THE PEOPLES ALGORITHM FOR SOLVING NON-LINEAR EQUATIONS

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ABSTRACT

An algorithm for solving non-linear equations has been developed. It is very simple to use and does not require much knowledge of numerical analysis. Besides it's simplicity, it does not have the problem on non-convergence associated with the Newton – Raphson method when small derivatives are encountered. The simplicity of the new algorithm was demonstrated using it to hand work three types of non - linear equations. One of the three problems was also hand worked by the Regular Falsi method. The same answer was obtained by the more difficult Regular - Falsi. Further, the new algorithm was programmed to solve the problem of calculating the internal rate of return from an investment and the computation of the mole fractions of vapour and liquid and their composition during a flash separation of Gas from liquid.

INTRODUCTON

There are several efficient algorithms for solving non-linear equations. The two commonly used algorithms are the Newton-Raphson and the Regular Falsi. The Newton-Raphson method which is fairly simple in application require a thorough knowledge of differentiation to handle complex functions frequently encountered in several aspects of learning. Further, the Newton-Raphson method may not converge when small derivatives are involved (Taffler,1969). The Regular Falsi is taught under the course of numerical methods at university level. It is not every body that needs to solve a non-linear equation that have the opportunity to take a calculus or a numerical method course.

Hence there is a need to develop a simple and efficient Algorithm to solve the frequently encountered non-linear equations in everyday learning. The new algorithm developed here requires only the elementary mathematics knowledge of formula evaluation to understand. The computations in this new Algorithm involve the evaluation of non-linear equation by a chosen variable. The results obtained from the evaluation are compared with the chosen variable. By increasing the variable systematically and repeating the comparison, it is possible to obtain the solution of a given non-linear equation, digit by digit.

Three types of non - linear equations were hands solved by use of the algorithm. This shows that the algorithm is capable of solving non-linear that can be solved by the difficult ancient methods. One of the problems hand solved by the new method, was also hand worked by the Regular Falsi. Given that a root of the equation lies between 3 and 4, it took 16 simple evaluations of the equation to arrive at the root. It took the Regular Falsi, 11 iteration to converge to the root. A single iterations of the Regular Falsi is at least equivalent to 2 simple evaluations of an equation. The new algorithm was programmed to calculate the internal rate of return from an investment, given the starting capital and cash flows for a given period of time. It was also programmed to solve the non-linear equation involved in the flash separation of Gas from Liquid in a complex hydrocarbon mixture.

The correct results obtained by use of the new Algorithm shows that it is just as useful as the more difficult methods. The simplicity of the new algorithm makes it to be called the Peoples Algorithm in this study.

DEVELOPMENT OF THE NEW ALGORITHM

Let $f(x)=0$ be a non – linear equation. Let r be a real number, then, r is a root of $f(x)=0$ if $f(r)=0$, (Spiegel, 1956). In numerical analysis, it is difficult to find r such that $f(r)=0$. This is due to round off error inherent in machine computation and error due to approximations in the algorithms used to solve the non-linear equations. The test performed in numerical analysis to check that r is a root of is:

$$f(r) \le \in \qquad (1)$$

Where $\in$ is a very small number, usually in the range 10^{-6} to 10^{-5}

Now, let us consider another real number λ that is not a real root of $f(x)=0$, then

$$f(\lambda)=R \qquad (2)$$

Where R is a Remainder.

In the method that follows, a small number usually 0.1 or 0.01 is added repeatedly to itself and each time the function f(x) = 0 is evaluated.

Let us denote the small number 0.1 by ϕ. Also,

let: $p=\sum_{i=1}^{n}\phi_i$

We found out, that p is the first digit of r (the positive real root of f(x) = 0) if

$|f(p)| \le \phi$, provided, r is such that:

$$0.1 \le r \le 1.0$$

After locating the first digit p, the second digit is located by adding $\phi/10$ to the first digit (p) repeatedly and evaluating the function. Each time an addition is made, a test is performed to see if the current value of the function is less than or equal to $\phi/10$. Let $u=\phi/10$ and

$q=\sum_{i=1}^{n}u_i$ then, q is the second digit if

$$|f(p+q)| \le u$$

The location of the next digits are done similarly.

Let $v=u/10$ and $s=\sum_{i=1}^{n}v_i$, then s is the third digit if $|f(p+q+s)| \le v$

Let $w=v/10$ and $t=\sum_{i=1}^{n}w_i$ then t is the fourth digit if $|f(p+q+s+t)| \le w$

This process is continued to locate other digits.

Example 1

A popular equation used in petrol-physics takes the form $S_w^m + aS_w^n + b = 0$

In this equation, laboratory and field measurement of S_w show that its value varies between 0.1 and 1.0. Given *that m = 2.4, n = 1.2, a = 0.2* and *b = -0.35*, obtain S_w by the use of People's Algorithm.

Solution

The search for the first digit p is done by evaluating $f(S_w) = 0$ with $\phi = 0.1$ and repeating the evaluation by constantly adding 0.1 to itself until $|f(S_w)| \le 0.1$. The calculation can be tabulated as shown in Table 1.

Table 1: Search for the First Digit *p* of $f(S\,w) = Sw^{2.4} + 0.2Sw^{1.2} - 0.35 = 0$

S_w	$f(S_w)$	
0.1	-0.3334	
0.2	-0.3000	
0.3	-0.2472	
0.4	-0.1725	
0.5	-0.0735,	$\lvert f(0.5)\rvert < 0.1$

The second digit (q) search, starts with

$$u=\frac{\phi}{10}=\frac{0.1}{10}=0.01$$

q is the required second digit if

$$|f(\phi+q)| \le u$$

The search for the second digit is shown in Table 2.

Table 2: Search for the Digit q of

$$f(\mathrm{S\,w}) = \mathrm{Sw}^{2.4} + 0.2Sw^{1.2} - 0.35 = 0$$

S_w	$f(S_w)$	
0.51	-0.06216	
0.52	-0.05058	
0.53	-0.03874	
0.54	-0.01424	
0.55	-0.00158,	|f(0.56)|<0.01

The third digit search starts with

$\frac{0.01}{10} = 0.001$ and the search is shown in Table 3.

Table 3: Search for the Third Digit s of

$$f(\mathrm{S\,w}) = \mathrm{Sw}^{2.4} + 0.2Sw^{1.2} - 0.35 = 0$$

S_w	$f(S_w)$	
0.561	-0.000296,	|f(0.561)|<0.001

The fourth digit search starts with $\frac{0.001}{10} = 0.0001$ and the search is shown in Table 4.

Table 4: Search for the Fourth Digit t of

$$f(\mathrm{S\,w}) = \mathrm{Sw}^{2.4} + 0.2Sw^{1.2} - 0.35 = 0$$

S_w	$f(S_w)$	
0.5611	-0.000168	
0.5612	-0.000040,	|f(0.5612)|<0.0001

The solution to $S_w^{2.4} + 0.2S_w^{1.2} - 0.35 = 0$

is $S_w = 0.5612$, correct to four decimal places.

The first digit of the positive real root of certain functions lies between 1 and 9. In such cases, a table of values of the function is used to locate the first digit of the root. Let d (1d" dd" 9) be the first digit of r located with the table of values. Then the next digit to *d* is located by beginning the search with the small number 0.1 or 0.01 depending on the occurrence of the digit zero (0) in the real root (r) of f(x) = 0.

Let $\phi = 0.1$ be the starting number for the search, it is repeatedly added to d, each time, evaluating the function f(x) = 0. A test is performed each time an evaluation is made to see if the current absolute value of the function is less than or equal to ϕ.

Let $p = \sum_{i=1}^{n} \phi_i$ and $e = d + p$. Then, p is the next digit to d if: $|f(e)| \le \phi$

Example 2

Use the People's Algorithm to find the first three digits of the positive root of

$f(x) = x + 0.5\sqrt{x} - 2 = 0$

Solution

A table of values of the function is shown in Table 5.

Table 5: Evaluation of

$$f(x) = x + 0.5\sqrt{x} - 2 = 0$$

x	0	0.5	1.0	1.5	2.0
f(x)	-2.00	-1.15	-0.50	0.11	0.71

The function changes sign between 1.0 and 1.5, thus a root lies between 1 and 1.5. We can start the search for the second digit by adding 0.1 repeatedly to 1.0 and testing to see if the absolute value of the function is less than or equal 0.1. The search for the second digit is shown in Table 6.

Table 6: Search for the Second Digit *p* of

$$f(x) = x + 0.5\sqrt{x} - 2 = 0$$

x	f(x)	
1.1	-0.3756	
1.2	-0.2523	
1.3	-0.1299	
1.4	-0.0084	|f(1.4)|<0.1

Here, p = 0.4.

To locate the next digit, add

$$v = \frac{u}{10} = \frac{0.1}{10} = 0.01$$

repeatedly to 1.4 and evaluate the function. After each addition and evaluation, a test is performed to see if $|f(x)| \leq 0.01$

The location of the next digit (q) is shown in next Table 7.

Table 7: Search for the Third Digit q, of

$$f(x) = x + 0.5\sqrt{x} - 2 = 0$$

x	f(x)	
1.41	-0.0037	|f(1.41)| <0.01

A positive root r of $f(x) = x + 0.5\sqrt{x} - 2 = 0$ is 1.41. This root is correct to 2 decimal places.

COMPARISON OF THE PEOPLES ALGORITHM WITH OTHER ALGORITHMS

The simplicity of the Peoples Algorithm makes it superior to other methods of finding the real roots of certain functions. Consider the problem of solving the equation $2^{x+2} = 3^x$

The Newton-Raphson method cannot be used to solve this problem by someone who has not taken a calculus course. The solution to this problem by use of People's Algorithm is shown in Example 3. For purpose of comparison, the Regular Falsi is also used to solve this problem.

Example 3

Solve $2^{x+2} = 3^x$ by use of:

(a) The People's Algorithm
(b) The Regular Falsi

Solution

(a) The equation can be arranged as

$$f(x) = 2^{x+2} - 3^x = 0$$

A table of values of the function f(x) in the range 0 - 4 is shown in Table 8.

Table 8: Values of $f(x) = 2^{x+2} - 3^x = 0$

x	0	1	2	3	4
f(x)	1	5	7	5	-14

The table shows that a root lies between 3 and 4. We can search for the second digit by repeated addition of 0.1 to 3. The layout is as shown in Table 9.

Table 9: Search for Second Digit of

$$f(x) = 2^{x+2} - 3^x = 0$$

x	f(x)	
3.1	4.1614	
3.2	3.1236	
3.3	1.8561	
3.5	-1.5105	←change of sign

The solution lies between 3.4 and 3.5. To locate the third digit, search with constant addition of 0.01 to 3.4. The search for the third digit of

$$f(x) = 2^{x+2} - 3^x = 0$$

is shown in Table 10.

Table 10: Search for Third Digit of

$$f(x) = 2^{x+2} - 3^x = 0$$

x	f(x)	
3.41	0.1553	
3.42	-0.0170	←change of sign

The solution lies between 3.41 and 3.42. To locate the fourth digit, search with constant addition of 0.001 to 3.41. The search is shown in Table 11

Table 11: Search for Fourth Digit of

$$f(x) = 2^{x+2} - 3^x = 0$$

x	f(x)	
3.411	0.1382	
3.412	0.1211	
3.413	0.1039	
3.414	0.0867	
3.415	0.0695	
3.416	0.0523	
3.417	0.0350	
3.418	0.0177	
3.419	0.00039	$\leftarrow \lvert f(3.149) \rvert < 0.001$

The solution is 3.419, and is correct to 3 decimal places.

(b) The Regular Falsi is

$$x_{i+1} = xL_i + \frac{(xH_i - xL_i)f(xL_i)}{f(xL_i) - f(xH_i)},$$

i = 0, 1, 2, 3, ...

Here, $xL_0 = 3$, xH0 = 4

$f(xL_0) = f(3) = 5$, $f(xH_0) = f(4) = -17$

$$x_1 = 3 + \frac{(4-3)5}{5-(-17)} = 3.2273$$

f(3.2273) = 2.8018, which is positive. Since the function changes sign between 3.2273 and 4, we know that the solution lies between 3.2273 and 4. Then,

xL1 = 3.2273, f(xL1) = 2.8018

xH1 = 4.0000, f(xH1) = -17

Note that xH is now held constant at 4.0000 for all other iterations. Then,

$$x_2 = 3.2273 + \frac{(4.0000 - 3.2273)(2.8018)}{2.8018 - (-17)} = 3.3366$$

The working is shown in Table 12.

Table12: Solution of by the Regular Falsi

i	*xL*	*xH*	*f(xL)*
0	3	4	5.0
1	3.2273	4	2.8018
2	3.3366	4	1.328127
3	3.38467	4	0.577429
4	3.404886	4	0.242144
5	3.413244	4	0.099727
6	3.416667	4	0.040794
7	3.418062	4	0.016641
8	3.418632	4	0.006780
9	3.418863	4	0.002761
10	3.418958	4	0.001124
11	3.418996	4	0.000458

By use of a tolerance of 0.001, the solution is 3.418996 ≈ 3.419 correct to 3 decimal places.

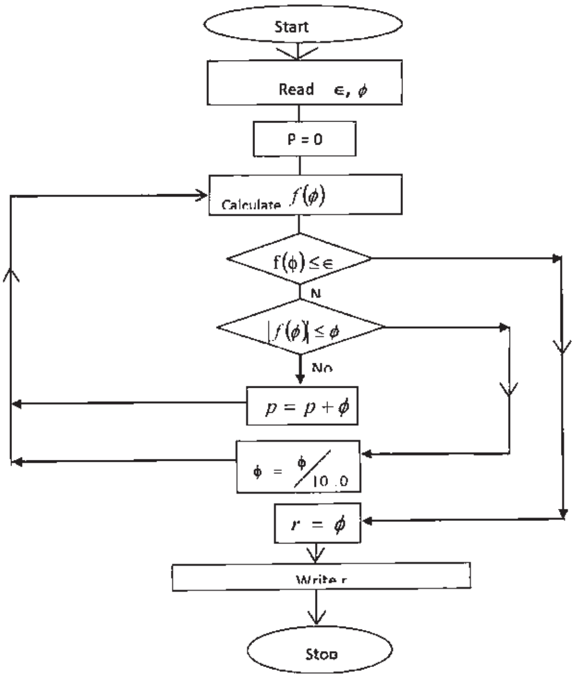

Figure 1: Flow Chart of the Peoples Algorithm, Zero(0) not in the Root

The flowchart for the People's Algorithm as discussed so far is shown in Figure 1.

This flowchart can only be used to find a positive root that does not contain the digit zero (0). The computer implementation of the flowchart as shown in Figure 1. It is used to solve a problem of the calculation of the mole fractions of liquid and vapour during flash separation of gas from petroleum liquid in Example 4.

Example 4

Petroleum is mainly a complex mixture of different types of hydrocarbons. During production it is desirable to separate the components which exist as gas from those that exist as liquid at a known temperature. It is possible to calculate the mole fraction of gas (NV) and the mole fraction of liquid (NL) using 1 mole of mole of the complex mixture as a basis. It is even possible to calculate the mole fraction of each component (hydrocarbon type) in liquid and vapour (gaseous) states by the following equations, Burcik (1961).

$$Z_i = x_i n_l + y_i n_v \tag{3}$$

$$y_i = K_i x_i \tag{4}$$

$$\sum x_i = \sum \frac{Z_i}{n_l + K_i n_v} \tag{5}$$

Where

Z_i = mole fraction of the ith component in the mixture

x_i = mole fraction of the ith component in the liquid

y_i = mole fraction of the ith component in the vapor

K_i = equilibrium constants of the ith component

Table 13 gives the composition and constants for a certain hydrocarbon mixture (Petroleum) existing at 200psia and 100 ^{o} F. Calculate the mole fraction of liquid n_l and mole fraction of vapour n_v and the composition of the liquid and vapour if separation is conducted at 200psia and 100 ^{o} F.

Table 13: Composition and Equilibrium Constant for a Hydrocarbon Mixture

Hydrocarbon Comp	Mole Fraction (Zi)	Equilibrium Constant (Ki)
C2H4	0.1500	14.1000
C2H6	0.0500	2.7800
C3H8	0.2500	0.9700
i-C4H10	0.0500	0.4600
n-C4H10	0.1500	0.3500
n-C5H12	0.2500	0.1160
n-C6H14	0.1000	0.0410

Solution

The program that follows is a main program that uses the Peoples Algorithm to calculate $\mathbf{n}_v$, $\mathbf{n}_l$, $\mathbf{x}_i$ and $\mathbf{y}_i$.

```
C  THIS PROGRAM USES THE PEOPLES ALGORITHM TO SOLVE A
C  PROBLEM OF SAPARATION DURING A FLASH PROCESS
   DIMENSION ZA(7),EK(7),XA(7),YA(7)
     OPEN(UNIT=6,FILE='MOLE.OUT',STATUS='NEW')
      DATA ZA/0.15,0.05,0.25,0.05,0.15,0.25,0.10/
      DATA EK/14.1,2.78,0.97,0.46,0.35,0.116,0.041/
     DATA CN,CNV,TOL,TOLL/1.0,0.1,0.01,0.00001/
 5  CNL = CN - CNV
    SUM = 0.0
    DO 10 I = 1,7
```

```
      XA(I) = ZA(I) / (CNL + EK(I) * CNV)
      SUM = SUM + XA(I)
 10   CONTINUE
      IF (ABS(SUM-1.0).LE.TOLL) GOTO 30
      IF (CNV.GE.1.0) GOTO 100
      IF (ABS(SUM-1.0).GT.TOL) GOTO 25
      TOL = TOL/10.0
 25   CNV = CNV + TOL
      GOTO 5
 30   DO 20 I = 1,7
      YA(I) = EK(I)*XA(I)
 20   CONTINUE
      WRITE(6,40)
      WRITE(6,50)(ZA(I),EK(I),XA(I),YA(I),I=1,7)
      WRITE(6,55)SUM
      WRITE(6,60)CNV
      WRITE(6,70)CNL
 40   FORMAT(2X,'MOLE
FRACTION',4X,'EQUILIBCONSTANT',14X,'XI',12X,'YI')
 50
FORMAT(5X,F8.4,10X,F8.4,12X,F10.6,4X,F10.6)
 55   FORMAT(///27X,'SIGMA X = ',F10.6)
 60   FORMAT(/27X,'MOLE FRACTION
WHICH IS VAPOUR = ',F10.6)
 70   FORMAT(/27X,'MOLE FRACTION
WHICH IS LIQUID = ',F10.6)
 100  STOP
      END
```

The general case of the Peoples Algorithm for locating the positive root of f (x) = 0 that can contain any of the digits (0,1,2,3,4,5,6,7,8,9) can be described as follows. The search for the first digit is started with 0.01. The number 0.01 is added repeatedly to itself, each time evaluating the function. A test is made after each evaluation to check if the function as evaluated with the current value of the summation is less than or equal to ten times 0.01. When this test is satisfied, the current value of the summation is the required first digit of the positive real root (r), of f(x) =0. Let us denote the starting value 0.01, by u1. Also, let:

$$z1 = \sum_{i=1}^{n} u1_i$$

Then, z1 is the first digit of the root (r) of f(x) =0 if

$$|f(z1)| \le 10u1$$

Let $u2 = \frac{u1}{10} = \frac{0.01}{10} = 0.001$ and $z2 = \sum_{i=1}^{n} u2_i$,

then z2 is the second digit of r, if:

Other digits are located similarly.

The FORTRAN programme that uses the peoples algorithm to calculate the internal rate of return of an investment is shown in Example 5. The search for other digits is terminated if:

$$|f(z1 + z2 + z3.......)| \le \in$$

where, is a specified tolerance. In the programme of Example 5, ∈ is replaced by TOLL

EXAMPLE 5

Let $a_1, a_2, a_3, \ldots, a_n$ be the net cash flows from an investment whose starting principle is P. The net present value (NPV) of such an investment is given by

$$\mathbf{NPV = - P +} \frac{a_1}{(1+y)} + \frac{a_2}{(1+y)^2} + \frac{a_3}{(1+y)^3} + \cdots + \frac{a_n}{(1+y)^n}$$

where y is the cost of capital. The rate which makes the NPV = 0 is called the discount cash flow rate (DCF rate) or internal rate of return (IRR). If the IRR is denote by **k** then :

$$\mathbf{NPV = - P +} \frac{a_1}{(1+k)} + \frac{a_2}{(1+k)^2} + \frac{a_3}{(1+k)^3} + \cdots + \frac{a_n}{(1+k)^n} = 0.$$

Given the starting capital P = 10000 and the net cash flows $\mathbf{a_1, a_2, a_3, \ldots, a_5}$, are : 2000, 4000, 4000, 2000, 1000 . Calculate the IRR(k) by use of The Peoples Algorithm.

Solution

Let $x = \frac{\mathbf{1}}{\mathbf{1+k}}$ then :

$$\mathbf{NPV = -p + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n = 0}$$

--------------(6)

Equation (1) is a non-linear equation in variable x.

Let us divide equation (6) by P, then

$$\mathbf{-1 + d_1x + d_2x^2 + d_3x^3 + \ldots + d_nx^n = 0}$$

Or $\mathbf{\sum d_i x^i - 1.0 = 0}$ **where** $\mathbf{d_i = a_i / p}$

The Fortran program that obtains the IRR is as follows: The main program supplies the value of the starting principle 10,000 and the net cash flows 2000,4000,4000, 2000,1000. It then calls on the function that uses the peoples algorithm to calculate the IRR.

The output that is attached to the program gave an IRR of 10.49577 percent

```
      DIMENSION A(40),XX(10)
      DATA N,P/5,10000./
      DATA A(1),A(2),A(3),A(4),A(5)/
     2000.,4000.,4000.,2000.,1000./
      DCF = ROFR(N,P,A)
      WRITE(*,1)DCF
    1 FORMAT(1X,'INTERNAL RATE OF RETURN IS = ',F10.5,' PERCENT')
      STOP
      END
C
C     THIS FUNCTION COMPUTES INTERNAL RATE OF RETURN FOR ALL
C     VALUES IN PERCENT BY USE OF THE PEOPLES ALGORITHM. THE MAIN
C     PROGRAMME SUPPLIES THE STARTING PRINCIPLE (P), THE NUMBER OF
C     DISCOUNTING PERIODS (N) AND CASH FLOWS A(1), I=1,N
C
C
      FUNCTION ROFR(N,P,A)
      DIMENSION A(N)
      WRITE(*,2)(A(I),I=1,N)
    2 FORMAT(1X,'CASH FLOWS ARE = ',/
     7F10.2//)
      TTOL = 0.1
      TOL = 0.01
      TOLL = 0.000001
      X = TOL
      DO 10 I = 1,N
   10 A(I) = A(I)/P
   20 S = 0.
      DO 30 I = 1,N
   30 S = S+A(I)*(X**I)
      IF (ABS((S-1)).LT.TOLL) GOTO 50
      IF (X.GE.1.0) GOTO 60
      IF (ABS((S-1.)).LE.TTOL) GOTO 70
      X = X+TOL
      GOTO 20
   70 TTOL = TTOL/10.
      TOL = TOL/10.
      X = X+TOL
      GOTO 20
   50 ROFR = (1./X-1.)*100
      RETURN
   60 WRITE (*,*) 'THE PEOPLES ALGORITHM COULD NOT CONVERGE'
      RETURN
      END
```

```
C:\Fortran77>SOLVE
CASH FLOWS ARE =
  2000.00   4000.00   4000.00   2000.00
1000.00
INTERNAL RATE OF RETURN IS =
10.49577 PERCENT
Stop - Program terminated.
c:\Fortran77>
```

CONCLUSION

A simple Algorithm for solving non-linear algebraic equations has been developed.

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