CHAOTIC DYNAMICS OF THE COUPLED SINGLE-WELL QUINTIC, HENON-HEILES AND HYDROGEN ATOM IN A UNIFORM MAGNETIC FIELD

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ABSTRACT

The behaviour of a body subjected to three coupled potentials, those of a single-well quintic potential, the Henon-Heiles potential and that of the hydrogen atom in a uniform magnetic field were investigated. The parameter under interest was the parameter that characterised the chaotic dynamics of the hydrogen atom in a uniform magnetic field. The body exhibited regular behaviour for values of the coupling parameter less than a threshold value, after which it abruptly became chaotic, with the degree of chaoticity increasing with the coupling parameter.

Keywords: Single-well Quintic, Henon-Heiles, Hydrogen Atom, Uniform Magnetic Field, Chaos

INTRODUCTION

The Henon-Heiles potential was proposed by Henon and Heiles (1964) as a model for the motion of a star in a plane around a galactic centre. The resulting motion became so important in the development of dynamical chaos because this simple system gave the spectrum of behaviour for chaotic dynamics in a two-dimensional system. There is only one constant of motion, the Hamiltonian of the system. For values of the energy less than a certain threshold, the system behaves like an integrable system, as the Poincare surface consists only of invariant curves. In other words, it is as if each of the two oscillators holds its energy. At higher energies, it is obvious that the system is not integrable as the two component oscillators no longer share energy in a predictable way. Thus, the Poincare surface from a single initial point generates a chaotic sea.

Various forms of the modified Henon-Heiles system had also been explored (Vesely and Podolsky, 2000; Brack *et al.*, 1999; Choudhury and Kalita, 2008). Kasperczuk, S. (1995) working on generalised Henon-Heiles systems used Melnikov's method to prove the existence of nondegenerate homoclinic orbits near two integrable cases.

The Henon-Heiles problem has transcended the boundaries of classical mechanics. Brack *et al.* (1993) observed that the quantum density of states of the Henon-Heiles Hamiltonian exhibited prominent low-frequency beats as a function of energy. These they interpreted in terms of interferences of the three simplest isolated classical periodic orbites by a calculation of their amplitudes in the Gutzwiller trace formula. They introduced a parameter governing the anharmonicity of the potential. Periodic orbit theory was applied to approximate the oscillating part of the resonance spectrum of the quantum spectrum of the Henon-Heiles potential up to twice the barrier energy (Kaidel *et al.*, 2005). Gupta and Deb (2006) studied the quantum dynamics of an electron moving under the Henon-Heiles potential in the presence of external time dependent laser fields of varying intensities by evolving in real time the unperturbed ground-state wave function of the Henon-Heiles oscillator. They also analysed the similarity between the Henon-Heiles potential and atoms/molecules in intense laser fields.

Chaoticists had also put a lot of attention on the hydrogen atom in a uniform magnetic field as an example of a chaotic system. Examples include the classical electronic motion of the atom near a metal surface (Simonovic, 1997), hydrogen atom in the presence of uniform magnetic and quadrupolar electric fields (Inarrea and Salas, 2002) and the hydrogen atom in weak electric and magnetic fields (Turbiner, 1983).

Wintgen and Friedrich (1986) studied the spectrum of the hydrogen atom in a uniform magnetic field by exact numerical calculations in a complete basis. Kravchenko *et al.* (1996)

developed a highly accurate series solution for a hydrogen atom in a uniform magnetic field of arbitrary strength, a power series in terms of the radius and the sine of the cone angle. In addition, Bachmann *et al.* (2000) extended the Feynman-Kleinert variational approach to calculate the temperature-dependent effective classical potential governing the quantum statistics of a hydrogen atom in a uniform magnetic field at all temperatures.

More recently, Popov and Karnakov (2014) studied the energy spectrum of atomic hydrogen in strong and ultra-strong magnetic fields in which the hydrogen electron started to move relativistically and quantum electrodynamics effects became important. Also, Amdouni and Eleuch (2014) analysed the relativistic corrections on the energy spectra of a hydrogen atom with realistic nucleus mass in a strong magnetic field.

The Duffing oscillator is a nonlinear second-order ordinary differential equation which has found application in many real-life situations. These include magneto-elastic mechanical systems (Guckenheimer and Holmes, 1983), sinusoidally excited buckled beam (Pezeshki and Dowell, 1987), nonlinear vibration of beams and plates (Ahmadian *et al.*, 2009), and flow-induced vibration (Srinil and Zanganeh, 2012).

Coupled Duffing oscillators have also been investigated for signal detection. Yue *et al.* (2006) detected periodic signals under the background of strong coloured noise by using two coupled Duffing oscillators. Wu *et al.* (2014) worked on the stochastic resonance of two coupled Duffing oscillators, applying this to weak signal detection (Wu *et al.*, 2014).

A search of the literature did not reveal any work done on these three systems having been coupled together before. This work was therefore undertaken to study the chaotic dynamics of a particle in such a potential well depending on how much of the hydrogen potential is added to the coupling.

METHODOLOGY

In this work, the oscillators under consideration have coordinates $_1q$ and $_2q$, with corresponding

momenta $_1p$ and $_2p$. In this respect, the Henon-Heiles potential, $_1V$, is given as,

$$V_1(q_1, q_2) = \frac{1}{2}(q_1^2 + q_2^2) + q_1^2 q_2 - \frac{1}{3}q_2^3 \quad (1)$$

The resulting Hamiltonian, $_1H$, is,

$$H_1 = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(q_1^2 + q_2^2) + q_1^2 q_2 - \frac{1}{3}q_2^3 \quad (2)$$

Referring to Friedrich and Wintgen (1989) treatment of the hydrogen atom in a uniform magnetic field (of strength B) described by the Hamiltonian,

$$H_2 = \frac{p^2}{2m_e} - \frac{e^2}{r} + \omega l_z + \frac{1}{2}m_e\omega^2(x^2 + y^2) \quad (3)$$

where H_2 is the Hamiltonian, z is the direction of the field, m_e is the reduced mass of the electron and the nucleus, and ω is half the cyclotron frequency, equal to $\frac{eB}{2m_e c}$, where e is the

electronic charge and c the speed of light. It has been shown that the dynamics is equivalent to that given by the potential:

$$V_2 = \frac{1}{2}\left(q_1^2 + q_2^2\right) - s\left(q_1^2 + q_2^2\right) + \frac{1}{8}q_1^2 q_2^2\left(q_1^2 + q_2^2\right) \quad (4)$$

where the scaled energy, $s = \frac{B}{B_0} = \frac{\hbar\omega}{\Re}$, determines

the degree of chaoticity of the system and $B_0 = m_e e^3 c / h^3$, the value of the magnetic field strength at which the oscillator energy equals the Rydberg energy, $\Re$. q_1, p_1, q_2 and p_2 are the coordinates and momenta of the equivalent system. The last term in Eq. (4) is the diamagnetic coupling term.

The system described by Eq. (4) has been been found by the authors to be fully chaotic for $s = -0.1$.

The double-well cubic potential which has attracted much attention (Flocken *et al.*, 1989; Visco and Sen, 1998; Turbiner, 2005; and Olusola *et al.*, 2010) has the form,

$$V(q) = m^2 q^2 + gq^4 \quad (5)$$

where $m^2 < 0$, and $g > 0$ an arbitrary constant.

Along these lines, the quintic potential can be

written in terms of the parameters,

$$V(q) = m^2 q^2 + gq^4 + hq^6 \qquad (6)$$

With the right choice of the parameters m, g and h, the potential could be a single well, a double well or triple well, single hump, double well with double hump or an inverted single well. In

particular, in this work we set $m^2 = \dfrac{1}{2}, \ g = \dfrac{1}{4}$

and $h = \dfrac{1}{6}$. This choice of values gives a single-

well potential and ensures the simplicity of the equations of motion due to the Hamilton equations for this conservative system.

We consider two nonlinearly coupled undamped and unforced Duffing oscillators, such that the resulting potential is,

$$V_3 = \frac{1}{2}(q_1^2 + q_2^2) + \frac{1}{4}(q_1^4 + q_2^4) + \frac{1}{6}(q_1^6 + q_2^6) - \frac{1}{2}(q_1^2 q_2^2) \quad (7)$$

The coupling term, $(q_1^2 q_2^2)/2$ has been chosen to ensure that the potential is symmetric for q_1 and q_2.

When coupled together, the resulting potential is,

$$V(q_1, q_2) = \frac{1}{2}(q_1^2 + q_2^2) + q_1^2 q_2 - \frac{1}{3}q_2^3 -$$

$$s\left(q_1^2 + q_2^2\right) + \frac{1}{8}q_1^2 q_2^2 \left(q_1^2 + q_2^2\right)$$

$$+ \frac{1}{4}(q_1^4 + q_2^4) + \frac{1}{6}(q_1^6 + q_2^6) - \frac{1}{2}(q_1^2 q_2^2) \quad (8)$$

The resulting Hamiltonian is,

$$H = \frac{1}{2}(p_1^2 + p_2^2) + \frac{1}{2}(q_1^2 + q_2^2) +$$

$$q_1^2 q_2 - \frac{1}{3}q_2^3 - s\left(q_1^2 + q_2^2\right) + \frac{1}{8}q_1^2 q_2^2 \left(q_1^2 + q_2^2\right)$$

$$+ \frac{1}{4}(q_1^4 + q_2^4) + \frac{1}{6}(q_1^6 + q_2^6) - \frac{1}{2}(q_1^2 q_2^2) \quad (9)$$

The Hamilton equations of motion are,

$$\frac{d}{dt}q_1 = \frac{\partial H}{\partial p_1} = p_1 \qquad (10)$$

$$\frac{d}{dt}q_2 = \frac{\partial H}{\partial p_2} = p_2 \qquad (11)$$

$$\frac{d}{dt}p_1 = -\frac{\partial H}{\partial q_1} = -(q_1 + 2q_1 q_2 - 2sq_1 + \frac{1}{2}q_1^3 q_2^2$$

$$+ \frac{1}{4}q_1 q_2^4 + q_1^3 + q_1^5 - q_1 q_2^2) \qquad (12)$$

$$\frac{d}{dt}p_2 = -\frac{\partial H}{\partial q_2} = -(q_2 + q_1^2 - q_2^2 - 2sq_2 + \frac{1}{2}q_2^3 q_1^2$$

$$+ \frac{1}{4}q_2 q_1^4 + q_2^3 + q_2^5 - q_2 q_1^2) \qquad (13)$$

The equations (10) to (13) were solved using the Fourth-order Runge-Kutta method with the check on errors such that the percentage error in the energy is less than 0.1% to obtain the bifurcation diagram. The *Lyapunov* spectrum was also taken to corroborate the bifurcation structure. Moreover, a few Poincare surfaces of section for chosen values of the coupling constant were plotted.

RESULTS AND DISCUSSION
Figure 1 is the bifurcation diagram of the system. The motion of the oscillators is regular for values of the coupling constant up to around 0.8. Thereafter, the system bursts into chaotic motion. The degree of chaoticity increases as the coupling parameter increases.

The Lyapunov diagram (Fig. 2) is in agreement with the bifurcation diagram as the system makes a sudden transition to chaotic motion. The Lyapunov diagram of a Hamiltonian system is such that the sum of all the Lyapunov exponents is zero. Unlike in the case of dissipative systems in which regular behaviour is signified by negative Lyapunov exponent, a regular region in Hamiltonian systems are characterised by their smoothness relative to the chaotic regime. Yet again, the degree of chaoticity increases as the coupling parameter increases.

Figure 3 shows the Poincare surface of section for some chosen values of the coupling parameter. It is observed that for values less than the threshold of 0.8, the system exhibits regular behaviour. It suddenly becomes chaotic at 0.8 and for a value as low as 1.0, the whole of the Poincare surface of section is completely covered by a single chaotic sea.
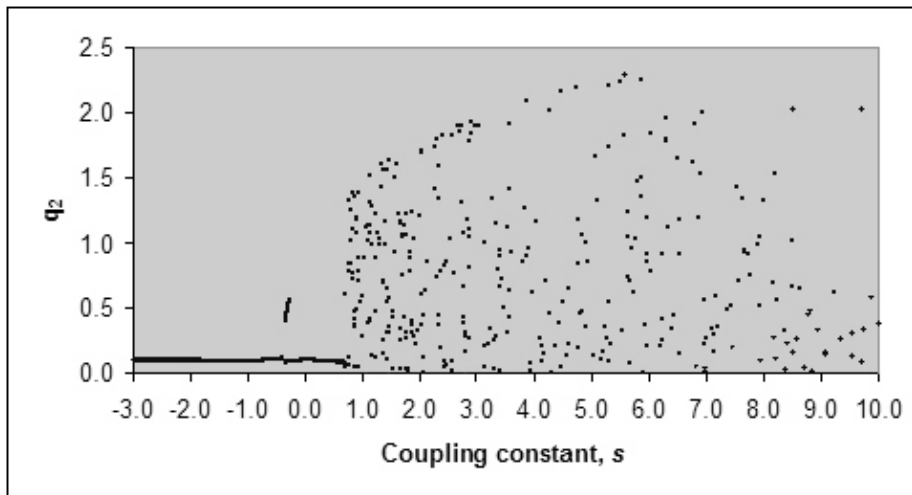
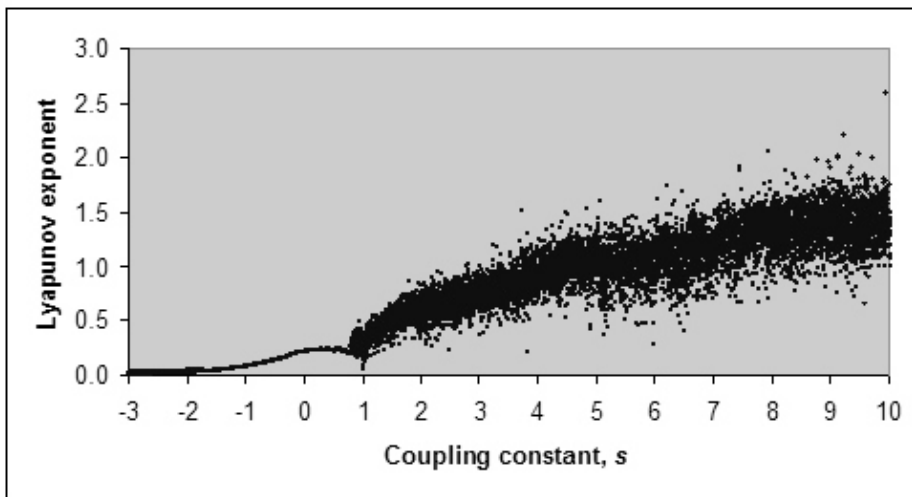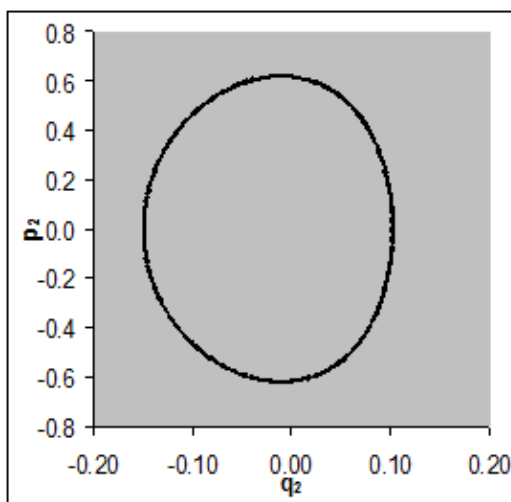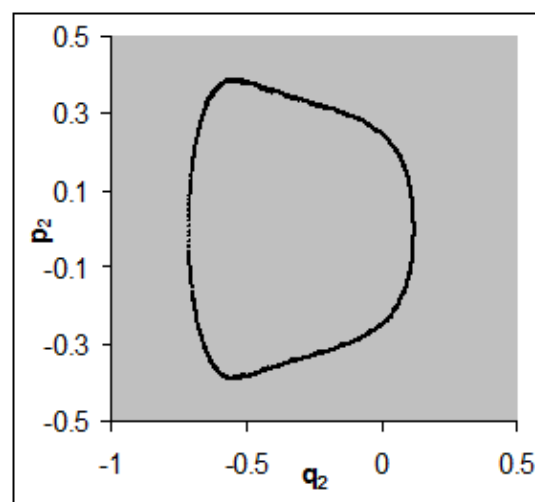Fig. 1: Bifurcation Diagram with $2q$ Versus s, the Coupling Parameter

Fig. 2: Lyapunov Exponent Versus s, the Coupling Parameter

(a) $s = -0.3$

(b) $s = 0.1$

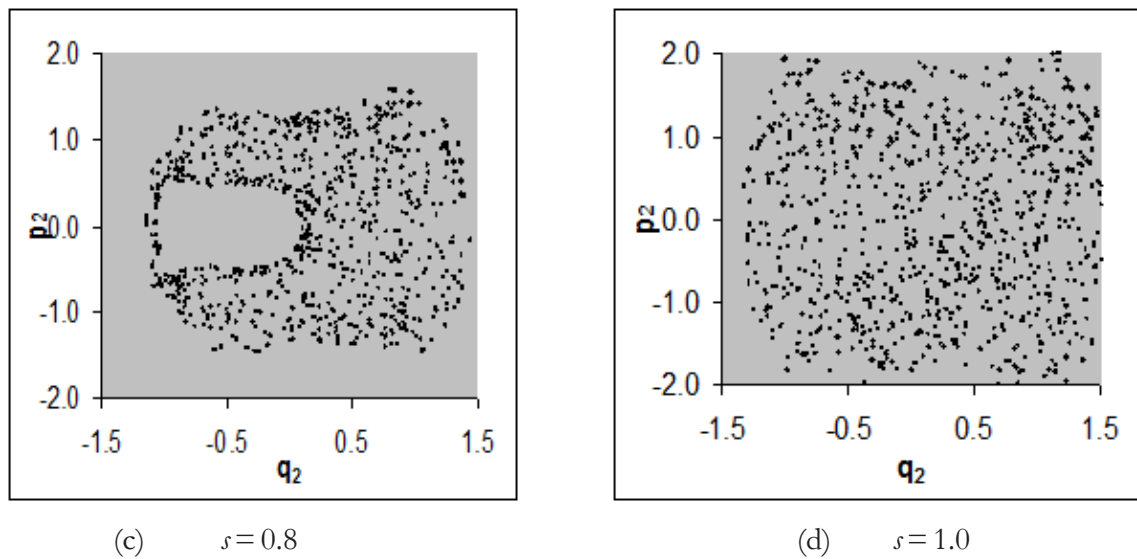(c) $s = 0.8$ (d) $s = 1.0$

Fig. 3: Poincare Surface of Section for Various Values of the Coupling Parameter

CONCLUSION

The results of this investigation has shown that two coupled oscillators subjected to the three potentials, Henon-Heiles, single-well quintic potential modified by the potential of the hydrogen atom in a uniform magnetic field, through a coupling parameter, exhibit regular motion until a threshold value of the coupling constant. Thereafter, chaotic motion sets in abruptly. The degree of chaoticity increases as the coupling constant increases. Thus, in the regular mode, each oscillator more or less keeps its energy, while in the chaotic regime, there is an inter-exchange of energy which cannot be predicted apriori.

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