THE DISTRIBUTIONAL PROPERTIES OF THE FAMILY OF LOGISTIC DISTRIBUTIONS*

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(*Received*: August 2007; *Accepted*: May 2008)

Abstract
The distributional properties of half logistic distribution and Type I generalized logistic distribution were studied, bringing out the L-moments (up to order four) of each of these. Skewness and Kurtosis were obtained.

Key words: Logistic distribution, L-moments

1. Introduction

Family of logistic distribution like standard logistic, half logistic, Type I generalized logistic, etc., were studied in Ph.D thesis of Olapade (2006); he obtained the properties of the family, using moment generating function and characteristic functions but did not consider the L-moment of the family of the distribution studied in this paper. Also, the L-skewness and L-kurtosis of the family of the distribution are obtained.
The standard probability density function of the logistic random variable x is given by:

$$f_x(x) = \frac{e^x}{\left(1+e^x\right)^2}, -\infty < x < \infty \qquad (1.1)$$

The cumulative distribution function (c.d.f) is given as

$$F_x(x) = \frac{1^x}{\left(1+e^{-x}\right)}, -\infty < x < \infty . \qquad (1.2)$$

The quartile (inverse distribution) function is given as:

$$x(F) = \ln\left(\frac{F}{1-F}\right), 0 \le F \le 1$$

and the L-moment of a given distribution as proposed by Hosking (1990) can be expressed as :

$$L_r = \int_0^1 x(F) P_{r-1}(F)\, dF,$$

Where $P_{r-1}(F) = \sum_{k=0}^{r-1} (-1)^{r-k-1} \binom{r-1}{k} \binom{r+k-1}{k} F^k$,

called Legendre polynomial of order $(r-1)$, Hosking (1990). The L-moment of the logistic distribution can be expressed as;

$$L_r = \int_0^1 \ln\left(\frac{F}{1-F}\right) P_{r-1}(F)\, dF$$

$$= \int_0^1 (\ln F) P_{r-1}(F)\, dF - \int_0^1 (\ln(1-F)) P_{r-1}(F)\, dF$$

$$= c\left[\int_0^1 F^k (\ln F)\, dF - \int_0^1 F^k (\ln(1-F))\, dF\right],$$

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* Presented in part at the First Faculty of Science Conference, Obafemi Awolowo University, Ile-Ife, July 3-5, 2007.

Where $c=\sum_{k=0}^{r-1}(-1)^{r-k-1}\binom{r-1}{k}\binom{r+k-1}{k}F^{k}$.

The following identities of integrals will help us to simplify the above equations

(i) $\int_0^1 x^k \ln(x)dx=\frac{-1}{(k+1)^2}$;

(ii) $\int_0^1 x^k \ln(1-x)dx=-\sum_{n=0}^{\infty}\frac{1}{n(n+k+1)}$.

Therefore, if $k=0$, $\lim_{j\to\infty}\sum_{n=0}^{j}\frac{1}{n(n+1)}=1$;

if $k=0$; $\lim_{j\to\infty}\sum_{n=1}^{j}\frac{1}{n(n+2)}=\frac{3}{4}$; if $k=2$, $\lim_{j\to\infty}\sum_{n=1}^{j}\frac{1}{n(n+3)}=\frac{11}{18}$;

the first four L-moment of the logistic distribution can be expressed as:

$$L_1=0$$
$$L_2=1$$
$$L_3=0$$
$$L_4=4.2$$

According to Hosking (1990), the L-skewness and L-kurtosis can be obtained by using

$$\tau_r=\frac{L_r}{L_2}, r>2.$$

Therefore for the logistic distribution,

L-skewness $=\tau_3=0$ and L-kurtosis $=\tau_4=4.2$

If the location (δ) and scale (ω) are included in the quartile function of logistic distribution as:

$$x(F)=\delta+\omega\ln\left(\frac{F}{1-F}\right), 0\le F\le 1$$

The condition given by Bickel and Lehmann (1976) stated suppose there has been defined a partial ordering, with F<G, meaning that G possesses the attribute under consideration more strongly than F, then the first condition required of a measure ϑ of this attribute is that $\vartheta(F)\le\vartheta(G)$, whenever F<G.

A second condition characterizes the behavior of $\vartheta(F)$ (which we shall also denote by $\vartheta(X)$ when X is a random variable with distribution F) under linear transformation. Thus, a measure of location should satisfy

$\vartheta(aX+b)=a\vartheta(X)+b$, for all a, b; and a measure of scale

$\vartheta(aX+b)=|a|\vartheta(X)$, for all $a\neq 0$ and all b.

Following this, we obtain out L-moment of the logistic distribution as:

$$L_1=\delta$$
$$L_2=\omega$$
$$L_3=0$$
$$L_4=4.2\omega$$

The Property of the Half logistic Distribution

One of the probability distributions which is a member of the family of the logistic distribution is half logistic distribution.

Its probability density function can be expressed as: $L_1=0$

$$f_Y(y) = \frac{2e^y}{(1+e^y)^2}, 0 < y < \infty.$$

The cumulative distribution function is

$$F_Y(y) = \frac{e^y - 1}{(1+e^y)}, 0 < y < \infty.$$

The inverse (quartile) distribution function of the half logistic distribution can be expressed as:

$$y(F) = \ln\frac{1+F}{1-F}.$$

The L-moment of the half logistic distribution can be expressed as:

$$L_r = \int_0^1 \ln\left(\frac{1+F}{1-F}\right) P_{r-1}(F)dF$$

$$= \int_0^1 (\ln(1+F))P_{r-1}(F)dF - \int_0^1 (\ln(1-F))P_{r-1}(F)dF$$

$$= c\left[\int_0^1 F^k \ln(1+F)dF - \int_0^1 F^k (\ln(1-F))dF\right].$$

The identity of integrals below will give clues on solving the L-moment, of the half logistic distribution:

(i) $\int_0^1 x^k \ln(1+x)dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n};$

(ii) $\int_0^1 x^k \ln(1-x)dx = -\sum_{n=1}^{\infty} \frac{1}{n(n+k+1)}.$

Therefore, we have the equation L_r above to be

$$L_r = c\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+k+1)} + \sum_{n=1}^{\infty} \frac{1}{n(n+k+1)}\right)$$

$$= \frac{c'}{1+k}\left(\sum_{n=1}^{\infty} \frac{1}{(2n-1)} - \sum_{n=1}^{\infty} \frac{1}{(2n+k)}\right).$$

If $k = 0$, $\lim_{j\to\infty} \sum_{n=1}^{j}\left(\frac{1}{(2n-1)} - \frac{1}{(2n)}\right) = \log_e 2;$

If $k = 1$, $\lim_{j\to\infty} \sum_{n=1}^{j}\left(\frac{1}{(2n-1)} - \frac{1}{(2n+1)}\right) = 1;$

If $k = 2$, $\lim_{j\to\infty} \sum_{n=1}^{j}\left(\frac{1}{(2n-1)} - \frac{1}{(2n+2)}\right) = \log_e 2 + \frac{1}{2}.$

If $k = 3$, $\lim_{j\to\infty} \sum_{n=1}^{j}\left(\frac{1}{(2n-1)} - \frac{1}{(2n+3)}\right) = \frac{4}{3}.$

The L-moment of the half logistic distribution can be expressed as:

$$L_1 = 2\log_e 2,$$

$$L_2 = -2\log_e 2 + 2 = 2(1 - \log_e 2),$$

$$L_3 = 2\log_e 2 - 6 + 4\left(\log_e 2 + \frac{1}{2}\right) = 6\log_e 2 - 4,$$

$$L_4 = 2\log_e 2 + 12 - 2(5)\left(\log_e 2 + \frac{1}{2}\right) + \frac{4}{3} = 15\frac{1}{3} - 22\log_e 2$$

The L-skewnesss and L-kurtosis of the half logistic distribution are

$$\text{L-skewness} = \tau_3 = \frac{6\log_e 2 - 4}{2(1-\log_e 2)}$$

and

$$\text{L-kurtosis} = \tau_4 = \frac{15\frac{1}{3} - 22\log_e 2}{2(1-\log_e 2)}.$$

If the location (δ) and scale (ω) parameter are included in the quartile function, we have:

$$x(F) = \delta + \omega \ln\left(\frac{F}{1-F}\right).$$

The L-moment of the half logistic can now be written as:

$$L_1 = \delta + \omega(2\log_e 2),$$

$$L_2 = \omega(-2\log_e 2 + 2) = 2\omega(1-\log_e 2),$$

$$L_3 = \omega\left(2\log_e 2 - 6 + 4\left(\log_e 2 + \frac{1}{2}\right)\right) = \omega(6\log_e 2 - 4),$$

$$L_4 = \omega\left(-2\log_e 2 + 12 - 2(15)\left(\log_e 2 + \frac{40}{3}\right)\right) = \omega\left(15\frac{1}{3} - 22\log_e 2\right)$$

The Distributional Property of the Type I generalized Logistic Distribution

The probability density function of a random variable X that has type I generalized logistic distribution is:

$$f_X(x) = \frac{be^{-x}}{(1+e^{-x})^{b+1}}, -\infty < x < \infty, b > 0.$$

The corresponding cumulative distribution function is:

$$F_X(x) = \frac{1}{(1+e^{-x})^{b}}, -\infty < x < \infty, b > 0.$$

The quartile function of the type I distribution function is

$$x(F) = \ln\left(\frac{F^{\frac{1}{b}}}{1-F^{\frac{1}{b}}}\right), 0 \le F \le 1.$$

The L-moment of the type I generalized logistic distribution can be expressed as

$$\int_0^1 \ln\left(\frac{F^{\frac{1}{b}}}{1-F^{\frac{1}{b}}}\right) P_{r-1}(F)dF = c\left(\int_0^1 F^k \ln F^{\frac{1}{b}} dF - \int_0^1 F^k\left(1-F^{\frac{1}{b}}\right)dF\right).$$

The identity of integrals below will give clues of solving the above equation.

Let $x^{\frac{1}{b}} = u,\ dx = \frac{b\,du}{x^{\frac{1}{b}-1}}$, if $x = 0,\ u = 0$ and $x = 1,\ u = 1$.

Therefore, we have

$$\int_0^1 u^{kb-1} \ln u\, du = \frac{-b}{(kb+b)^2}.$$

Also, $\int_0^1 x^k \ln\left(1-x^{\frac{1}{b}}\right)dx = b\int_0^1 u^{kb+b-1}\ln(1-u)du = -b\sum_{n=1}^{\infty}\frac{1}{n(bk+b+n)}$.

$$\sum_{n=1}^{\infty}\frac{1}{n(bk+b+n)} = \sum_{n=1}^{\infty}\left(\frac{1}{n(bk+b)} - \frac{1}{b(k+n)(n+bk+b)}\right)$$

$$= \frac{1}{(bk+b)}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{(n+bk+b)}\right).$$

If $k=0$, $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{(n+b)}\right)=\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)$.

If $k=1$, $b=$ only value,

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{(n+2b)}\right)=\sum_{n=1}^{2b}\left(\frac{1}{n}\right).$$

If $k=2$, $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{(n+3b)}\right)=\sum_{n=1}^{3b}\left(\frac{1}{n}\right)$.

If $k=3$, $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{(n+4b)}\right)=\sum_{n=1}^{4b}\left(\frac{1}{n}\right)$.

We can assume for any value of k , $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{(n+bk+b)}\right)=\sum_{n=1}^{(k+1)b}\left(\frac{1}{n}\right)$.

But k and b must be positive integers so the L-moment of the type I generalized logistic Distribution can be expressed as

$$c\left(\frac{1}{b(k+1)^2}+\frac{1}{b(k+1)}\sum_{n=1}^{(k+1)b}\left(\frac{1}{n}\right)\right)$$

The value of b is obtained, using the maximum likelihood method of estimation, and it is assumed to be approximated to the nearest integer.

If $b=1$, the distribution gives the standard logistic distribution.

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