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ON QUADRATIC HARMONIC NUMBER SUMS

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ABSTRACT

In this note, we extend a result of Sofo and Hassani concerning the evaluation of a certain type of Euler sums.

Keywords: Harmonic number; Riemman zeta function; Euler sum

INTRODUCTION

Sofo and Hassani (2012) gave a closed form evaluation for the series

$$\mathsf{S}(q,1)=\sum_{r=1}^{\infty}\frac{H_r^2}{r(r+q)},$$

for $q\in\mathsf{Z}^+$, where H_j is the j^{th} harmonic number defined by

$$H_j=\sum_{s=1}^{j}\frac{1}{s}.$$

Our goal in this paper is to extend their result to the evaluation of

$$\mathsf{S}(q,m)=\sum_{r=1}^{\infty}\frac{H_r^2}{r^m(r+q)},$$

for $q,m\in\mathsf{Z}^+$.

Throughout this paper $\zeta(s)$ denotes the Riemann zeta function, defined by

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},\quad \mathrm{Re}(s)>1,$$

and the generalized r^{th} harmonic number of order s denoted by $H_{r,s}$ is defined by

$$H_{r,s}=\sum_{n=1}^{r}\frac{1}{n^s}.$$

Studies containing results related to quadratic and higher order sums have also been carried out recently by Sofo (2016), Si *et al.* (2017) and Chen and Chen (2020).

REQUIRED THEOREM AND LEMMA

Our result is given in the next section. We will however need the following known results in the proof of the main theorem and also for subsequent evaluations.

Lemma 2.1 (Partial fraction decomposition)

$$\frac{(-1)^m q^{m-1}}{r^m(r+q)}=\sum_{k=2}^{m}(-1)^k\frac{q^{k-2}}{r^k}-\frac{1}{r(r+q)},\quad m\in\mathsf{Z}^+.$$

Theorem 2.1 (Euler) *Let s be a positive integer greater than unity. Then,*

$$2\sum_{r=1}^{\infty}\frac{H_r}{r^s}=(s+2)\zeta(s+1)-\sum_{j=1}^{s-2}\zeta(j+1)\zeta(s-j).$$

Theorem 2.2 (Borwein *et al.*, 1995) *Let s be a positive integer greater than unity. Then,*

$$\begin{aligned}\sum_{s=1}^{\infty}\frac{H_r^2}{r^s}-\sum_{s=1}^{\infty}\frac{H_{r,2}}{r^s} &= -\left(\frac{s}{2}+1\right)\sum_{k=1}^{s-1}\zeta(k+1)\zeta(s-k+1)\\ &\quad+\frac{1}{3}\sum_{k=2}^{s-2}\zeta(s-k)\sum_{j=1}^{k-1}\zeta(j+1)\zeta(k-j+1)\\ &\quad+\frac{1}{3}(s+1)(s+3)\zeta(s+2)+\zeta(2)\zeta(s).\end{aligned}$$

We note that we have corrected an error in the version of this theorem that was given in Borwein *et. al* (1995).

Theorem 2.3 (Borwein *et al.*, 1995) *Let n and s be positive integers with $s>1$ and such that* $(n+s)$ mod 2=1. *Then*

$$\begin{aligned}2\sum_{r=1}^{\infty}\frac{H_{r,n}}{r^s} &= \zeta(n+s)\left\{1-(-1)^n\binom{n+s-1}{n}-(-1)^n\binom{n+s-1}{s}\right\}\\ &\quad+\left(1-(-1)^n\right)\zeta(n)\zeta(s)\\ &\quad+(-1)^n 2\sum_{j=1}^{\lfloor n/2\rfloor}\binom{n+s-2k-1}{s-1}\zeta(2j)\zeta(n+s-2j)\\ &\quad+(-1)^n 2\sum_{j=1}^{\lfloor s/2\rfloor}\binom{n+s-2k-1}{n-1}\zeta(2j)\zeta(n+s-2j),\end{aligned}$$

where $\zeta(1)$ should be interpreted as 0 wherever it occurs.

Theorem 2.4 (Sofo and Hassani, 2012) *Assume that $q\geq 1$ is an integer, and let*

$$\mathsf{F}(q)=H_{q-1}\zeta(2)+H_{q-1}H_{q-1,2}-H_{q-1,3}+H_{q-1}^3.$$

Also for $j \neq q$ let

$$\mathrm{A}(q,j) = \frac{H_{j-1}}{qj^2} + \frac{1}{2j(q-j)}\left(H_{j-1}^2 + H_{j-1,2}\right)$$

We have

$$\mathrm{S}(q,1) = \sum_{r=1}^{\infty} \frac{H_r^2}{r(r+q)} = \frac{3\zeta(3) + \mathrm{F}(q)}{q} - \sum_{j=1}^{q-1} \mathrm{A}(q,j).$$

MAIN RESULT AND COROLLARIES

We now state our main result.

Theorem 3.1 *Let q and m be positive integers. Then,*

$$\begin{aligned} 2(-1)^m q^{m-1} S(q,m) &= 2(-1)^m q^{m-1} \sum_{r=1}^{\infty} \frac{H_r^2}{r^m(r+q)} \\ &= \sum_{k=2}^{m} \left\{ 2(-1)^k q^{k-2} \sum_{r=1}^{\infty} \frac{H_{r,2}}{r^k} \right\} \\ &\quad - \sum_{k=2}^{m} (-1)^k q^{k-2}(k+2) \sum_{p=1}^{k-1} \zeta(p+1)\zeta(k-p+1) \\ &\quad + \frac{2}{3} \sum_{k=2}^{m} (-1)^k q^{k-2} \sum_{p=2}^{k-2} \zeta(k-p) \sum_{j=1}^{p-1} \zeta(j+1)\zeta(p-j+1) \\ &\quad + \frac{2}{3} \sum_{k=2}^{m} (-1)^k q^{k-2}(k+1)(k+3)\zeta(k+2) \\ &\quad + 2\zeta(2) \sum_{k=2}^{m} (-1)^k q^{k-2} \zeta(k) - 2S(q,1). \end{aligned}$$

Proof. Multiply through Lemma 2.1 by H_r^2, sum over $r \geq 1$ while taking into account Theorems 2.1, 2.2 and 2.4.

Corollary 3.1

$$\sum_{r=1}^{\infty} \frac{H_r^2}{r^2(r+1)} = \frac{17}{4}\zeta(4) - 3\zeta(3), \qquad (1)$$

$$5\sum_{r=1}^{\infty} \frac{H_r^2}{r^2(r+5)} = \frac{17}{4}\zeta(4) - \frac{3\zeta(3)}{5} - \frac{5\zeta(2)}{12} - \frac{8737}{8640}, \qquad (2)$$

Proof. With $m = 2$ in Theorem 3.1 and using the result

$$\sum_{r=1}^{\infty} \frac{H_{r,n}}{r^n} = \zeta(n)^2 + \zeta(2n), \qquad (3)$$

we have

$$q\mathrm{S}(q,2) = \frac{17}{4}\zeta(4) - \mathrm{S}(q,1),$$

from which (1) follows. The identity (2) follows from the fact that

$$S(1,1) = 3\zeta(3) \text{ and } S(5,1) = \frac{3\zeta(3)}{5} + \frac{5\zeta(2)}{12} + \frac{8737}{8640},$$

(see Sofo and Hassani, 2012).

Corollary 3.2

$$\sum_{r=1}^{\infty} \frac{H_r^2}{r^2} = \mathrm{S}(1,2) + \mathrm{S}(1,1) = \frac{17}{4}\zeta(4).$$

This is a well-known result that has been rediscovered by several authors (see for example Borwein *et al.*, 1995).

Proof. A consequence of the fact that

$$\frac{q}{r^2(r+q)} + \frac{1}{r(r+q)} = \frac{1}{r^2}.$$

Corollary 3.3

$$2\sum_{r=1}^{\infty} \frac{H_r^2}{r^3(r+1)} = 7\zeta(5) - 2\zeta(2)\zeta(3) + \zeta(2)^2 - 11\zeta(4) + 6\zeta(3), \qquad (4)$$

$$\begin{aligned} 50\sum_{r=1}^{\infty} \frac{H_r^2}{r^3(r+5)} &= 5[7\zeta(5) - 2\zeta(2)\zeta(3)] + \zeta(2)^2 \\ &\quad - 11\zeta(4) + \frac{6\zeta(3)}{5} + \frac{5\zeta(2)}{6} + \frac{8737}{4320}. \end{aligned} \qquad (5)$$

Proof. With $m = 3$ in Theorem 3.1 and using again (3) and also Theorem 2.3 with $n = 2$ and $s = 3$, we have

$$2q^2 S(q,3) = q\big(7\zeta(5) - 2\zeta(2)\zeta(3)\big) + \big(\zeta(2)^2 - 11\zeta(4)\big) + 2S(q,1),$$

of which identities (4) and (5) are particular cases.

Corollary 3.4

$$6\sum_{r=1}^{\infty}\frac{H_r^2}{r^4(r+1)} = -30\zeta(2)\zeta(4) - 12\zeta(3)^2 + 2\zeta(2)^3 + 68\zeta(6) + 6\zeta(2)\zeta(3) - 21\zeta(5) - 3\zeta(2)^2 + 33\zeta(4) - 18\zeta(3), \quad (6)$$

$$750\sum_{r=1}^{\infty}\frac{H_r^2}{r^4(r+5)} = 25\{-30\zeta(2)\zeta(4) - 12\zeta(3)^2 + 2\zeta(2)^3 + 68\zeta(6)\} + 5\{6\zeta(2)\zeta(3) - 21\zeta(5)\} - 3\zeta(2)^2 + 33\zeta(4) - \frac{18\zeta(3)}{5} - \frac{5\zeta(2)}{2} - \frac{8737}{1440}. \quad (7)$$

Proof. Choosing $m = 4$ in Theorem 3.1 and using the known result (see for example Flajolet *et al.*, 1998)

$$\sum_{r=1}^{\infty}\frac{H_{r,2}}{r^4} = \zeta(3)^2 - \frac{1}{3}\zeta(6),$$

we obtain

$$6q^3 S(q,4) = q^2\{-30\zeta(2)\zeta(4) - 12\zeta(3)^2 + 2\zeta(2)^3 + 68\zeta(6)\} + q\{6\zeta(2)\zeta(3) - 21\zeta(5)\} - 3\zeta(2)^2 + 33\zeta(4) - 6S(q,1),$$

from which we deduce (6) and (7).

Corollary 3.5

$$6\sum_{r=1}^{\infty}\frac{H_r^2}{r^5(r+1)} = -30\zeta(3)\zeta(4) + 36\zeta(7) - 6\zeta(2)\zeta(5) + 6\zeta(3)\zeta(2)^2 + 12\zeta(3)^2 - 2\zeta(2)^3 - 68\zeta(6) + 30\zeta(2)\zeta(4) - 6\zeta(2)\zeta(3) + 21\zeta(5) + 3\zeta(2)^2 - 33\zeta(4) + 18\zeta(3), \quad (8)$$

$$3750\sum_{r=1}^{\infty}\frac{H_r^2}{r^5(r+5)} = 125\{-30\zeta(3)\zeta(4) + 36\zeta(7) - 6\zeta(2)\zeta(5) + 6\zeta(3)\zeta(2)^2\} + 25\{12\zeta(3)^2 - 2\zeta(2)^3 - 68\zeta(6) + 30\zeta(2)\zeta(4)\} + 5\{-6\zeta(2)\zeta(3) + 21\zeta(5)\} + 3\zeta(2)^2 - 33\zeta(4) + \frac{18\zeta(3)}{5} + \frac{5\zeta(2)}{2} + \frac{8737}{1440}. \quad (9)$$

Proof. The choice $m = 5$ in Theorem 3.1 and the use of Theorem 2.3 with $n = 2$ and $s = 5$ leads to

$$6q^4 S(q,5) = q^3\big[-30\zeta(3)\zeta(4) + 36\zeta(7) - 6\zeta(2)\zeta(5) + 6\zeta(3)\zeta(2)^2\big] + q^2\big(12\zeta(3)^2 - 2\zeta(2)^3 - 68\zeta(6) + 30\zeta(2)\zeta(4)\big) + q\big(-6\zeta(2)\zeta(3) + 21\zeta(5)\big) + 3\zeta(2)^2 - 33\zeta(4) + 6S(q,1),$$

from which we get (8) and (9).

CONCLUSION

We extended the result of Sofo and Hassani (2012) for $S(q,1)$, by deriving an explicit formula for the evaluation of

$$S(q,m) = \sum_{r=1}^{\infty}\frac{H_r^2}{r^m(r+q)},$$

for $q,m \in \mathbb{Z}^+$.

From the statement of Theorem 3.1, it is not possible to reduce $S(q,m)$ to zeta values alone for $m > 5$, since it is considered highly improbable that the linear Euler sum $\sum_{r\ge1} H_{r,2}/r^n$ can be expressed in terms of zeta values alone, Borwein *et. al.* (1995), for even $n > 4$.

REFERENCES

Borwein, D and Borwein, J.M. (1995). On an Intriguing Integral and Some Series Related to ζ(4). Proceedings of the American Mathematical Society, 123 (4): 1191—1198.

Borwein, D., Borwein, J.M and Girgensohn, R. (1995). Explicit Evaluation of Euler Sums, Proceedings of the Edingburgh Mathematical Society, 38 (2): 277—294.

Chen K. and Chen W. (2020) . Infinite series containing generalized harmonic functions. Notes on Number Theory and Discrete Mathematics, 26 (2):85-104

Flajolet, P and Salvy, B. (1998) Euler Sum and Contour Integral Representations, Experimental Mathematics, 7 (1): 15-35.

Si X., Xu C. and Zhang M. (2017) Quadratic and cubic harmonic number sums. Journal of Mathematical Analysis and Applications, 447 (1): 419-434.

Sofo, A and Hassani, M. (2012). Quadratic Harmonic Number. Applied Maths. E-Notes, 12: 110-117.

Sofo A. (2016). New Results Containing Quadratic Harmonic Numbers. Journal of Classical Analysis, 9 (2): 117-125