SOME PROPERTIES OF NORMAL MOMENT DISTRIBUTION

A. OLOSUNDE[1,+] and M.O. OJO[2]
1. Department of Statistics, University of Agriculture, Abeokuta, Nigeria.
2. Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria.

(*Received*: July 2006; *Accepted*: May 2008)

Abstract

This paper provides an introductory overview of a portion of distribution theory in which we propose a new family of an extended form of a normal distribution called normal moment distribution; some of its properties are obtained. The cumulative distribution function which is not in close form but the table of the approximate values are presented in the appendix for some selected á (which is the shape parameters such that á = 0 corresponds to the standard normal density). Finally, relationship between normal moment distribution and some well known distributions are established.

1. Introduction

The normal moment distribution which can be regarded as an extended form of family of the well known normal distribution is proposed.

Definition

Let X be a continuous random variable such that

$$E(X^{2\alpha}) = k(\alpha)\int_{-\infty}^{\infty} x^{2\alpha} e^{-\frac{x^2}{2}} = 1, \qquad -\infty < x < \infty, \alpha \geq 0 \tag{1}$$

where k(á) is the normalizing factor, evaluating the integral we have

$$k(\alpha) = \frac{1}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+\frac{1}{2})} \tag{2}$$

Thus we define the probability density function f(x) of the normal moment distribution as

$$f(x) = \frac{1}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+\frac{1}{2})} x^{2\alpha} e^{\frac{x^2}{2}}, \quad -\infty < x < \infty, \alpha \geq 0 \tag{3}$$

where á is the shape parameter and Ã(.) is gamma function. Then we say that X is a normal moment random variable with parameter á; for brevity we shall say that X is NM(á). (3) is a proper density function. In this paper we shall study the standardized type i.e. where ì = 0 and ó = 1.

The importance of the statistical theory and application of the normal distribution is well known. Since the proposed distribution is more flexible than the normal distribution, it is anticipated that this class of the distribution will give a better fit for some sets of data for a selected values of the shape parameter á, indeed when á = 0, the proposed distribution reduces to standard normal distribution.

The moments as well as the tables of the approximate values of the cumulative distribution functions for selected values of á are obtained and finally, relationships between this distribution and some commonly encountered distributions are established.

2. The Moments, The Mode and The Cumulative Distributiou Function

In what follows we derive the rth moment of the distribution

Theorem 2.1

If X is NM(á) then the even rth moment is given as

+ *corresponding author* (email: adeseye777@yahoo.co.uk).

$$E(X^{2\alpha}) = \int_{-\infty}^{\infty} x^{2r} f(x)dx = \frac{2^r\Gamma(\alpha + r + \frac{1}{2})}{\Gamma(\alpha + \frac{1}{2})} \tag{4}$$

$when\, r = 1$

$$E(X^2) = 2\alpha + 1 \tag{5}$$

$when\ r = 2$

$$E(X^4) = 4\alpha^2 + 8\alpha + 3 \tag{6}$$

$thus$

$$E(X) = 0 \tag{7}$$

$$\text{var}(X) = 2\alpha + 1 \tag{8}$$

$$\gamma_1 = 0 \tag{9}$$

$$\gamma_2 = 2\alpha + 3 \tag{10}$$

The expressions γ1 and γ2 are the skewness and kurtosis respectively for the standard normal moment distribution. The value of X that maximize the probability density function (the mode) of the normal moment distribution i.e.

$$\frac{df(x)}{dx} = 0\ is\ \pm\sqrt{2\alpha}$$

The cumulative distribution function (c.d.f) is given as

$$F(x) = \int_{-\infty}^{x} f(w)dw, \qquad -\infty < x < \infty, \alpha \geq 0, \sigma > 0 \tag{11}$$

where

$$f(w) = \frac{1}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha + \frac{1}{2})} w^{2\alpha} e^{-\frac{w^2}{2}}, \qquad -\infty < w < \infty, \alpha \geq 0 \tag{12}$$

solving this we note that the integral is not in closed form, therefore tables of the approximate values are partially reproduced and provided in the appendix for some selected values of α.

3. Relationships with other Known Distributions

We prove four theorems that established relationships between the normal moment distribution and some well known distributions.

Theorem 3.1

Let W be a continuously distributed random variable having the even normal moment distribution. Then W = √y has the chi-square distribution with r = 2α+1 degree of freedom.

Proof:

$$F(w) = \frac{1}{2^{\alpha+\frac{1}{2}}\Gamma(\alpha + \frac{1}{2})} \int_{-\infty}^{\infty} w^{2\alpha} e^{-\frac{w^2}{2}}, \qquad -\infty < w < \infty, \alpha \geq 0$$

for Y = W^2 and α = (r-1)/2 and solving this, we have

$$f(y) = \frac{1}{2^{r/2}\Gamma(r/2)} y^{\frac{r}{2}-1} e^{-\frac{y}{2}}, \qquad 0 < y < \infty \tag{13}$$

This is chi-square p.d.f. with
$\mu = r$
$\sigma^2=2r$
where r is the degree of freedom and the moment generating function is

$$\frac{1}{(1-2t)^{\frac{r}{2}}}, t < \frac{1}{2} \tag{14}$$

Theorem 3.2
Let S and T be two stochastically independent continuously distributed random variable each having the even normal moment distribution symmetric about 0 with parameters $\alpha 1$ and $\alpha 2$ respectively, then the random variable

$$F = \frac{T^2/(\alpha_1 + \frac{1}{2})}{S^2/(\alpha_2 + \frac{1}{2})}, \qquad 0 < f < \infty \tag{15}$$

has an f-distribution with $r1 = 2\alpha1+1$ and $r2 = 2\alpha2+1$ degree of freedom.
Proof:

$$g(t) = k(\alpha_1)t^{2\alpha_1}e^{-t^2/2} \tag{16}$$

and

$$g(s) = k(\alpha_2)s^{2\alpha_1}e^{-s^2/2} \tag{17}$$

The joint distribution function of S and T since they are stochastically independent is

$$\Phi(s,t) = k(\alpha_1)k(\alpha_2)t^{2\alpha_1}s^{2\alpha_2}e^{-(t^2+s^2)/2}, \quad 0 < s < \infty, \quad 0 < t < \infty. \tag{18}$$
$$0 = elsewhere.$$

From equation (15) above we define a new random variable and propose finding the p.d.f. g1(f) of F, the equation

$$f = \frac{t^2/(\alpha_1 + \frac{1}{2})}{s^2/(\alpha_2 + \frac{1}{2})}, \qquad s = p \tag{19}$$

$$since\ t\ is$$

$$\sqrt{f\left(\frac{(\alpha_1 + \frac{1}{2})}{(\alpha_2 + \frac{1}{2})}\right)p^2}, \qquad s = p \tag{20}$$

The joint p.d.f. of g(f,p) of the random variables f and s = p is

$$g(f,p)=\frac{f^{\alpha_1-\frac{1}{2}}}{2^{\alpha_1+\alpha_2}\Gamma(\alpha_1+\frac{1}{2})\Gamma(\alpha_2+\frac{1}{2})}\left(\frac{\alpha_1+\frac{1}{2}}{\alpha_2+\frac{1}{2}}\right)^{(\alpha_1+\frac{1}{2})}p^{2\alpha_1+2\alpha_2+1}e^{-\frac{p^2}{2}\left[1+f\left(\frac{\alpha_1+\frac{1}{2}}{\alpha_2+\frac{1}{2}}\right)\right]} \qquad (21)$$

let

$$z=\frac{p^2}{2}\left[1+f\left(\frac{\alpha_1+\frac{1}{2}}{\alpha_2+\frac{1}{2}}\right)\right] \qquad (22)$$

it implies that

$$p=\sqrt{\frac{2z}{1+f\left(\frac{\alpha_1+\frac{1}{2}}{\alpha_2+\frac{1}{2}}\right)}} \qquad (23)$$

and

$$\frac{dp}{dz}=\frac{1}{\sqrt{2z\left[1+f\left(\frac{\alpha_1+\frac{1}{2}}{\alpha_2+\frac{1}{2}}\right)\right]}} \qquad (24)$$

Then the marginal probability density function g1 of F is

$$\frac{f^{r_1/2-1}(\frac{r_1}{r_2})^{r_1/2}\Gamma\left(\frac{r_1+r_2}{2}\right)}{\Gamma(\frac{r_1}{2})\Gamma\left(\frac{r_2}{2}\right)\left[1+f\left(\frac{r_1}{r_2}\right)\right]^{(r_1+r_2)/2}}, \qquad 0<f<\infty \qquad (25)$$

This is the probability density function of f-distribution with degree of freedom r_1 and r_2.

Theorem 3.3

Let X be a random variable of continuous type having the normal moment distribution symmetric about 0 and α=1.5. Then Y = -2ln(x/2) has Gumbel probability density function proposed by Ojo and Adeyemi (2003) having the shape parameter λ=2.

Proof:

The probability density function of normal moment distribution f(x) is

$$k(\alpha)x^{2\alpha}e^{-\frac{x^2}{2}}, \qquad -\infty<x<\infty,\ \alpha\geq 0 \qquad (26)$$

then

with $\alpha=\frac{1}{2}$ *and* $Y=-2\ln(\frac{x}{2})$

we have

$$f(y)=4e^{-2y}e^{-2e^{-y}}, \qquad -\infty<y<\infty,\ \lambda=2 \qquad (27)$$

The corresponding moment generating function is

$$2^t\Gamma(2-t) \tag{28}$$

This is the moment generating function of Gumbel distribution as obtained by Adeyemi and Ojo (3) when shape parameter $\lambda=2$.

Theorem 3.4

Let X_1 and X_2 be two stochastically independent random variables which are N(0,1) and normal moment distribution respectively, then $Y_1=x_1/x_2$ has generalized Cauchy type distribution.

Proof:

The joint density function of X_1 and X_2 is given as

$$f(x_1,x_2)=\frac{k(\alpha)}{\sqrt{2\pi}}x_2^{2\alpha}e^{-(x_1^2+x_2^2)/2} \tag{29}$$

define new random variables:

$$y_1 = x_1/x_2 \text{ and } y_2 = x_2$$

Then the joint probability density function of y_1 and y_2 is

$$f(y_1,y_2)=\frac{k(\alpha)}{\sqrt{2\pi}}y_2^{2\alpha}e^{-\frac{y_2^2}{2}(1+y_1^2)}, \tag{30}$$

The marginal probability density function of y_1 say $g_1(y_1)$ and solving, we have

$$\frac{2k(\alpha)}{\sqrt{2\pi}}\int_0^\infty y_2^{2\alpha+1}e^{-y_2^2(1+y_1^2)}dy_2 \tag{31}$$

define

$$z=y_2^2(1+y_1^2) \tag{32}$$

that is

$$y_2=\sqrt{\frac{2z}{1+y_1^2}}, \tag{33}$$

and

$$\frac{dy_2}{dz}=\frac{1}{\sqrt{2z(1+y_1^2)}} \tag{34}$$

substituting in (30) above and solving the preceding integral, we have

$$\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\sqrt{\pi}}\frac{1}{(1+y_1^2)^{\alpha+1}} \tag{35}$$

This is Cauchy type distribution, note that when $\alpha=0$ the distribution reduces to Cauchy distribution. $g_1(y_1)$ does not converge for $\alpha=0$, where the $\mu 2r$ moment is

$$\frac{\Gamma(r+\frac{1}{2})\Gamma(\alpha-r+\frac{1}{2})}{\Gamma(\alpha+1)} \tag{36}$$

Acknowledgement

The authors are grateful to the reviewers for their useful comments.

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APPENDIX

Table of the cumulative distribution function of the even normal moment distribution

Below are the values of x for a selected α, note that

$$\int_{-\infty}^{\rho} \frac{1}{\Gamma(\alpha+\frac{1}{2})2^{\alpha+\frac{1}{2}}} \exp[-(x^2)/2] = 0.5$$

x	α 0.01	0.5	1	1.5	2
0.05	0.0186086	0.000625	0.000017	0.000001	0.000001
0.1	0.0376891	0.0024938	0.000133	0.000006	0.000002
0.15	0.05687	0.005594	0.000446	0.000056	0.000002
0.2	0.076	0.009901	0.001051	0.000099	0.000008
0.25	0.095122	0.0153834	0.0020393	0.000239113	0.000025
0.3	0.114038	0.0220013	0.00349508	0.000491	0.000064
0.35	0.13235	0.029706	0.00549653	0.000901	0.000134
0.4	0.5116	0.0384418	0.00811369	0.00151717	0.000257
0.45	0.169266	0.0481465	0.0114076	0.00239629	0.000457
0.5	0.187009	0.0587515	0.0154298	0.00359549	0.00076
0.55	0.204348	0.0701836	0.0202212	0.00517389	0.0012021
0.6	0.221248	0.0823649	0.0258121	0.00719058	0.00182
0.65	0.237676	0.952142	0.0322219	0.0097032	0.002656
0.7	0.253603	0.108648	0.0394586	0.012766	0.003758
0.75	269005	0.12258	0.0475196	0.0164309	0.005172
0.8	0.283859	0.136925	0.0563914	0.0207416	0.006951
0.85	0.298149	0.151598	0.0660503	0.0257372	0.009144
0.9	0.311861	0.166512	0.0764631	0.0314488	0.0118044
0.95	0.324984	0.181584	0.0875878	0.0378991	0.0149798
1	0.337512	0.196735	0.099374	0.045102	0.0187171
1.05	0.34944	0.211885	0,111765	0.0530623	0.0230589
1.1	0.36077	0.226963	0.124697	0.0617753	0.0280428
1.15	0.371503	0.241897	0.138101	0.0712267	0.0337
1.2	0.381645	0.256624	0.151907	0.0813931	0.0400559
1.25	0.391205	0.271083	0.166039	0.0922422	0.0471267
1.3	0.400193	0.285221	0.18042	0.103733	0.0549214
1.35	0.408621	0.298989	0.194975	0.115818	0.0634401
1.4	0.416506	0.312344	0.209625	0.128442	0.0726742
1.45	0.423863	0.32525	0.224296	0.141543	0.0826059
1.5	0.43071	0.337674	0.238916	0.155057	0.0932091
1.55	0.437067	0.349591	0.253415	0.168912	0.104449
1.6	0.442953	0.360981	0.267727	0.183037	0.116283
1.65	0.448391	0.37183	0.281791	0.197358	0.128662
1.7	0.453401	0.382127	0.295551	0.2118	0.14153
1.75	0.458006	0.391867	0.308956	0.226289	0.154825
1.8	0.462227	0.401051	0.321959	0.240753	0.16848

1.85	0.466087	0.40968	0.334523	0.25512	0.182427
1.9	0.469608	0.417763	0.346613	0.269325	0.196594
1.95	0.472812	0.425309	0.358202	0.283303	0.210906
2	0.47572	0.432332	0.369268	0.296997	0.225292
2.05	0.478352	0.438848	0.379794	0.310353	0.239678
2.1	0.480729	0.444875	0.38977	0.323324	0.253993
2.15	0.482871	0.450431	0.39919	0.335866	0.268169
2.2	0.484794	0.455539	0.408052	0.347944	0.282141
2.25	0.486519	0.46022	0.416361	0.359528	0.29585
2.3	0.48806	0.464497	0.424124	0.370593	0.309239
2.35	0.489435	0.468394	0.443135	0.381121	0.322258
2.4	0.490658	0.471933	0.438056	0.391099	0.334862
2.45	0.491743	0.475138	0.444256	0.400519	0.347012
2.5	0.492704	0.478032	0.44997	0.40938	0.358676
2.55	0.493532	0.480637	0.455218	0.417684	0.369828
2.6	0.494298	0.482976	0.460023	0.425436	0.380445
2.65	0.494954	0.48507	0.464408	0.432648	0.390514
2.7	0.495529	0.486939	0.468397	0.439333	0.400025
2.75	0.496031	0.488603	0.472013	0.445508	0.408974
2.8	0.49647	0.490079	0.475282	0.451191	0.417362
2.85	0.49685	0.491386	0.478227	0.456404	0.425194
2.9	0.497181	0.499254	0.480872	0.461169	0.43248
2.95	0.497467	0.493555	0.48324	0.465509	0.439232
3	0.497714	0.494446	0.485355	0.46945	0.445468
3.05	0.497926	0.495225	0.487236	0.4473016	0.451205
3.1	0.498109	0.495906	0.488905	0.476232	0.456465
3.15	0.498265	0.496498	0.490382	0.479123	0.461269
3.2	0.498399	0.497012	0.491684	0.481713	0.465643
3.25	0.498513	0.497457	0.492829	0.484027	0.469611
3.3	0.49861	0.497841	0.493832	0.486086	0.473197
3.35	0.498692	0.498172	0.494709	0.487913	0.476429
3.4	0.498761	0.498456	0.495474	0.489529	0.47933
3.45	0.49882	0.498699	0.496138	0.490954	0.481926
3.5	0.49887	0.498906	0.496713	0.492207	0.484241
3.55	0.498911	0.499083	0.4949721	0.493305	0.486299
3.6	0.498946	0.499233	0.497638	0.494264	0.488122
3.65	0.498975	0.49936	0.498006	0.495099	0.489732
3.7	0.498999	0.499468	0.498321	0.495823	0.491148
3.75	0.499019	0.499558	0.498589	0.496451	0.492391
3.8	0.499036	0.499634	0.498818	0.496992	0.493478
3.85	0.49905	0.499698	0.499013	0.497458	0.494426
3.9	0.499061	0.499751	0.499177	0.497858	0.495249
3.95	0.49907	0.499795	0.499316	0.498199	0.495962
4	0.499078	0.499832	0.499433	0.49849	0.496578
4.05	0.499084	0.499863	0.499531	0.498738	0.497108
4.1	0.499089	0.499888	0.499613	0.498948	0.497563
4.15	0.499094	0.499909	0.499682	0.499125	0.497952
4.2	0.499097	0.499926	0.499739	0.499275	0.498283
4.25	0.4991	0.49994	0.499787	0.4994	0.498565

4.3	0.499102	0.499952	0.499826	0.499505	0.498804
4.35	0.499104	0.499961	0.499858	0.499593	0.499006
4.4	0.499105	0.499969	0.499885	0.499666	0.499177
4.45	0.499106	0.499975	0.499907	0.499727	0.49932
4.5	0.499107	0.49998	0.499925	0.499777	0.499439
4.55	0.499108	0.499984	0.499939	0.499819	0.499539
4.6	0.499109	0.499987	0.499951	0.499853	0.499622
4.65	0.49911	0.49999	0.499961	0.499881	0.499691
4.7	0.49911	0.499992	0.499969	0.499904	0.499748
4.75	0.49911	0.499994	0.499975	0.499923	0.499795
4.8	0.49911	0.499995	0.49998	0.499938	0.499834
4.85	0.49911	0.499996	0.499984	0.49995	0.499866
4.9	0.49911	0.499997	0.499988	0.49996	0.499892
4.95	0.49911	0.499998	0.49999	0.499968	0.499913
5	0.49911	0.499998	0.49999	0.499975	0.49993