab^a}{y^{a+1}\left[1-(1-\lambda)\left(\left(\dfrac{b}{y}\right)^a\right)\right]^2}, & \theta = 1 \end{cases} \tag{3.19}$$

$$F(y) = \begin{cases} \dfrac{\theta^{1-(b/y)^a}-1}{(\theta-1)\left[1-(1-\lambda)\left(1-\left(\dfrac{\theta^{1-(b/y)^a}-1}{\theta-1}\right)\right)\right]}, \theta > 0, & \theta \neq 1 \\ \dfrac{1-(b/y)^a}{[1-(1-\lambda)]\left(\left(\dfrac{b}{y}\right)^a\right)}, & \theta = 1 \end{cases} \tag{3.19a}$$

Extended PT1 20 (EPT1-20)

EPT1-20 with two additional shape parameters (θ > 0, λ > 0) is obtained as an extension of EPT1-16 using the functions in 1.4 through 1.4b as generators but now with additional parameter, λ > 0, and EPT1-16 as base distribution. It is therefore referred to in this study as a PTM with respect to EPT1-16. The PDF and CDF of this extended distribution are respectively given below.

$$f(y) = \begin{cases} \dfrac{\lambda y^{\lambda-1} ab^a \log(\theta)}{y^{\lambda(a+1)}(\theta-1)} \theta^{1-(b/y^{\lambda})^a}, \theta > 0, & \theta \neq 1 \\ \dfrac{\lambda y^{\lambda-1} ab^a}{y^{\lambda(a+1)}}, & \theta = 1 \end{cases} \tag{3.20}$$

$$F(y) = \begin{cases} \dfrac{\theta^{1-(b/y^{\lambda})^a}-1}{\theta-1}, \theta > 0, & \theta \neq 1 \\ 1-\left(\dfrac{b}{y^{\lambda}}\right)^a, & \theta = 1 \end{cases} \tag{3.20a}$$

Extended PT1 21 (EPT1-21)

EPT1-21 is a PTM with an additional shape parameter (θ > 0) obtained as an extension of PT1 (base distribution) using the functions in 1.4 through 1. 4b as generators. The PDF, CDF, and SF of this extended distribution are respectively given below.

$$f(y) = \theta y^{\theta-1} \frac{ab^a}{y^{\theta(a+1)}} \tag{3.21}$$

$$F(y) = 1-\left(\frac{b}{y^{\theta}}\right)^a \tag{3.21a}$$

$$\bar{F}(y) = \left(\frac{b}{y^{\theta}}\right)^a \tag{3.21b}$$

Extended PT1 22 (EPT1-22)

EPT1-22 with two additional shape parameters (θ > 0, λ > 0) is obtained as an extension of EPT1-21 using the functions in 1.2 through 1.2c as generators but now with additional parameter, λ >

0, and EPT1-21 as base distribution. It is therefore referred to in this study as a PHM with respect to EPT1-21. The PDF and CDF of this extended distribution are respectively given below.

$$f_Y(x) = \lambda\theta y^{\theta-1}\frac{ab^a}{y^{\theta(a+1)}}\left(\frac{b}{y^\theta}\right)^{a(\lambda-1)} \quad (3.22)$$

$$F_Y(x) = 1 - \left(\frac{b}{y^\theta}\right)^{a\lambda} \quad (3.22a)$$

Validity of PDFs

When $b = 1$, the PDFs of EPT1-1, EPT1-2, EPT1-3, and EPT1-4 in 3.1, 3.2, 3.3, and 3.4 respectively become:

$$f(y) = \frac{a\theta}{y^{a\theta+1}} \quad (4.1)$$

$$f(y) = \lambda\theta a(y)^{-(a\theta+1)}\left(1-(y)^{-a\theta}\right)^{\lambda-1} \quad (4.2)$$

$$f(y) = \frac{\lambda\theta a((y)^{-a})^{\theta-1}}{y^{a+1}[1-(1-\lambda)((y)^{-a\theta})]^2} \quad (4.3)$$

$$f(y) = \frac{a\theta\lambda y^{\lambda-1}\left(y^{-\lambda a}\right)^{\theta-1}}{y^{\lambda(a+1)}} \quad (4.4)$$

The PDFs above are all non-negative, therefore 2.1 is satisfied.

$$\int_1^\infty \frac{a\theta}{y^{a\theta+1}}dy = -y^{-a\theta}\Big|_1^\infty = 1 \quad (4.5)$$

$$\int_1^\infty \lambda\theta a(y)^{-(a\theta+1)}\left(1-(y)^{-a\theta}\right)^{\lambda-1}dy = \left(1-y^{-a\theta}\right)^\lambda\Big|_1^\infty = 1 \quad (4.6)$$

$$\int_1^\infty \frac{\lambda\theta a((y)^{-a})^{\theta-1}}{y^{a+1}[1-(1-\lambda)((y)^{-a\theta})]^2}dy = \frac{-\lambda y^{a\theta}(y^{-a})^\theta}{y^{a\theta}+\lambda-1}\Big|_1^\infty = 1 \quad (4.7)$$

$$\int_1^\infty \frac{a\theta\lambda y^{\lambda-1}\left(y^{-\lambda a}\right)^{\theta-1}}{y^{\lambda(a+1)}}dy = -\left(y^{-a\lambda}\right)^\theta\Big|_1^\infty = 1 \quad (4.8)$$

From 4.5, 4.6, 4.7, 4.8, it can be seen that 2.2 is also satisfied. Hence the four PDFs are true PDFs

Flexibility of Extended Pareto Type 1 Distributions

The plots of density and hazard functions of EPT1-1 is shown in Figure 1. No new shape was introduced. Not all generalized distributions introduce new shapes. Figure 2 represents possible density and hazard shapes of EPT1-2. The two functions can both have a downward sloping shape and unimodal shapes with a tail to the right. EPT1-2 introduced a new shape. The possible density and Hazard plots of EPT1-3 are illustrated in Figure 3. Figure 3 reveals decreasing density and hazard functions and unimodal shape with a long tail to the right. EPT1-3 also introduced a new shape. Figure 4, plots of density and hazard functions of EPT1-4, show decreasing density functions and downward sloping, upward sloping, and constant hazard shapes. New shapes observed (constant, increasing and unimodal shapes) established flexibility of distributions

DISCUSSION

EPT1-1 is a PHM with one additional parameter. Comparison of equations 3.1 through 3.1d and equations 1.1 through 1.1d shows that EPT1-1 does not introduce any new shape. The Alpha Power Pareto distribution studied by Ihtisham (2019) corresponds to EPT1-16 when $b = 1$. EPT1-6 is the exponentiated Pareto distribution by Gupta *et al.* (1998). EPT1-7 is the Kumaraswamy Pareto distribution proposed by Bourguignon *et al.* (2021). EPT1-11 is the Marshall-Olkin Pareto discussed by Bdair and Haj Ahmad (2021).

When $b = 1$ in EPT1-19, the PDF and the CDF corresponds to those of Marshall-Olkin alpha power Pareto by Almetwally and Haj Ahmed (2020). EPT1-19 was derived by sequentially applying alpha power transformation and Marshall-Olkin generalization methods. When the power transformation generalization method is used together with any other generalization method to generate generalized distribution with two additional parameters, the order of usage appears to be inconsequential as the same distributions are generated. This can be seen when functions of EPT1-4 are compared with those of EPT1-22. Functions of EPT1-4 were obtained by first using functions of a PHM as generators followed by those of PTM whereas functions of EPT1-22 were obtained by using functions of PTM as generators followed by those of PHM. EPT1-4 and EPT1-22 are the same.

The density and hazard shapes of EPT1-1 in Figure 1 indicated that flexibility was not added as no new shapes were introduced. However, EPT1-2 an extension of EPT1-1 with an extra parameter introduced a new shape as observed in Figure 2. EPT1-3 also introduced a new shape as seen in Figure 3. EPT1-4 can have a constant, increasing and downward hazard shapes represented in

Figure 4. EPT1-4 reduces to EPT1-21 when $\theta = 1$. At this value of θ ($b=1$, $a=0.5$, $\theta=1$, $\lambda=1.5$) in Figure 4, an increasing hazard rate was observed which is a possible shape of EPT1-21. This therefore establishes flexibility of EPT1-21.

CONCLUSION

Extended Pareto Type 1 distributions were derived through introduction of parameters into the probability distributions of the Pareto Type 1 distribution. Application of any method produced extended Pareto Type 1 distribution with an additional parameter. Sequentially using two methods produced those with two additional shape parameters. Comparisons between density and hazard shapes of some extended distributions with those of the parent distribution showed introduction of new shapes establishing added flexibility. Some of the extended distributions are in existence, others are recommended for future studies.

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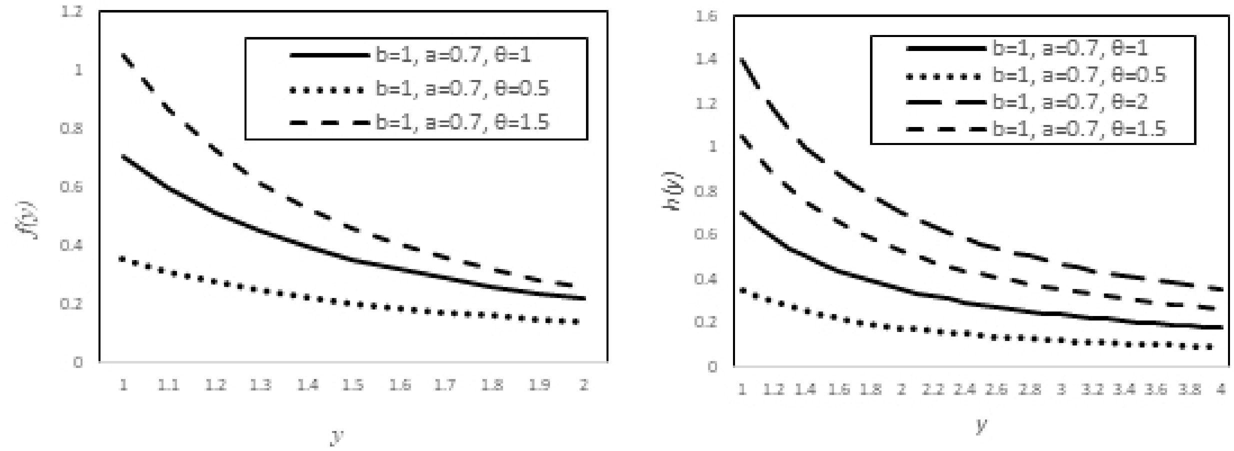

Figure 1: Plots of possible shapes of PDFs and HFs of EPT1-1.

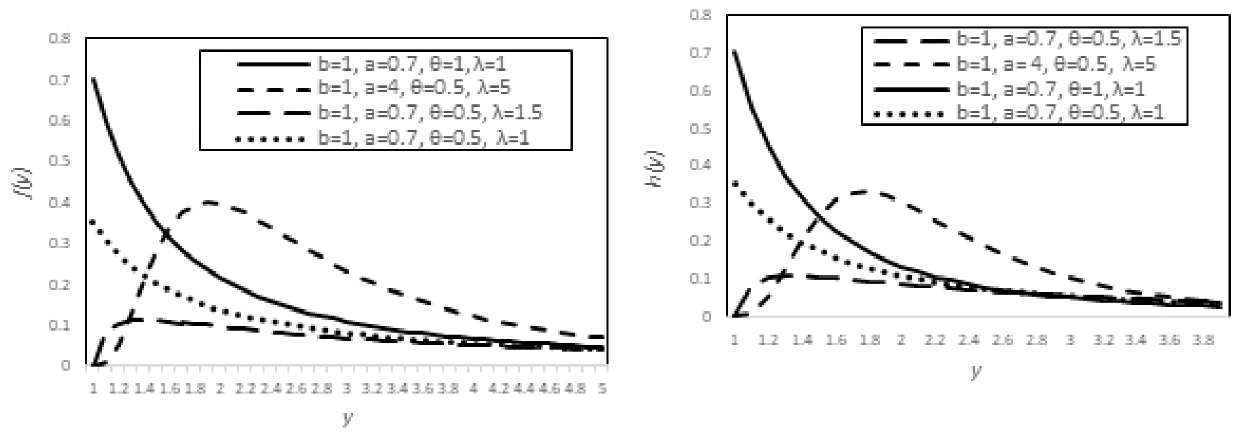

Figure 2: Plots of possible shapes of PDFs and HFs of EPT1-2.

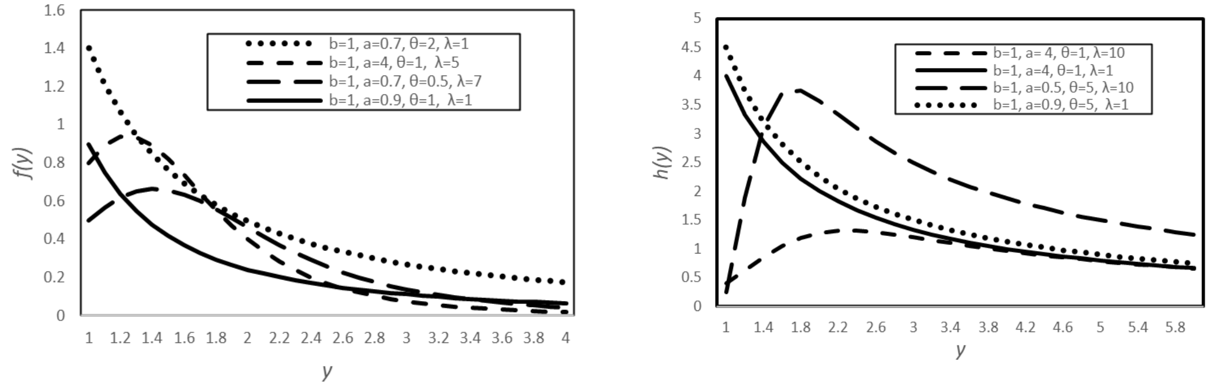

Figure 3: Plots of possible shapes of PDFs and HFs of EPT1-3.

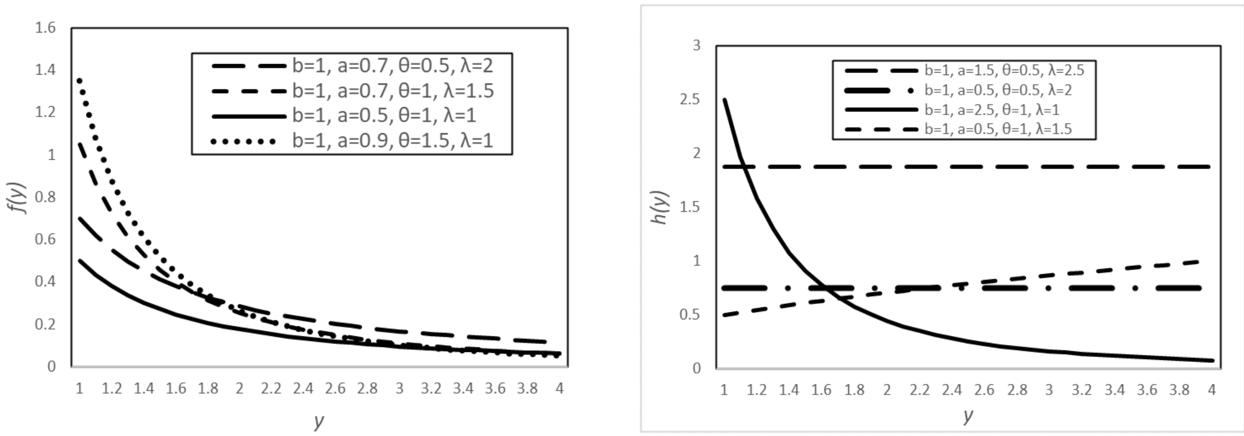

Figure 4: Plots of possible shapes of PDFs and HFs of EPT1-4.

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