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On the Existence of Solutions of Multi-Term Fractional Order Volterra Integro-Differential Equations

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Abstract

In this paper, the problem of multi-term fractional order Volterra integro-differential equations is considered. The multi-term fractional order derivative part of the multi-term fractional order Volterra integro-differential equations is converted to its equivalent integral equation and Schauder's fixed point theorem is applied to establish the existence of solutions for the multi-term fractional order Volterra integro-differential equations under some mild conditions. Furthermore, examples were given to test the applicability of the proposed theorem.

Keywords: Existence of Solution, Fractional Integro-Differential Equation, Schauder's Fixed Point Theorem, Volterra Integro-Differential Equation.
MSC2010: 26A33, 34A08, 45D05, 47H10.

1 Introduction

Fractional calculus is a branch of mathematical analysis which deals with the investigation and applications of integral and derivatives of arbitrary order. Therefore, it is the generalization of classical calculus, involving derivatives and integrals of real or complex order [1]. Many physical events are more desirably explained by fractional derivatives because it considers the evolution of system into account. However, it is sometimes difficult to find exact solutions for some of these equations. Hence, the need for a numerical approach. In the past few decades, a number of numerical approaches for approximation of solutions to this class of equations have found applications in various field of sciences, engineering and social sciences such as chaotic systems [2], Fluid Mechanics [3], Viscoelasticity [4], Optimal Control problems [5], Biology [6], Physics [7], Bioengineering [8], Finance [9], Social Sciences [10], Economics [11], Optics [12], Chemical Reactions [13] and Rheology [14]. Furthermore, numerous scholars have proposed and studied the existence and uniqueness of solutions of these equations such as [15] and [16] studied the existence of solution of multi-term fractional order Fredholm integro-differential equation and Uniqueness and convergence of solution of multi-term fractional order Fredholm Integro-differential equation, respectively. While [17] studied the uniqueness of solution of multi-term fractional order Volterra Integro-differential equation

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with convergence analysis, [18] proved the existence results for fractional integro-differential equations with nonlocal condition via resolvent operators. Also, the existence of solutions of various types of fractional differential equations and fractional integro-differential equations of boundary value problems and initial value problems were explored recently by [18–36], and so many more. A special class of these equations are the Volterra type, it has been used to describe heat transfer issues, nano hydrodynamics, mass diffusion processes, neutron diffusion, biological species coexisting together with diminishing and increasing rate of growth, and electromagnetic theory [38]

In this study, we considered the multi-term fractional order Volterra integro-differential equation of the form

$$D^{\alpha}y(x) = \sum_{i=0}^{n} u_i(x)D^{\gamma_i}y(x) + g\left(x\right) + \int_0^s k\left(s,t\right)G\left(y\left(t\right)\right)dt \tag{1.1}$$

subject to the initial condition

$$\begin{aligned} y^{(k)}\left(0\right) &= d_k,\ k = 0,1,2,...,m-1, \\ m-1 &< \alpha \leq m,\ 0<\gamma_0<\gamma_1<\cdots<\gamma_i<\alpha,\ m,n\in\mathbb{N}, \end{aligned} \tag{1.2}$$

where D is the differential operator defined in Caputo sense, $y : Q = [0,1] \longrightarrow \mathbb{R}$ is a continuous function which needs to be determined, $u_i,\ g : Q \longrightarrow \mathbb{R}$ are given continuous functions, $K : Q \times Q \longrightarrow \mathbb{R}$ is the kernel of integration which is also continuous, $G : \mathbb{R} \longrightarrow \mathbb{R}$ is a Lipschitz function.

2 Preliminaries

Definition 2.1. *(**Compact Map** [39]). Let X and Y be Banach spaces and let $\Omega \subseteq X$. A map $F : \Omega \longrightarrow Y$ is said to be compact if it is continuous and $F(\Omega)$ is relatively compact (i.e, for every $(x_n)_n \subseteq \Omega$, there exists a subsequence $\left(x_{n_j}\right)_j$ of $(x_n)_n$ such that $F\left(x_{n_j}\right)_j$ is convergent).*

Definition 2.2. *(**Uniformly Bounded** [1]). A set M is called uniformly bounded if there exists a constant $K > 0$ such that $\|m\|_{\infty} \leq K$ for every $m \in M$.*

Definition 2.3. *(**Equicontinuous** [1]). A set M is called equicontinuous if, for every $\epsilon > 0$, there exists some $\delta > 0$ such that, for all $m \in M$ and all $x_1, x_2 \in [a,b]$ with $|x_1 - x_2| < \delta$, we have $|m(x_1) - m(x_2)| < \epsilon$.*

Theorem 1. (**Arzelà-Ascoli** [1]). Let M be a subset of $C[a,b]$ equipped with the norm $(\|\cdot\|_{\infty})$. Then M is relatively compact in $C[a,b]$ *if and only if*, M is equicontinuous and uniformly bounded.
Theorem 2. (**Schauder's Fixed Point Theorem** [39]). Let C be a nonempty, closed, bounded and convex subset of a Banach space X and let $T : C \longrightarrow C$ be compact. Then T has a fixed point.

Proposition 2.4. *(**Riemann-Liouville Fractional Integral** [1]). Reimann-Liouville fractional integral of order α of a function y is defined as*

$$I^{\alpha}y\left(x\right) = \frac{1}{\Gamma\left(\alpha\right)}\int_0^x \left(x-s\right)^{\alpha-1}y\left(s\right)ds, \qquad x>0,\ \alpha\in\mathbb{R}^+, \tag{2.1}$$

where $\mathbb{R}^+$ is the set of positive real numbers.

Proposition 2.5. *(**Riemann-Liouville Fractional Derivative** [1]).Reimann-Liouville fractional derivative of order α of a function yis defined as*

$$\begin{aligned} {}^{RL}D^{\alpha}y\left(x\right) &= D^m I^{m-\alpha}y\left(x\right), \qquad m-1<\alpha\leq m,\ m\in\mathbb{N} \\ &= \frac{d^m}{dt^m}\left(\frac{1}{\Gamma\left(m-\alpha\right)}\int_0^x \left(x-s\right)^{m-\alpha-1}y\left(s\right)ds\right). \end{aligned} \tag{2.2}$$

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Proposition 2.6. *(**Caputo Fractional Derivative** [1]). The fractional derivative of $y(x)$ in the Caputo sense is defined by*

$$\begin{aligned} {}^{C}D^{\alpha}y(x) &= I^{m-\alpha}D^{m}y(x) \\ &= \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-s)^{m-\alpha-1}\frac{d^m y(s)}{ds^m}ds, \quad m-1<\alpha\le m. \end{aligned} \tag{2.3}$$

with the foloowing properties

(i) $I^{\alpha}D^{\alpha}y(x) = y(x) - \sum_{k=0}^{m-1}\frac{y^{(k)}(0)}{k!}x^k$, $m-1<\alpha\le m$,

(ii) $I^{\alpha}D^{\gamma}y(x) = I^{\alpha-\gamma}y(x)$, $0<\gamma<\alpha$, and $m-1<\alpha\le m, m\in\mathbb{N}$,

(iii) $I^{\alpha}y(x) = \frac{x^{\alpha}}{\Gamma(\alpha+1)}$, where $y(x)=1,\ x\in[0,1]$

3 Preliminary results

Throughout this work, we denote by

i $\|\cdot\|_{\infty}$ the sup norm on $C(Q,\mathbb{R})$, i.e for $u\in C(Q,\mathbb{R})$, $\|u\|_{\infty} = \sup_{x\in Q}|u(x)|$.

ii $\Lambda := \sum\limits_{i=0}^{n}\frac{\|u_i\|_{\infty}}{\Gamma(\alpha-\gamma_i+1)}$.

Lemma 1. Let $y: Q\longrightarrow\mathbb{R}$ and $g: Q\longrightarrow\mathbb{R}$ be continuous functions. Then, a function y is a solution to the fractional integro-differential equation (1.1) – (1.2) *if and only if,*

$$y(x) = \sum_{k=0}^{m-1}\frac{d_k}{k!}x^k + \sum_{i=0}^{n}u_i(x)I^{\alpha-\gamma_i}y(s) + I^{\alpha}g(s) + I^{\alpha}\left(\int_0^s k(s,t)G(y(t))dt\right). \tag{3.1}$$

Proof. Applying (2.1) on (1.1) and using property (i), (ii) and (iii) we have,

$$\begin{aligned} I^{\alpha}(D^{\alpha}y(x)) &= I^{\alpha}\left(\sum_{i=0}^{n}u_i(x)D^{\gamma_i}y(x)\right) + I^{\alpha}(g(x)) + I^{\alpha}\left(\int_0^s k(s,t)G(y(t))dt\right) \\ &= \sum_{i=0}^{n}u_i(x)I^{\alpha}(D^{\gamma_i}y(x)) + I^{\alpha}(g(x)) + I^{\alpha}\left(\int_0^s k(s,t)G(y(t))dt\right) \\ &= \sum_{k=0}^{m-1}\frac{y^{(k)}(0)}{k!}x^k + \sum_{i=0}^{n}u_i(x)I^{\alpha-\gamma_i}y(s) + I^{\alpha}g(s) + \\ &\quad I^{\alpha}\left(\int_0^s k(s,t)G(y(t))dt\right) \\ &= \sum_{k=0}^{m-1}\frac{d_k}{k!}x^k + \sum_{i=0}^{n}u_i(x)I^{\alpha-\gamma_i}y(s) + I^{\alpha}g(s) + I^{\alpha}\left(\int_0^s k(s,t)G(y(t))dt\right). \end{aligned}$$

Thus, y solves (1.1) – (1.2) *if and only if*, y solves (3.1) □

Lemma 2. Let X and Y be normed linear spaces and let $f: A\longrightarrow Y$ be a Lipschitz map, $A\subseteq X$. Then f sends bounded sets to bounded sets.

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Proof. For any set $E \subseteq A$, if there exists $M > 0 : \|x\|_X \leq M$ for all $x \in E$, then we show that there exists $\tilde{M} > 0 : \|f(x)\|_Y \leq \tilde{M}$ for all $x \in E$. Since f is Lipschitz, that is, there exists $L > 0$ such that

$$\|f(x) - f(y)\|_Y \leq L \|x - y\|_X \text{ for all } x, y \in A.$$

Let x_0 be a fixed element of A.Then, if there exists $M > 0 : \|x\|_X \leq M$ for all $x \in E$, then,

$$\begin{aligned} \|f(x)\|_Y &= \|f(x) - f(x_0) + f(x_0)\|_X \\ &\leq \|f(x) - f(x_0)\| + \|f(x_0)\| \\ &\leq L\|x - x_0\| + \|f(x_0)\| \\ &\leq L(\|x\| + \|x_0\|) + \|f(x_0)\| \\ &\leq L(M + \|x_0\|) + \|f(x_0)\|. \end{aligned}$$

Thus, $\|f(x)\|_Y \leq \tilde{M}$ for all $x \in E$, where $\tilde{M} = L(M + \|x_0\|) + \|f(x_0)\|$. Hence, $f(E)$ is bounded. Thus, f sends bouned sets to bounded sets □

Lemma 3. For any $0 \leq a < b$,

$$b^\alpha - a^\alpha \leq (b-a)^\alpha, \alpha \in (0,1).$$

Proof. Define $f : [0, b) \longrightarrow \mathbb{R}$ by

$$f(t) = (b-t)^\alpha - b^\alpha + t^\alpha, \ t \in [0, b), \tag{3.2}$$

Then from equation (3.2)

$$\begin{aligned} f(t) &= (b-t)^\alpha - b^\alpha + t^\alpha \\ &\geq 0 \end{aligned}$$

Case I. Suppose $\alpha = 0$, then for any $0 \leq a < b$ we have from equation (3.2)

$$\begin{aligned} f(a) &= 1 \\ &\geq 0 \end{aligned}$$

Case II (a). Suppose $\alpha > 0$, and $f(t)$ is increasing, then we have from equation (3.2)

$$f'(t) = -\alpha\left((b-t)^{\alpha-1} - t^{\alpha-1}\right) \geq 0.$$

Thus,

$$(b-t)^{\alpha-1} - t^{\alpha-1} \leq 0.$$

Hence,

$$t \geq \frac{b}{2}.$$

(b) Suppose $\alpha > 0$, and $f(t)$ is decreasing, then we have from equation (3.2)

$$f'(t) = -\alpha\left((b-t)^{\alpha-1} - t^{\alpha-1}\right) \leq 0.$$

Thus,

$$(b-t)^{\alpha-1} - t^{\alpha-1} \geq 0.$$

Hence,

$$t \leq \frac{b}{2}.$$

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So, f is decreasing on $\left[0, \frac{b}{2}\right]$ and increasing on $[\frac{b}{2}, b)$. Hence, $\inf_{t\in[0,b)} f(t) = f(0) = 0$ and $\lim_{t\longrightarrow b^-} f(t) = f(b) = 0$, i.e, $\inf_{t\in[0,b)} f(t) = f(0)$ or $f(b)$. Thus, $\inf_{t\in[0,b)} f(t) = 0$. Therefore, for any $0 \le a < b$, $f(a) \ge \inf_{t\in[0,b)} f(t) = 0$, i.e,

$$(b-a)^{\alpha} - b^{\alpha} + a^{\alpha} \ge 0,$$

that is,

$$b^{\alpha} - a^{\alpha} \le (b-a)^{\alpha}.$$

Case III (a). Suppose $\alpha < 0$, and $f(t)$ is increasing, then we have from equation (3.2)

$$f'(t) = -\alpha\left((b-t)^{\alpha-1} - t^{\alpha-1}\right) \ge 0.$$

Thus,

$$(b-t)^{\alpha-1} - t^{\alpha-1} \ge 0.$$

Hence,

$$t \le \frac{b}{2}.$$

(b) Suppose $\alpha < 0$, and $f(t)$ is decreasing, then we have from equation (3.2)

$$f'(t) = -\alpha\left((b-t)^{\alpha-1} - t^{\alpha-1}\right) \le 0$$

Thus,

$$(b-t)^{\alpha-1} - t^{\alpha-1} \le 0.$$

Hence,

$$t \ge \frac{b}{2}.$$

So, f is increasing on $\left[0, \frac{b}{2}\right]$ and decreasing on $[\frac{b}{2}, b)$. Hence, $\inf_{t\in[0,b)} f(t) = f(0) = 0$ and $\lim_{t\longrightarrow b^-} f(t) = f(b) = 0$, i.e, $\inf_{t\in[0,b)} f(t) = f(0)$ or $f(b)$. Thus, $\inf_{t\in[0,b)} f(t) = 0$. Therefore, for any $0 \le a < b$, $f(a) \ge \inf_{t\in[0,b)} f(t) = 0$, i.e,

$$(b-a)^{\alpha} - b^{\alpha} + a^{\alpha} \ge 0,$$

that is,

$$b^{\alpha} - a^{\alpha} \le (b-a)^{\alpha}.$$

□

4 Main results

Throughout this work, we make the following hypotheses:

$\mathbf{h}_1$ there exists a constant $M > 0$ such that for any $y_1, y_2 \in C(Q, \mathbb{R})$ we have

$$|G(y_1(x)) - G(y_2(x))| \le M \|y_1 - y_2\|_{\infty} \quad x \in Q$$

$\mathbf{h}_2$ there exists a constant $\hat{K}$ such that

$$\hat{K} = \sup_{x,t\in[0,1]} \int_0^x |k(x,t)|\, dt < \infty$$

Theorem 3. (*Existence of Solution*). Assume that (h_1) and (h_2) holds, if

$$\left(\Lambda + \frac{\hat{K}M}{\Gamma(\alpha+1)}\right) < 1, \tag{4.1}$$

then there exists a solution $y \in C(Q,\mathbb{R})$ to problem (1.1) – (1.2).

Proof. Let T be an operator such that $T : C(Q,\mathbb{R}) \longrightarrow C(Q,\mathbb{R})$ defined by

$$\begin{aligned}(Ty)(x) &= \sum_{k=0}^{m-1} \frac{d_k}{k!}x^k + \sum_{i=0}^{n} \frac{1}{\Gamma(\alpha-\gamma_i)} \int_0^x (x-s)^{\alpha-\gamma_i-1} u_i(s)y(s)ds \\ &+\frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} g(s)\, ds \\ &+\frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(\int_0^s k(s,t)\, G(y(t))\, dt\right) ds. \end{aligned} \tag{4.2}$$

Our goal is to apply Schauder's fixed point theorem. To do that, we will show that T satisfies the following;

i T is Lipschitz,

ii T sends bounded sets to bounded sets,

iii T sends bounded sets to equicontinuous sets.

First, we note that T is well definded. Indeed, since $x \longmapsto \sum_{k=0}^{m-1} \frac{d_k}{k!}x^k$, $x \longmapsto \sum_{i=0}^{n} u_i(x)\left(I^{\alpha-\gamma_i}y\right)(x)$, $x \longmapsto (I^{\alpha}g)(x)$, $x \longmapsto \int_0^x k(x,t)\, G(y(t))\, dt$ are continuous, the right hand of (4.2) is well defined and $x \longmapsto (Ty)(x)$ is continuous. Thus, for $y \in C(Q,\mathbb{R})$, $Ty \in C(Q\mathbb{R})$.

Let $a, b \in C(Q,\mathbb{R})$, for any $x \in [0,1]$, we have by setting $E = |(Ta)(x) - (Tb)(x)|$

$$\begin{aligned} E &= \left|\sum_{i=0}^{n} \frac{1}{\Gamma(\alpha-\gamma_i)} \int_0^x (x-s)^{\alpha-\gamma_i-1} u_i(s)\left(a(s)-b(s)\right) ds \right. \\ &\left. +\frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(\int_0^s k(s,t)\left(G(a(t)) - G(b(t))\right) dt\right) ds\right| \\ &\leq \left|\sum_{i=0}^{n} \frac{1}{\Gamma(\alpha-\gamma_i)} \int_0^x (x-s)^{\alpha-\gamma_i-1} u_i(s)\left(a(s)-b(s)\right) ds\right| \\ &+\left|\frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(\int_0^s k(s,t)\left(G(a(t)) - G(b(t))\right) dt\right) ds\right| \\ &\leq \sum_{i=0}^{n} \frac{1}{\Gamma(\alpha-\gamma_i)} \int_0^x (x-s)^{\alpha-\gamma_i-1} |u_i(s)|\, |(a(s)-b(s))|\, ds \\ &+\frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} \left(\int_0^s |k(s,t)|\, |G(a(t)) - G(b(t))|\, dt\right) ds \\ &\leq \sum_{i=0}^{n} \|u_i\|_\infty \|a-b\|_\infty \frac{1}{\Gamma(\alpha-\gamma_i)} \int_0^x (x-s)^{\alpha-\gamma_i-1} ds \\ &+\hat{K}M\, \|a-b\|_\infty \frac{1}{\Gamma(\alpha)} \int_0^x (x-s)^{\alpha-1} ds. \end{aligned}$$

By property (iii) we have

$$|(Ta)(x) - (Tb)(x)| \leq \left(\Lambda + \frac{\hat{K}M}{\Gamma(\alpha+1)}\right) \|a-b\|_\infty, \text{ for all } x \in [0,1]$$

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Thus,

$$\|(Ta)-(Tb)\|_{\infty} \leq \Phi\,\|a-b\|_{\infty}\,,$$

where

$$\Phi = \Lambda + \frac{\hat{K}M}{\Gamma(\alpha+1)} \in \mathbb{R}.$$

It follows that T is Lipschitz continuous.

Next, by *Lemma 2*, we can conclude that operator T (i.e T being Lipschitz) maps bounded sets to bounded sets in $C(Q,\mathbb{R})$.

Next, we show that T maps bounded sets to equicontinuous sets of $C(Q,\mathbb{R})$. Let $y \in B_{\epsilon} = \{y \in C(Q,\mathbb{R}) : \|y\|_{\infty} \leq \epsilon\}$, let $\sup_{x\in Q}|G(y(x))| \leq M\|y\|_{\infty} + G(0)$ and let $x_1, x_2 \in [0,1]$ with $x_1 < x_2$. Setting $E = (Ty)(x_2) - (Ty)(x_1)$, then we have from (4.2)

$$\begin{aligned}
|E| \;=\; & \Big| \textstyle\sum_{k=0}^{m-1} \frac{d_k}{k!}\left(x_2^k - x_1^k\right) \\
& + \textstyle\sum_{i=0}^{n} \frac{1}{\Gamma(\alpha-\gamma_i)} \Big(\int_0^{x_2} (x_2-s)^{\alpha-\gamma_i-1} u_i(s)y(s)ds \\
& - \textstyle\int_0^{x_1} (x_1-s)^{\alpha-\gamma_i-1} u_i(s)y(s)ds \Big) \\
& + \tfrac{1}{\Gamma(\alpha)} \Big(\textstyle\int_0^{x_2} (x_2-s)^{\alpha-1} g(s)\,ds - \int_0^{x_1} (x_1-s)^{\alpha-1} g(s)\,ds \Big) \\
& + \tfrac{1}{\Gamma(\alpha)} \Big(\textstyle\int_0^{x_2} (x_2-s)^{\alpha-1} \left(\int_0^s k(s,t)\,G(y(t))\,dt\right) \\
& - \textstyle\int_0^{x_1} (x_1-s)^{\alpha-1} \left(\int_0^s k(s,t)\,G(y(t))\,dt\right) \Big) ds \Big| \\
\leq\; & \left| \sum_{k=0}^{m-1} \frac{d_k}{k!}\left(x_2^k - x_1^k\right) \right| \\
& + \Bigg| \sum_{i=0}^{n} \frac{1}{\Gamma(\alpha-\gamma_i)} \Bigg(\int_0^{x_2} (x_2-s)^{\alpha-\gamma_i-1} u_i(s)y(s)ds \\
& - \int_0^{x_1} (x_1-s)^{\alpha-\gamma_i-1} u_i(s)y(s)ds \Bigg) \Bigg| \\
& + \left| \frac{1}{\Gamma(\alpha)} \left(\int_0^{x_2} (x_2-s)^{\alpha-1} g(s)\,ds - \int_0^{x_1} (x_1-s)^{\alpha-1} g(s)\,ds \right| \right) \\
& + \Bigg| \frac{1}{\Gamma(\alpha)} \Bigg(\int_0^{x_2} (x_2-s)^{\alpha-1} \left(\int_0^s k(s,t)\,G(y(t))\,dt \right) \\
& - \int_0^{x_1} (x_1-s)^{\alpha-1} \left(\int_0^s k(s,t)\,G(y(t))\,dt \right) \Bigg) ds \Bigg|.
\end{aligned}$$

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$$
\begin{aligned}
|E| \leq\ & \left|\sum_{k=0}^{m-1} \frac{d_k}{k!}\left(x_2^k - x_1^k\right)\right| \\
& + \left|\sum_{i=0}^{n} \frac{1}{\Gamma(\alpha-\gamma_i)}\left(\int_0^{x_2}(x_2-s)^{\alpha-\gamma_i-1} u_i(s)y(s)\right.\right. \\
& - \int_0^{x_1}(x_2-s)^{\alpha-\gamma_i-1} u_i(s)y(s) + \int_0^{x_1}(x_2-s)^{\alpha-\gamma_i-1} u_i(s)y(s) \\
& \left.\left. - \int_0^{x_1}(x_1-s)^{\alpha-\gamma_i-1} u_i(s)y(s)\right)ds\right| \\
& + \left|\frac{1}{\Gamma(\alpha)}\left(\int_0^{x_2}(x_2-s)^{\alpha-1} g(s) - \int_0^{x_1}(x_2-s)^{\alpha-1} g(s)\right.\right. \\
& \left.\left. + \int_0^{x_1}(x_2-s)^{\alpha-1} g(s) - \int_0^{x_1}(x_1-s)^{\alpha-1} g(s)\right)ds\right| \\
& + \left|\frac{1}{\Gamma(\alpha)}\left(\int_0^{x_2}(x_2-s)^{\alpha-1}\left(\int_0^{s} k(s,t)\, G(y(t))\, dt\right)\right.\right. \\
& - \int_0^{x_1}(x_2-s)^{\alpha-1}\left(\int_0^{s} k(s,t)\, G(y(t))\, dt\right) \\
& + \int_0^{x_1}(x_2-s)^{\alpha-1}\left(\int_0^{s} k(s,t)\, G(y(t))\, dt\right) \\
& \left.\left. - \int_0^{x_1}(x_1-s)^{\alpha-1}\left(\int_0^{s} k(s,t)\, G(y(t))\, dt\right)\right)ds\right|
\end{aligned}
$$

$$
\begin{aligned}
|E| \leq\ & \sum_{k=0}^{m-1} \frac{|d_k|}{k!}\left(x_2^k - x_1^k\right) \\
& + \sum_{i=0}^{n} \frac{1}{\Gamma(\alpha-\gamma_i)}\left(\int_{x_1}^{x_2}(x_2-s)^{\alpha-\gamma_i-1} |u_i(s)|\,|y(s)|\, ds\right. \\
& + \int_0^{x_1}(x_2-s)^{\alpha-\gamma_i-1} |u_i(s)|\,|y(s)|\, ds \\
& \left. - \int_0^{x_1}(x_1-s)^{\alpha-\gamma_i-1} |u_i(s)|\,|y(s)|\, ds\right) \\
& + \frac{1}{\Gamma(\alpha)}\left(\int_{x_1}^{x_2}(x_2-s)^{\alpha-1} |g(s)|\, ds\right. \\
& \left. + \int_0^{x_1}(x_2-s)^{\alpha-1} |g(s)|\, ds - \int_0^{x_1}(x_1-s)^{\alpha-1} |g(s)|\, ds\right) \\
& + \frac{1}{\Gamma(\alpha)}\left(\int_{x_1}^{x_2}(x_2-s)^{\alpha-1}\left(\int_0^{s} |k(s,t)|\,|G(y(t))|\, dt\right) ds\right. \\
& + \int_0^{x_1}(x_2-s)^{\alpha-1}\left(\int_0^{s} |k(s,t)|\,|G(y(t))|\, dt\right) ds \\
& \left. - \int_0^{x_1}(x_1-s)^{\alpha-1}\left(\int_0^{s} |k(s,t)|\,|G(y(t))|\, dt\right) ds\right).
\end{aligned}
$$

The above inequality can be written as

$$
|(Ty)(x_2) - (Ty)(x_1)| \leq A + B + C + D, \tag{4.3}
$$

where

$$
A = \sum_{k=0}^{m-1} \frac{|d_k|}{k!}\left(x_2^k - x_1^k\right)
$$

$$
\begin{aligned}
B &= \sum_{i=0}^{n} \frac{1}{\Gamma(\alpha-\gamma_i)}\left(\int_{x_1}^{x_2} (x_2-s)^{\alpha-\gamma_i-1}\left|u_i(s)\right|\left|y(s)\right| ds\right. \\
&\quad \left. + \int_0^{x_1} (x_2-s)^{\alpha-\gamma_i-1}\left|u_i(s)\right|\left|y(s)\right| ds - \int_0^{x_1} (x_1-s)^{\alpha-\gamma_i-1}\left|u_i(s)\right|\left|y(s)\right| ds\right)
\end{aligned}
$$

$$
\begin{aligned}
C &= \frac{1}{\Gamma(\alpha)}\left(\int_{x_1}^{x_2} (x_2-s)^{\alpha-1}\left|g(s)\right| ds + \int_0^{x_1} (x_2-s)^{\alpha-1}\left|g(s)\right| ds\right. \\
&\quad \left. - \int_0^{x_1} (x_1-s)^{\alpha-1}\left|g(s)\right| ds\right)
\end{aligned}
$$

$$
\begin{aligned}
D &= \frac{1}{\Gamma(\alpha)}\left(\int_{x_1}^{x_2} (x_2-s)^{\alpha-1}\left(\int_0^{s}\left|k(s,t)\right|\left|G(y(t))\right| dt\right) ds\right. \\
&\quad + \int_0^{x_1} (x_2-s)^{\alpha-1}\left(\int_0^{s}\left|k(s,t)\right|\left|G(y(t))\right| dt\right) ds \\
&\quad \left. - \int_0^{x_1} (x_1-s)^{\alpha-1}\left(\int_0^{s}\left|k(s,t)\right|\left|G(y(t))\right| dt\right) ds\right).
\end{aligned}
$$

On Simplifying A by cosidering $\left(x_2^k - x_1^k\right) \leq (x_2 - x_1)$ for $0 \leq k \leq m-1$, since $x_2, x_1 \in [0,1]$ and $x_1 < x_2$ and taking $d_{k^*} = \max_{0\leq k\leq m-1}\{d_k\}$, we have

$$
\begin{aligned}
A &= \sum_{k=0}^{m-1} \frac{|d_k|}{k!}\left(x_2^k - x_1^k\right) \\
&\leq \left(\frac{|d_1|}{1!} + \frac{|d_2|}{2!} + \ldots + \frac{|d_{m-1}|}{(m-1)!}\right)(x_2 - x_1) \\
&= \frac{|d_{k^*}|}{k^*!}(m-1)(x_2 - x_1).
\end{aligned}
$$

On simplifying B and by propeerty (iii) and *Lemma 3* we have

$$
\begin{aligned}
B &= \sum_{i=0}^{n} \frac{1}{\Gamma(\alpha-\gamma_i)}\left(\int_{x_1}^{x_2} (x_2-s)^{\alpha-\gamma_i-1}\left|u_i(s)\right|\left|y(s)\right| ds\right. \\
&\quad \left. + \int_0^{x_1} (x_2-s)^{\alpha-\gamma_i-1}\left|u_i(s)\right|\left|y(s)\right| ds - \int_0^{x_1} (x_1-s)^{\alpha-\gamma_i-1}\left|u_i(s)\right|\left|y(s)\right| ds\right) \\
&\leq \sum_{i=0}^{n} \frac{\|u_i\|_\infty \|y\|_\infty}{\Gamma(\alpha-\gamma_i+1)}\left((x_2-x_1)^{\alpha-\gamma_i} + \left(x_2^{\alpha-\gamma_i} - (x_2-x_1)^{\alpha-\gamma_i}\right) - x_1^{\alpha-\gamma_i}\right) \\
&= \Lambda\epsilon\left(x_2^{\alpha-\gamma_i} - x_1^{\alpha-\gamma_i}\right) \\
&\leq \Lambda\epsilon\,(x_2-x_1)^{\alpha-\gamma_i}.
\end{aligned}
$$

Simplifying C, we use property (iii) and *Lemma 3* as follows

$$
\begin{aligned}
C &= \frac{1}{\Gamma(\alpha)}\left(\int_{x_1}^{x_2} (x_2-s)^{\alpha-1}\left|g(s)\right| ds\right. \\
&\quad \left. + \int_0^{x_1} (x_2-s)^{\alpha-1}\left|g(s)\right| ds - \int_0^{x_1} (x_1-s)^{\alpha-1}\left|g(s)\right| ds\right) \\
&\leq \frac{\|g\|_\infty}{\Gamma(\alpha+1)}\left((x_2-x_1)^{\alpha} + (x_2^{\alpha} - (x_2-x_1)^{\alpha}) - x_1^{\alpha}\right) \\
&\leq \frac{\|g\|_\infty}{\Gamma(\alpha+1)}(x_2-x_1)^{\alpha}.
\end{aligned}
$$

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To simplify D, we use property (iii) and *Lemma 3* as follows

$$\begin{aligned} D &= \frac{1}{\Gamma(\alpha)}\left(\int_{x_1}^{x_2}(x_2-s)^{\alpha-1}\left(\int_0^s |k(s,t)|\,|G(y(t))|\,dt\right)ds \right.\\ &\quad +\int_0^{x_1}(x_2-s)^{\alpha-1}\left(\int_0^s |k(s,t)|\,|G(y(t))|\,dt\right)ds\\ &\quad \left.-\int_0^{x_1}(x_1-s)^{\alpha-1}\left(\int_0^s |k(s,t)|\,|G(y(t))|\,dt\right)ds\right)\\ &\leq \frac{\hat{K}\left(M\|y\|_\infty+|G(0)|\right)}{\Gamma(\alpha+1)}\left((x_2-x_1)^\alpha+\left(x_2^\alpha-(x_2-x_1)^\alpha-x_1^\alpha\right)\right)\\ &\leq \frac{K^*\left(M\epsilon+|G(0)|\right)}{\Gamma(\alpha+1)}(x_2-x_1)^\alpha. \end{aligned}$$

Thus, equation (4.3) is

$$\begin{aligned} |(Ty)(x_2)-(Ty)(x_1)| &\leq \frac{|d_{k^*}|}{k^*!}(m-1)(x_2-x_1)+\sum_{i=0}^n\frac{\|u_i\|_\infty\|y\|_\infty}{\Gamma(\alpha-\gamma_i+1)}(x_2-x_1)^{\alpha-\gamma_i}\\ &\quad +\frac{\|g\|_\infty}{\Gamma(\alpha+1)}(x_2-x_1)^\alpha+\frac{K^*\left(M\epsilon+|G(0)|\right)}{\Gamma(\alpha+1)}(x_2-x_1)^\alpha. \end{aligned}$$

We see that the right hand side of the above equation is independent of y and tends to zero as $x_2-x_1\longrightarrow 0$. This leads to $|(Ty)(x_2)-(Ty)(x_1)|\longrightarrow 0$ as $x_2\longrightarrow x_1$ uniformly in y. Therefore, the set $\{Ty : y\in B_\epsilon\}$ is equicontinuous and finally, we need to show that there exists a closed convex bounded subset C of X such that $Tc\subseteq C$.

Consider $B_\epsilon=\{y\in C(Q,\mathbb{R}) : \|y\|_\infty\leq\epsilon\}$, we will show that for some $\epsilon>0$, $TB_\epsilon\subseteq B_\epsilon$. For contradiction, suppose that $TB_\epsilon\nsubseteq B_\epsilon$ for all $\epsilon>0$.

Let n be a positive integer, then there exists $y_n\in B_n$ such that $\|Ty_n\|_\infty>n$.
Consider

$$\begin{aligned} |(Ty_n)(x)| &\leq \left|\sum_{k=0}^{m-1}\frac{d_k}{k!}x^k\right|\left|\sum_{i=0}^n\frac{1}{\Gamma(\alpha-\gamma_i)}\int_0^x(x-s)^{\alpha-\gamma_i-1}u_i(s)y_n(s)ds\right|\\ &\quad +\left|\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}g(s)\,ds\right|\\ &\quad +\left|\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left(\int_0^s k(s,t)\,G(y_n(t))\,dt\right)ds\right|\\ &\leq \sum_{k=0}^{m-1}\sup_{x\in[0,1]}\frac{|d_k|}{k!}x^k\\ &\quad +\sum_{i=0}^n\frac{1}{\Gamma(\alpha-\gamma_i)}\int_0^x(x-s)^{\alpha-\gamma_i-1}\sup_{s\in Q}|u_i(s)|\sup_{s\in Q}|y_n(s)|\,ds\\ &\quad +\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\sup_{s\in Q}|g(s)|\,ds\\ &\quad +\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left(\int_0^s\sup_{s\in Q}|k(s,t)|\sup_{t\in Q}|G(y_n(t))|\,dt\right)ds\\ &\leq \sum_{k=0}^{m-1}\frac{|d_k|}{k!}+\sum_{i=0}^n\frac{\|u_i\|_\infty\|y_n\|_\infty}{\Gamma(\alpha-\gamma_i+1)}\\ &\quad +\frac{\|g\|_\infty}{\Gamma(\alpha+1)}+\frac{\hat{K}\left(Mn+|G(0)|\right)}{\Gamma(\alpha+1)},\ \text{ for all } x\in[0,1]\,text. \end{aligned}$$

Thus,

$$\|(Ty_n)\|_{\infty} \leq \sum_{k=0}^{m-1} \frac{|d_k|}{k!} + \Lambda n + \frac{\|g\|_{\infty}}{\Gamma(\alpha+1)} + \frac{\hat{K}(Mn + |G(0)|)}{\Gamma(\alpha+1)}.$$

Observe that if

$$\begin{aligned} n &< \|(Ty_n)\|_{\infty} \\ &< \sum_{k=0}^{m-1} \frac{|d_k|}{k!} + \Lambda n + \frac{\|g\|_{\infty}}{\Gamma(\alpha+1)} + \frac{\hat{K}(Mn + |G(0)|)}{\Gamma(\alpha+1)}. \end{aligned}$$

Dividing through by n we have,

$$1 < \sum_{k=0}^{m-1} \frac{|d_k|}{nk!} + \Lambda + \frac{\|g\|_{\infty}}{n\Gamma(\alpha+1)} + \frac{\hat{K}M}{\Gamma(\alpha+1)}.$$

Letting $n \longrightarrow \infty$ we obtain

$$1 < \Lambda + \frac{\hat{K}M}{\Gamma(\alpha+1)}.$$

Which is a contradiction to equation (4.1). Hence, for some n_0, $TB_{n_0} \subseteq B_{n_0}$.

Let $C := B_{n_0}$ and let $\hat{T} := T|c$, i.e, $T : C \longrightarrow C$ with $\hat{T}y = Ty$. By Arzelà-Ascoli thus, for any $(y_n)_n \subseteq C$, since C is bounded $(y_n)_n$ is bounded and by $\left(\hat{T}y_n\right)_n \equiv (Ty_n)_n$ is equicontinuous. Then there exists a subsequence $\left(\hat{T}y_{n_j}\right)_j$ of $\left(\hat{T}y_n\right)_n$ which is convergent. Hence, $\hat{T}$ is compact. By of Schauder's fixed point theorem, there exists a fixed point y of T in $C(Q, \mathbb{R})$. Then y is a solution of equation (1.1) − (1.2). □

5 Numerical Illustration

Example 1 [42]. Consider the Volterra fractional integro-differential equation

$$D^{\frac{1}{2}} y(x) = \frac{8}{3\sqrt{\pi}} x^{\frac{3}{2}} - \frac{2}{\sqrt{\pi}} x^{\frac{1}{2}} - \frac{3}{12} x^5 + \frac{4}{12} x^4 + \int_0^x xty(t)\,dt. \tag{5.1}$$

Subject to $y(0) = 0$ with exact solution $y(x) = x^2 - x$.

Solution: Equation (5.1) can be written as

$$\begin{aligned} |(Ty_2)(x) - (Ty_1)(x)| &= \left| \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^x (x-s)^{-\frac{1}{2}} \left(\int_0^s st\,(y_2(t) - y_1(t))\,dt \right) ds \right| \\ &\leq \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_0^x (x-s)^{-\frac{1}{2}} \left(\int_0^s st\,|y_2(t) - y_1(t)|\,dt \right) ds \\ &\leq \frac{\|y_2 - y_1\|_{\infty}}{\Gamma\left(\frac{1}{2}\right)} \int_0^x (x-s)^{-\frac{1}{2}} s^3 ds \end{aligned}$$

By *Lemma 2* we have,

$$\|Ty_2 - Ty_1\|_{\infty} \leq \left(\frac{\Gamma(4)\,x^{3.5}}{\Gamma(4.5)} \right) \|y_2 - y_1\|_{\infty}$$

Thus,

$$\|Ty_2 - Ty_1\|_{\infty} \leq (0.515\,83)\,\|y_2 - y_1\|_{\infty}. \tag{5.2}$$

Since $0.515\,83 < 1$, we say that the problem satisfies the condition of *Theorem 3*.

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Example 2. Consider the multi-term fractional order integro-differential equation

$$D^2y(x) - x^2D^{\frac{3}{2}}y(x) - \sqrt{x}D^{\frac{1}{2}}y(x) - \sqrt[3]{x}y(x) = 6\sqrt{\pi}x - 8\sqrt{x^7} - \frac{16}{5}x^3 - \sqrt[3]{x^{10}}\sqrt{\pi} + \int_0^x xt^2 y(t)\,dt. \tag{5.3}$$

Subject to $y(0) = y'(0) = 0$ with exact solution $y(x) = \sqrt{\pi}x^3$.

Solution: Equation (5.3) can be written as

$$\begin{aligned}
|(Ty_2)(x) - (Ty_1)(x)| \leq\ & \frac{1}{\Gamma(2)\Gamma(1.5)}\int_0^x (x-s)\, s^{\frac{5}{2}}\, |y_2(s) - y_1(s)|\, ds \\
& + \frac{1}{\Gamma(2)\Gamma(2.5)}\int_0^x (x-s)\, s^2\, |y_2(s) - y_1(s)|\, ds \\
& + \frac{1}{\Gamma(2)}\int_0^x (x-s)\, s^{\frac{1}{3}}\, |y_2(s) - y_1(s)|\, ds \\
& + \frac{1}{3\Gamma(2)}\int_0^x (x-s)\, |y_2(s) - y_1(s)|\, ds \\
\leq\ & \frac{\|y_2 - y_1\|_\infty}{\Gamma(2)\Gamma(1.5)}\int_0^x (x-s)\, s^{\frac{5}{2}} ds + \frac{\|y_2 - y_1\|_\infty}{\Gamma(2)\Gamma(2.5)}\int_0^x (x-s)\, s^2 ds \\
& + \frac{\|y_2 - y_1\|_\infty}{\Gamma(2)}\int_0^x (x-s)\, s^{\frac{1}{3}} ds \\
& + \frac{\|y_2 - y_1\|_\infty}{3\Gamma(2)}\int_0^x (x-s)\, ds.
\end{aligned}$$

By *Lemma 2* we have

$$|(Ty_2)(x) - (Ty_1)(x)| \leq \left(\frac{\Gamma(3.5)\, x^{4.5}}{\Gamma(5.5)\Gamma(1.5)} + \frac{\Gamma(3)\, x^4}{\Gamma(2.5)\Gamma(5)} + \frac{\Gamma\left(\frac{4}{3}\right) x^{\frac{7}{3}}}{\Gamma\left(\frac{10}{3}\right)} + \frac{\Gamma(1)\, x^2}{3\Gamma(3)}\right)\|y_2 - y_1\|_\infty.$$

Thus,

$$\|Ty_2 - Ty_1\|_\infty \leq (0.62243)\, \|y_2 - y_1\|_\infty.$$

Since $0.62243 < 1$, we say that the problem satisfies the condition of the *Theorem 3*.

6 Conclussion

The problem of multi-term fractional order Volterra integro-differential equation is successfully converted to its equivalent integral form using Riemann-Liouville fractional integral, a lemma is extablished to demonstrate the solution of the multi-term fractional order Volterra integro-differential equation. We used Schauder's fixed point theorem in establishing the existence of the solution. Moreover, examples were considered to test the applicability of the proposed existence theorem for the solution of multi-term fractional order Volterra integro-differential equations.

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