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Variable exponent Picone identity and $p(x)$ sub-Laplacian first eigenvalue for general vector fields

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Abstract

Picone identity is a powerful tool for proving qualitative properties of differential operators with ubiquitous applications in the analysis of partial differential equations, so generalizing it for different types of differential equations has become a desired venture. $p(x)$-Laplacian is a non-homogeneous quasilinear partial differential operator arising from various mathematical model with non-standard growth. However, in this paper, we establish a new generalized nonlinear variable exponent Picone identities for $p(x)$-sub-Laplacian. As applications we prove uniqueness, simplicity, monotonicity and isolatedness of the first nontrivial Dirichlet eigenvalue of $p(x)$-sub-Laplacian with respect to the general vector fields. Further applications yield Hardy type inequalities and Caccioppoli estimates with variable exponents.

Keywords: Picone Identity, $p(x)$-Sub-Laplacian, Principal Eigenvalue, Hardy Inequality, Caccioppoli Estimate.
MSC2010: 35B05 35H20, 35P30, 47J10.

1 Preliminaries

1.1 Introduction

This paper is concerned with variable exponent Picone identity in the context of sub-Riemannian geometry. We derive a nonlinear Picone identity which allows us to study some qualitative properties of the principal eigenvalue of $p(x)$-sub-Laplacian with respect to the general vector fields on smooth manifolds. As by-products, we also derive Hardy type inequalities and Caccioppoli estimates with variable exponents. These results are appearing for the first time, even in the Euclidean setting. In recent years, several authors have devoted their researches towards the study of variable exponent elliptic equations and systems with $p(x)$-growth condition in Euclidean setting with many interesting results [1–6]. Models involving $p(x)$-growth condition arise from physical processes such as nonlinear elasticity theory, electrorheological fluids, image processing, etc [7–9]. It has been observed that $p(x)$-Laplacian is similar in many respect to the classical p-Laplacian (p-constant) but it lacks certain vital properties such as homogeneity. This therefore makes the nonlinearity so much complicated and many of known approaches to p-Laplacian can no longer hold for $p(x)$-Laplacian. It is interesting to consider $p(x)$-Laplacian in the sub-elliptic setting and investigate which of the known results for p-constant hold for variable exponents.

1.2 $p(x)$-Sub-Laplace operator and eigenvalues

Let M be an n-dimensional smooth manifold equipped with a volume form dx and $\{X_k\}_{k=1}^N$, $n \geq N$, be a family of vector fields defined on M. Consider the operator

$$\mathscr{L}_X := \sum_{k=1}^{N} X_k^* X_k,$$

which is a second-order differential operator usually called canonical sub-Laplacian. This operator is related to the operator for the sum of squares of vector fields and it is well known to be locally hypoelliptic if the commutators of the vector fields $\{X_k\}_{k=1}^N$ generate the tangent space of M as the Lie algebra, due to Hörmander's pioneering work [10]. We denote the horizontal gradients for general vector fields by

$$\nabla_X = (X_1, \cdots, X_N) \quad \text{and} \quad \nabla_X^* = (X_1^*, \cdots, X_N^*),$$

where X_k and its formal adjoint X_k^* are respectively given by

$$X_k = \sum_{j=1}^{n} a_{kj}(x) \frac{\partial}{\partial x_j} \quad \text{and} \quad X_k^* = -\sum_{j=1}^{n} \frac{\partial}{\partial x_j}(a_{kj}(x)), \quad k = 1, \cdots, N.$$

There are numbers of examples of sub-manifolds where vector fields can be defined. For examples, we list among others, the Carnot groups, Heisenberg groups, Engel groups, and Grushin plane (which does not even posses a group structure). Interested readers can see the book [11] for more examples and detail discussions on the sub-Laplacian and its various extensions in each case. In the case $M = \mathbb{R}^n$, then dx is the Lebesgue measure, $\nabla_X = \nabla$ and $\mathscr{L}_X = \Delta$ are the usual Euclidean gradient and Laplacian, respectively.

Let $p : \bar{\Omega} \to \mathbb{R}$ be a continuous function and $p(x) > 1$ for $x \in \bar{\Omega} \subset M$. We define the $p(x)$-sub-Laplacian for general vector fields on M by the formula

$$\mathscr{L}_p u := \nabla_X^*(|\nabla_X u|^{p(x)-2} \nabla_X u),$$

where u is a smooth function. If $p(x) = p$ (p=constant), the operator $\mathscr{L}_p u$ becomes the p-sub-Laplacian, $\nabla_X^*(|\nabla_X u|^{p-2}\nabla_X u)$ and $|x|$ stands for the Euclidean length of $x = (x_1, \cdots, x_n)$.

As mentioned earlier, various partial differential equations with variable exponent growth condition have appeared in literature (see [1–6] for instance), but there is scarcity of such mathematical models in the subelliptic setting. In this paper however we shall consider the indefinite weighted Dirichlet eigenvalue problem for $p(x)$-sub-Laplacian on $\Omega \subset M$, $p(x) > 1$,

$$\begin{cases} -\nabla_X^*(|\nabla_X u|^{p(x)-2}\nabla_X u) = \lambda g(x)|u|^{p(x)-2}u, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

and discuss some properties of the eigenvalue $\lambda \in \mathbb{R}^+$ and the corresponding eigenfunction $u(x)$ in certain Sobolev spaces with variable exponents [12–14]. It is well known in the classical setting ($p(x) = p$-constant and $M = \mathbb{R}^n$) that Problem (1.1) possesses a closed set of nondecreasing sequence of nonnegative eigenvalues $\{\lambda_k\}$ which grows to $+\infty$ as $k \to +\infty$, and that the first nonzero eigevalue is simple and isolated. Due to some complication in the nonlinearities in $p(x)$-Laplacian and inhomogeneity of the corresponding variable exponent norm, some of the results in the classical case may not hold or rather under restrictive assumptions. In [4], the authors studied (1.1) (with $g(x) = 1$, $M = \mathbb{R}^n$) and showed the existence of infinitely many eigenvalues and established some sufficient condition for the infimum of the spectrum (called the principal eigenvalue),

$$\lambda_{1,p} = \inf_{u \neq 0} \frac{\int_\Omega |\nabla u|^{p(x)} dx}{\int_\Omega |u|^{p(x)} dx}, \quad p(x) > 1,$$

to be zero and positive, respectively. The properties that $\lambda_{1,p} > 0$ is very useful in analysis and applications. Motivated by [4], we are able to assume the existence of $\lambda_{1,p} > 0$ for (1.1) and proved its uniqueness, monotonicity, simplicity and isolatedness. The variable exponent Picone identity (discussed in Section 2) plays a crucial role in our proofs.

1.3 Picone identities

Picone identity is a very useful tool in the study of qualitative properties of solutions of differential equations, and for this, several linear and nonlinear Picone type identities have been derived to handle differential equations of various type. Picone identity was originally developed by Mauro Picone in 1910 to prove Sturm Comparison principle and oscillation theory for a system of differential equations. This identity was later extended to partial differential equation involving Laplacian by Allegretto [15] and p-Laplacian by Allegretto and Huang [16] to establish among others, existence and nonexistence of positive solutions, Sturmian comparison principle, Liouville type theorems, Hardy inequalities and some profound results involving p-Laplace equations and systems. Precisely, Allegretto [15] proved that, for nonnegative differentiable functions u and v with $v \neq 0$, the following formula

$$|\nabla u|^2 + \frac{u^2}{v^2}|\nabla v|^2 - 2\frac{u}{v}\nabla u \nabla v = |\nabla u|^2 - \nabla\left(\frac{u^2}{v}\right)\nabla v \geq 0 \tag{1.2}$$

holds. Allegretto and Huang [16] extended (1.2) to handle p-Laplace equations and eigenvalue problems involving p-Laplacian. Their identity reads as follows, for $u \geq 0$, $v > 0$, then

$$|\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}|\nabla v|^{p-2}\nabla v \nabla u = R_p(u, v), \tag{1.3}$$

where

$$R_p(u, v) := |\nabla u|^p - \nabla\left(\frac{u^p}{v^{p-1}}\right)|\nabla v|^{p-2}\nabla v \geq 0.$$

Several extensions and generalization of Picone identity have been established in order to handle more general elliptic operators. Tyagi [17] and Bal [18] established nonlinear versions of (1.2) and its p-Laplace analogue (1.3), respectively, with several applications, (see also [19–21]). For other interesting extension of Picone type identities one can find [22, 23] (for Finsler p-Laplacian with application to Caccioppoli inequality), [24–26] (for general vector fields and p-sub-Laplacian with applications to Grushin plane, Heisenberg group, Stratified Lie groups), [27] (for p-sub-Laplacian on Heisenberg group and applications to Hardy inequalities), [28,29] (for nonlinear Picone identities for anisotropic p-sub-Laplacian and p-biLaplacian with applications to horizontal Hardy inequalities and weighted eigenvalue problem on Stratified Lie groups).

Allegretto [30] established variable exponent Picone type identity for differentiable functions

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$v > 0$, $0 \leq u \in C_0^\infty(\Omega)$, $\Omega \subset \mathbb{R}^n$ with $n \geq 1$ and continuous $p(x) > 1$ as follows:

$$\begin{aligned} &\frac{|\nabla u|^{p(x)}}{p(x)} - \nabla\left[\frac{u^{p(x)}}{p(x)v^{p(x)-1}}\right]|\nabla v|^{p(x)-2}\nabla v \\ &\qquad = \frac{|\nabla u|^{p(x)}}{p(x)} - \left(\frac{u}{v}\right)^{p(x)-1}|\nabla v|^{p(x)-2}\nabla v\nabla u + \frac{p(x)-1}{p(x)}\left(\frac{u}{v}|\nabla v|\right)^{p(x)} \\ &\qquad\quad + \frac{1}{p(x)}\frac{u^{p(x)}}{v^{p(x)-1}}|\nabla v|^{p(x)-2}\left[\frac{1}{p(x)} - \ln\left(\frac{u}{v}\right)\right]\nabla v\nabla p(x) \geq 0 \end{aligned} \tag{1.4}$$

on the assumption that $\nabla v\nabla p(x) = 0$. He used the inequality to prove Barta theorem and some other results. Later, Yoshida [31] (see also [32, 33]) established similar Picone identities for quasi-linear and half-linear elliptic equations involving $p(x)$-Laplacian and pseudo $p(x)$-Laplacian, and consequently developed Sturmian comparison theory. Most recently, Feng and Han [34], motivated by Allegretto [30] proved a modified form of (1.4) and showed that

$$|\nabla u|^{p(x)} - \nabla\left(\frac{u^{p(x)}}{v^{p(x)-1}}\right)|\nabla v|^{p(x)-2}\nabla v \geq 0 \tag{1.5}$$

if $\nabla v\nabla p(x) = 0$ a.e in Ω, with equality if and only if $\nabla(u/v) = 0$ in Ω. They proved monotonicity of principal eigenvalue $\lambda_{1,p}$ and a variable exponent Barta inequality for $p(x)$-Laplacian in the form

$$\lambda_{1,p} \geq \inf_{x\in\Omega}\left[\frac{\Delta_p v}{v^{p(x)-1}}\right], \quad \Omega \subset \mathbb{R}^n,$$

where $\Delta_p := -\nabla(|\nabla v|^{p(x)-2}\nabla v)$, on the assumption that $\nabla v\nabla p(x) = 0$.

1.4 Variable exponent functional spaces

In order to discuss generalized solutions, we need some concepts from the theory of variable Lebesgue and Sobolev spaces. Detailed description of these spaces can be found in [12–14].

Let $\Omega \subset M$ be an open domain and $E(\Omega)$ denotes the set of all equivalence classes of measurable real-valued functions defined on Ω being equal almost everywhere.

Definition 1.1. *The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined as*

$$L^{p(\cdot)}(\Omega) = \left\{u \in E(\Omega) : \int_\Omega |u(x)|^{p(\cdot)}dx < \infty\right\}$$

equipped with the (Luxemburg) norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{t > 0 : \int_\Omega \left|\frac{u(x)}{t}\right|^{p(x)} dx \leq 1\right\}.$$

Consider the functional (also called the ρ-modular) on $L^{p(\cdot)}(\Omega)$, which is the mapping $\rho_{p(\cdot)}(u) : L^{p(\cdot)}(\Omega) \to \mathbb{R}$, and defined by

$$L^{p(\cdot)}(\Omega) := \int_\Omega |u(x)|^{p(x)}dx.$$

The following proposition contains vital results in the study of variable Lebesgue space. We suppose a continuous function $p : \bar{\Omega} \to \mathbb{R}^+$, $p(x) > 1$ is such that

$$1 < p^- := ess \inf_{x\in\bar{\Omega}} p(x) \leq p(x) \leq p^+ := ess \sup_{x\in\bar{\Omega}} p(x) < \infty.$$

Proposition 1.2. *[12–14]*
Denote $\|u\|_{p(x)} := \|u\|_{L^{p(x)}(\Omega)}$. For any $u, u_m \in L^{p(x)}(\Omega)$, where $m = 1, 2, \cdots$, the following statements are true:

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1. $\|u\|_{p(x)} < 1 (= 1 \text{ or } > 1)$ *if and only if* $\rho_{p(x)}(u) < 1 (= 1 \text{ or } > 1)$;

2. *If* $\|u\|_{p(x)} \leq 1$ *then* $\|u\|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p^-}$;

3. *If* $\|u\|_{p(x)} > 1$ *then* $\|u\|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p^+}$;

4. $\|u_m - u\|_{p(x)} \to 0$ *if and only if* $\rho_{p(x)}(u_m - u) \to 0$;

5. $\min\{\|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+}\} \leq \rho_{p(x)}(u) \leq \max\{\|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+}\}$.

The following generalized Hölder's inequality can be used to define equivalent norms.

Proposition 1.3. *(Hölder's inequality [12, 13])*
Let $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ *a.e. on* Ω*, then for all* $u \in L^{p(x)}(\Omega)$ *and* $v \in L^{p'(x)}(\Omega)$ *we have* $uv \in L^1(\Omega)$ *and*

$$\int_\Omega |u(x)v(x)|dx \leq \left(1 + \frac{1}{p^-} - \frac{1}{p^+}\right) \|u\|_{p(x)}\|v\|_{p'(x)}.$$

Definition 1.4. *The variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ *is defined as*

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : |\nabla_X u| \in L^{p(\cdot)}(\Omega)\}$$

equipped with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla_X u\|_{L^{p(\cdot)}(\Omega)}.$$

The space $W_0^{1,p(\cdot)}(\Omega)$ *is defined as the closure of* $C_0^\infty(\Omega)$ *in* $W^{1,p(\cdot)}(\Omega)$ *with respect to the norm*

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla_X u\|_{L^{p(\cdot)}(\Omega)}.$$

It can be easily proved that $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ are all separable and reflexive Banach spaces in their respectful norms if $1 < \inf p(x) < \sup p(x) < \infty$ on Ω.

1.5 Plan of the paper

In this paper, we derive new generalized variable exponent Picone type identities for general vector fields in the sub-Riemannian settings. The derived generalized identity contains some known Picone type identities in various settings as will be discussed in Section 2. Consequently, we give several applications to qualitative properties of the principal eigenvalue of $p(x)$-sub-Laplacian. Here, we are concerned with uniqueness, simplicity, monotonicity and isolatedenss of the Dirichlet principal eigenvalue. These are discussed in Section 3. Lastly, motivated by [23,24], we derive as a consequent of Picone identity, sub-elliptic variable exponents Caccioppoli estimates in the form

$$\int_\Omega \phi^{p(x)}|\nabla_X v|^{p(x)}dx \leq (p^+)^{p^+} \int_\Omega v^{p(x)}|\nabla_X \phi|^{p(x)}dx$$

for every nonnegative test function $\phi \in C_0^\infty(\Omega)$, where v is a sub-solution in $\Omega \subset M$ and $p^+ := ess \sup p(x)$. On the other hand, for v positive $p(x)$-superharmonic functions, we obtain a new version of logarithmic Caccioppolli inequality

$$\int_\Omega |\phi\nabla_X \log v|^p dx \leq \left(\frac{p^+}{p^- - 1}\right)^{p^+} \int_\Omega |\nabla_X \phi|^p dx.$$

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2 Nonlinear variable exponent Picone identity

Here we give the statement and the proof of the nonlinear Picone identity with variable exponent, which is the main result of this section. First, we state some hypotheses as adopted in this section (and ofcourse throughout the paper) and Young's inequality in the forms that will be applied here and later.

Let M be an n-dimensional smooth manifold and Ω any domain in M, $p(x) > 1$ is a continuous function on $\bar{\Omega}$, $p'(x) = p(x)/(p(x) - 1)$ is Hölder conjugate to $p(x)$.

Lemma 2.1. *(Classical Young's inequality) Let $s \geq 0$, $t \geq 0$, and $p(x) > 1$ such that $1/p(x) + 1/p'(x) = 1$. There holds the inequality*

$$st \leq \frac{s^{p(x)}}{p(x)} + \frac{t^{p'(x)}}{p'(x)} \tag{2.1}$$

with equality if and only if $s^{p(x)} = t^{p'(x)}$.

Inequality (2.1) is the classical Young's inequality which can be varied in the following form.

Lemma 2.2. *(Modified Young's inequality) Let $\Phi(x), \Psi(x) \geq 0$, $p(x) > 1$ such that $1/p(x) + 1/p'(x) = 1$ and $\varepsilon : \Omega \to \mathbb{R}^+$ be a continuous and bounded function. There holds the inequality*

$$\Phi\Psi^{p(x)-1} \leq \frac{\Phi^{p(x)}}{p(x)\varepsilon(x)^{p(x)-1}} + \frac{p(x)-1}{p(x)}\varepsilon(x)\Psi^{p(x)} \tag{2.2}$$

for a.e. $x \in \Omega$ Furthermore, there is equality in (2.2) if and only if $\Phi = \varepsilon(x)\Psi$.

Proof. Applying the classical Young's inequality (2.1) with

$$s = \frac{\Phi}{\varepsilon(x)^{\frac{p(x)-1}{p(x)}}} \quad \text{and} \quad t = \left(\Psi\varepsilon(x)^{\frac{1}{p(x)}}\right)^{p(x)-1},$$

we have

$$\begin{aligned}\Phi\Psi^{p(x)-1} &= \left(\frac{\Phi}{\varepsilon(x)^{\frac{p(x)}{p(x)-1}}}\right)\left(\Psi\varepsilon(x)^{\frac{1}{p(x)}}\right)^{p(x)-1} \\ &\leq \frac{\Phi^{p(x)}}{p(x)\varepsilon(x)^{p(x)-1}} + \frac{p(x)-1}{p(x)}\left(\Psi\varepsilon(x)^{\frac{1}{p(x)}}\right)^{p(x)}.\end{aligned}$$

□

The next is the variable exponent Picone identity which is the main theorem in this section.

Theorem 2.3. *Let $u \geq 0$ and $v > 0$ be nonconstant differentiable functions a.e. in Ω. Suppose $p : \bar{\Omega} \to (0, \infty)$ is a C^1-function for $p(x) > 1$, and $f : (0, \infty) \to (0, \infty)$ is a C^1-function satisfying $f(y) > 0$ and $f'(y) \geq (p(x) - 1)\left[f(y)^{\frac{p(x)-2}{p(x)-1}}\right]$ for $y > 0$. Define*

$$\begin{aligned}L(u, v) = &|\nabla_X u|^{p(x)} - \frac{u^{p(x)}\ln u}{f(v)}|\nabla_X v|^{p(x)-2}\nabla_X v\nabla_X p(x) \\ &- p(x)\frac{u^{p(x)-1}}{f(v)}|\nabla_X v|^{p(x)-2}\nabla_X v\nabla_X u + \frac{u^{p(x)}f'(v)}{(f(v))^2}|\nabla_X v|^{p(x)}\end{aligned} \tag{2.3}$$

and

$$R(u, v) = |\nabla_X u|^{p(x)} - \nabla_X\left(\frac{u^{p(x)}}{f(v)}\right)|\nabla_X v|^{p(x)-2}\nabla_X v. \tag{2.4}$$

Then

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1. $L(u,v) = R(u,v)$.
2. *Moreover* $L(u,v) \geq 0$ *if* $\nabla_X v \nabla_X p(x) \equiv 0$.
3. *Furthermore,* $L(u,v) = 0$ *a.e. in* Ω *if and only if* $\nabla_X (u/v) = 0$ *a.e. in* Ω.

Proof. By direct computation we have

$$\begin{aligned}
R(u,v) &= |\nabla_X u|^{p(x)} - \left(\frac{\nabla_X (u^{p(x)})}{f(v)} - \frac{u^{p(x)} \nabla_X (f(v))}{(f(v))^2} \right) |\nabla_X v|^{p(x)-2} \nabla_X v \\
&= |\nabla_X u|^{p(x)} - \frac{u^{p(x)} \ln u \nabla_X p(x) + p(x) u^{p(x)-1} \nabla_X u}{f(v)} |\nabla_X v|^{p(x)-2} \nabla_X v \\
&\qquad + \frac{u^{p(x)} f'(v)}{(f(v))^2} |\nabla_X v|^{p(x)} \\
&= L(u,v),
\end{aligned}$$

which proves (1) of the theorem.

Next we verify $L(u,v) \geq 0$. Rewriting the expression for $L(u,v)$ as follows

$$\begin{aligned}
L(u,v) &= |\nabla_X u|^{p(x)} - p(x) \frac{u^{p(x)-1}}{f(v)} |\nabla_X v|^{p(x)-1} |\nabla_X u| + \frac{u^{p(x)} f'(v)}{(f(v))^2} |\nabla_X v|^{p(x)} \\
&\quad + p(x) \frac{u^{p(x)-1}}{f(v)} \left(|\nabla_X v| |\nabla_X u| - \nabla_X v \nabla u \right) - \frac{u^{p(x)} \ln u}{f(v)} |\nabla_X v|^{p(x)-2} \nabla_X v \nabla_X p(x) \\
&= p(x) \left(\frac{|\nabla_X u|^{p(x)}}{p(x)} + \frac{p(x)-1}{p(x)} \left[\frac{(u|\nabla_X v|)^{p(x)-1}}{f(v)} \right]^{\frac{p(x)}{p(x)-1}} \right) + \frac{u^{p(x)} f'(v)}{(f(v))^2} |\nabla_X v|^{p(x)} \\
&\quad - (p(x)-1) \left[\frac{(u|\nabla_X v|)^{p(x)-1}}{f(v)} \right]^{\frac{p(x)}{p(x)-1}} - p(x) \frac{u^{p(x)-1}}{f(v)} |\nabla_X v|^{p(x)-1} |\nabla_X u| \\
&\quad + p(x) \frac{u^{p(x)-1}}{f(v)} \left(|\nabla_X v| |\nabla_X u| - \nabla_X v \nabla u \right) - \frac{u^{p(x)} \ln u}{f(v)} |\nabla_X v|^{p(x)-2} \nabla_X v \nabla_X p(x) \\
&= L_1(u,v) \quad + \quad L_2(u,v) \quad + \quad L_3(u,v) \quad + \quad L_4(u,v),
\end{aligned}$$

where

$$\begin{aligned}
L_1(u,v) := p(x) &\left(\frac{|\nabla_X u|^{p(x)}}{p(x)} + \frac{p(x)-1}{p(x)} \left[\frac{(u|\nabla_X v|)^{p(x)-1}}{f(v)} \right]^{\frac{p(x)}{p(x)-1}} \right) \\
&\qquad - p(x) \frac{u^{p(x)-1}}{f(v)} |\nabla_X v|^{p(x)-1} |\nabla_X u|,
\end{aligned}$$

$$L_2(u,v) := \frac{u^{p(x)} f'(v)}{(f(v))^2} |\nabla_X v|^{p(x)} - (p(x)-1) \left[\frac{(u|\nabla_X v|)^{p(x)-1}}{f(v)} \right]^{\frac{p(x)}{p(x)-1}},$$

$$L_3(u,v) := p(x) \frac{u^{p(x)-1}}{f(v)} \left(|\nabla_X v| |\nabla_X u| - \nabla_X v \nabla u \right),$$

$$L_4(u,v) := -\frac{u^{p(x)} \ln u}{f(v)} |\nabla_X v|^{p(x)-2} \nabla_X v \nabla_X p(x).$$

Applying the Young's inequality (2.1), choosing $s = |\nabla_X u|$ and $t = \frac{(u|\nabla_X v|)^{p(x)-1}}{f(v)}$, we obtain

$$p(x)\frac{u^{p(x)-1}}{f(v)}|\nabla_X v|^{p(x)-1}|\nabla_X u| \leq p(x)\left(\frac{|\nabla_X u|^{p(x)}}{p(x)} + \frac{p(x)-1}{p(x)}\left[\frac{(u|\nabla_X v|)^{p(x)-1}}{f(v)}\right]^{\frac{p(x)}{p(x)-1}}\right),$$

implying that $L_1(u,v) \geq 0$ with equality if and only if there is equality in the Young's inequality, that is, $s = t^{\frac{1}{p(x)-1}}$.

Applying the assumption $f'(y) \geq (p(x)-1)\left[f(y)^{\frac{p(x)-2}{p(x)-1}}\right]$, we have

$$\frac{u^{p(x)}f'(v)}{(f(v))^2}|\nabla_X v|^{p(x)} \geq (p(x)-1)\left[\frac{(u|\nabla_X v|)^{p(x)-1}}{f(v)}\right]^{\frac{p(x)}{p(x)-1}},$$

which implies that $L_2(u,v) \geq 0$ with equality if and only if $f'(y) = (p(x)-1)\left[f(y)^{\frac{p(x)-2}{p(x)-1}}\right]$. Clearly, $L_3(u,v) \geq 0$ by reverting to the inequality $|\nabla_X v||\nabla_X u| - \nabla_X v \nabla_X u \geq 0$. By the virtue of the assumption that $\nabla_X v \nabla_X p(x) \equiv 0$, we have also $L_4(u,v) \equiv 0$. Putting all of these together we obtain that $L(u,v) \geq 0$ a.e. in Ω.

Observe that $L(u,v) = 0$ holds if and only if

$$|\nabla_X u| = \frac{u}{f(v)^{\frac{1}{p(x)-1}}}|\nabla_X v|, \tag{2.5}$$

$$f'(y) = (p(x)-1)\left[f(y)^{\frac{p(x)-2}{p(x)-1}}\right], \tag{2.6}$$

and

$$|\nabla_X v||\nabla_X u| = \nabla_X v \nabla_X u. \tag{2.7}$$

Upon solving for (2.6) we get $f(v) = v^{p(x)-1}$. If $\nabla_X(u/v) = 0$ then there exists a positive constant, say $\alpha > 0$ such that $u = \alpha v$, then equality (2.7) holds. Combining $f(v) = v^{p(x)-1}$ and $u = \alpha v$, then (2.5) holds. We can now conclude that $L(u,v) = 0$ implies $\nabla_X(u/v) = 0$. Indeed, if $L(u,v)(x_0) = 0$, $x_0 \in \Omega$, there are two cases to consider, namely; the case $u(x_0) \neq 0$ and the case $u(x_0) = 0$.
(a) If $u(x_0) \neq 0$, then $L(u,v) = 0$ for all $x_0 \in \Omega$, that is, $L_1(u,v) = 0$, $L_2(u,v) = 0$ and $L_3(u,v) = 0$, and we conclude that (2.5), (2.6) and (2.7) hold, which when combined gives $u = \alpha v$ a.e. for some constant $\alpha > 0$ and $\nabla_X(u/v) = 0$ for all $x_0 \in \Omega$.
(b) If $u(x_0) = 0$, we denote $\Omega^* = \{x \in \Omega : u(x) = 0\}$, and suppose $\Omega^* \neq \Omega$. Here $u(x_0) = \alpha v(x_0)$ implies $\alpha = 0$ since $u(x_0) = 0$ and $v(x_0) > 0$. By the first case (Case (a)) we know that $u(x) = \alpha v(x)$ and $u(x) \neq 0$ for all $x \in \Omega \setminus \Omega^*$, then it is impossible that $\alpha = 0$. This contradiction implies that $\Omega^* = \Omega$.

□

Remark 2.4. *Theorem 2.3 generalizes many known results. For examples:*

1. *If $M = \mathbb{R}^n$ and $f(v) = v^{p(x)-1}$ in (2.3) and (2.4). Then, we obtain the variable exponent Picone identity of Allegretto [30] and Feng and Han [34].*

2. *If $p(x) = p$, $f(v) = v^{p-1}$ in (2.3) and (2.4), then our result covers Allegretto and Huang's [16] ($M = \mathbb{R}^n$), Niu, Zhang and Wang [27] (Heisenberg group), Ruzhansky, Sabitbek and Suragan [24] (for general vector fields).*

3. *If we allow $p(x) = p$ in (2.3) and (2.4), we then recover Bal [18] in the Euclidean setting and Suragan and Yessirkegenov [29] in the setting of stratified Lie groups.*

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3 Applications

Eigenvalue problem for $p(x)$-sub-Laplacian

Let $\Omega \subset M$ be a bounded domain with smooth boundary $\partial\Omega$. We suppose a continuous function $p : \bar{\Omega} \to \mathbb{R}^+$, $p(x) > 1$ is such that

$$1 < p^- := ess \inf_{x\in\bar{\Omega}} p(x) \le p(x) \le p^+ := ess \sup_{x\in\bar{\Omega}} p(x) < \infty.$$

Now consider the indefinite weighted Dirichlet eigenvalue problem for $p(x)$-Laplacian

$$\begin{array}{ll} -\nabla_X^*(|\nabla_X u|^{p(x)-2}\nabla_X u) = \lambda g(x)|u|^{p(x)-2}u, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{array} \tag{3.1}$$

where Ω is as defined above, $g(x)$ is a positive bounded function and $p : \bar{\Omega} \to (1, \infty)$ is a continuous function for $x \in \bar{\Omega}$.

Definition 3.1. *Let $\lambda \in \mathbb{R}^+$ and $u \in W_0^{1,p(x)}(\Omega)$, the pair (u, λ) is called a solution of (3.1) if*

$$\int_\Omega |\nabla_X u|^{p(x)-2}\langle \nabla_X u, \nabla_X \phi\rangle dx - \lambda \int_\Omega g(x)|u|^{p(x)-2}u\phi dx = 0 \tag{3.2}$$

for all $\phi \in W_0^{1,p(x)}(\Omega)$. If (u, λ) is a solution of (3.1), we call λ an eigenvalue, and u an eigenfunction corresponding to λ.

Similarly, by the sup-solution and sub-solution of (3.1), we mean the pair (u, λ) such that

$$\int_\Omega |\nabla_X u|^{p(x)-2}\langle \nabla_X u, \nabla_X \phi\rangle dx - \lambda \int_\Omega g(x)|u|^{p(x)-2}u\phi dx \ge 0 \tag{3.3}$$

and

$$\int_\Omega |\nabla_X u|^{p(x)-2}\langle \nabla_X u, \nabla_X \phi\rangle dx - \lambda \int_\Omega g(x)|u|^{p(x)-2}u\phi dx \le 0 \tag{3.4}$$

for all $\phi \in W_0^{1,p(x)}(\Omega)$, respectively.

Denote the principal eigenvalue of (3.1) (the least positive eigenvalue) by $\lambda_{1,p} := \lambda_{1,p}(\Omega)$, clearly for the solution (u, λ) and $u \neq 0$, we get

$$\lambda_{1,p} = \inf_{u\in\in W_0^{1,p(x)}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla_X u|^{p(x)}dx}{\int_\Omega g(x)|u|^{p(x)}dx}.$$

In the case $p(x) = p(\text{constant})$, it is well known that $\lambda_{1,p}(\Omega)$ given above is the first eigenvalue of p-Laplacian (with $g(x) = 1$, $\Omega \subset \mathbb{R}^n$), which must be positive. But this is not true for general $p(x)$ in the sense that $\lambda_{1,p}$ may be zero [14]. Nevertheless, Fan, Zhang and Zhao in [4] have proved the existence of infinitely many eigenvalues $p(x)$-Laplacian and established sufficient conditions for $\lambda_{1,p}(\Omega) > 0$ (see also Franzina and Lindqvist [35]). Motivated by [4], we are able to assume the existence of $\lambda_{1,p} > 0$ in the rest of this section.

In the rest of this section we are concerned with the indefinite weighted Dirichlet eigenvalue problem (3.1) and discuss some properties of its solutions v satisfying $\nabla_X v \nabla_X p(x) \equiv 0$ by the application of Picone identity in Theorem 2.3. We remark that the results of this paper are classical in the sense that they have been established using different methods such variational approach (see [1, 4, 6] for instance) where the condition $\nabla_X v \nabla_X p(x) \equiv 0$ is not required.

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3.1 Variable exponent Hardy type inequality

Proposition 3.2. *Let $\Omega \subset M$ be an open bounded domain. Suppose that a function $v \in C_0^\infty(\Omega)$ satisfies $\nabla_X v \nabla_X p(x) \equiv 0$ and*

$$\begin{array}{ll} -\mathscr{L}_p v = \mu a(x) f(v) & \text{in } \Omega, \\ v > 0 & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{array} \tag{3.5}$$

where $f : \mathbb{R}^+ \to \mathbb{R}^+$ is C^1 and satisfies $f'(y) \geq (p(x)-1)\left[f(y)^{\frac{p(x)-2}{p(x)-1}}\right]$, $\mu > 0$ is a constant, $a(x)$ is a positive continuous function. Then there holds

$$\int_\Omega |\nabla_X u|^{p(x)} dx \geq \mu \int_\Omega a(x)|u|^{p(x)} dx$$

for any $0 \leq u \in C_0^1(\Omega)$.

Proof. Since $v > 0$ and solves (3.5) in Ω, that is, $v \in W_0^{1,p(x)}(\Omega)$. For a given a $\epsilon > 0$, we set $\phi = \frac{|u|^{p(x)}}{f(v+\epsilon)}$. By the definition of solution (3.2) we compute

$$\begin{aligned} \mu \int_\Omega a(x) f(v) \frac{|u|^{p(x)}}{f(v+\epsilon)} dx &\leq \int_\Omega |\nabla_X v|^{p-2} \nabla_X v \nabla_X \left(\frac{|u|^{p(x)}}{f(v+\epsilon)}\right) dx \\ &= \int_\Omega \left[|\nabla_X u|^{p(x)} - R(u, v+\epsilon)\right] dx \\ &= \int_\Omega |\nabla_X u|^{p(x)} dx - \int_\Omega L(u, v+\epsilon) dx. \end{aligned}$$

Taking the limit as $\epsilon \to 0^+$, applying Fatou's Lemma and Lebesgue dominated convergence theorem respectively on the left hand side and right hand side of the last expression, we obtain

$$0 \leq \int_\Omega |\nabla_X u|^{p(x)} - \mu \int_\Omega a(x)|u|^{p(x)} dx - \int_\Omega L(u, v) dx.$$

Therefore we have

$$0 \leq \int_\Omega |\nabla_X u|^{p(x)} dx - \mu \int_\Omega a(x)|u|^{p(x)} dx$$

since $L(u, v) \geq 0$ almost everywhere in Ω. This therefore completes the proof. □

Corollary 3.3. *Suppose there exists $\lambda > 0$ and a strictly positive sup-solution of (3.1) such that $\nabla_X p(x) \nabla_X v = 0$. Then*

$$\int_\Omega |\nabla_X u|^{p(x)} dx \geq \lambda \int_\Omega g(x)|u|^{p(x)} dx \tag{3.6}$$

for all $u \in W_0^{1,p(x)}(\Omega)$.

Proof. Applying Proposition 3.2 by setting $a(x) \equiv g(x)$, $\mu = \lambda$ and $f(v) = |v|^{p(x)-2} v$, then one arrives at the conclusion (3.5) at once. □

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3.2 Principal frequency and domain monotonicity

Proposition 3.4. *Let there exists λ and a strictly positive sup-solution $v \in W_0^{1,p(x)}(\Omega)$ of (3.1) such that $\nabla_X p(x) \nabla_X v = 0$. Then we have*

$$\int_\Omega |\nabla_X u|^{p(x)} dx \geq \lambda \int_\Omega g(x)|u|^{p(x)} dx \tag{3.7}$$

and

$$\lambda_{1,p}(\Omega) \geq \lambda \tag{3.8}$$

for all $u \in W_0^{1,p(x)}(\Omega)$.

Proof. Suppose there exists $\lambda > 0$, since v is strictly positive sup-solution of (3.1) in Ω, we have

$$\int_\Omega |\nabla_X v|^{p(x)-2} \langle \nabla_X v, \nabla_X \phi \rangle dx \geq \lambda \int_\Omega g(x)|v|^{p(x)-2} v \phi dx \tag{3.9}$$

for all $\phi \in W_0^{1,p(x)}(\Omega)$. For a given small $\epsilon > 0$, setting $\phi = \frac{|u|^{p(x)}}{(v+\epsilon)^{p(x)-1}}$ into (3.9). Then, following the proof of the Proposition 3.2, we arrive at (3.7).

Now, let $u_1 \in W_0^{1,p(x)}(\Omega)$ be the eigenfunction corresponding to the principal eigenvalue $\lambda_{1,p}(\Omega)$. We have

$$\int_\Omega |\nabla_X u_1|^{p(x)-2} \langle \nabla_X u_1, \nabla_X \phi \rangle dx = \lambda_{1,p} \int_\Omega g(x)|u_1|^{p(x)-2} u_1 \phi dx \tag{3.10}$$

for any $\phi \in W_0^{1,p(x)}(\Omega)$. Choosing $\epsilon > 0$ (small) we can define via Picone identity that

$$0 \leq L(u_1, v+\epsilon) = R(u_1, v+\epsilon), \quad v > 0. \tag{3.11}$$

Integrating (3.11) over Ω and then using (3.9) with $\phi = \frac{|u_1|^{p(x)}}{f(v+\epsilon)}$ and (3.10) with $\phi = u_1$, we obtain

$$\begin{aligned} 0 &\leq \int_\Omega L(u_1, v+\epsilon) dx = \int_\Omega R(u_1, v+\epsilon) dx \\ &= \int_\Omega |\nabla_X u_1|^{p(x)} dx - \int_\Omega \nabla_X \left(\frac{|u_1|^{p(x)}}{f(v+\epsilon)} \right) |\nabla_X v|^{p(x)-2} \nabla_X v dx \\ &= \int_\Omega |\nabla_X u_1|^{p(x)} dx + \int_\Omega \frac{|u_1|^{p(x)}}{f(v+\epsilon)} \nabla_X^* (|\nabla_X v|^{p(x)-2} \nabla_X) v dx \\ &\leq \lambda_{1,p}(\Omega) \int_\Omega g(x)|u_1|^{p(x)} dx - \lambda \int_\Omega g(x) \frac{|u_1|^{p(x)}}{f(v+\epsilon)} |v|^{p(x)-2} v dx. \end{aligned}$$

As usual, taking the limit as $\epsilon \to 0^+$, applying Fatou's Lemma and Lebesgue dominated convergence theorem, setting $f(v) = v^{p(x)-1}$, we arrive at

$$0 \leq (\lambda_{1,p}(\Omega) - \lambda) \int_\Omega g(x)|u_1|^p dx,$$

which implies $\lambda_{1,p}(\Omega) \geq \lambda$. □

As a corollary to the last proposition, we show strict monotonicity of the principal eigenvalue with respect to domain monotonicity. Let $\lambda_{1,p}(\Omega) > 0$ be the principal eigenvalue of $\mathscr{L}_p$ on Ω.

Corollary 3.5. *Suppose $\Omega_1 \subset \Omega_2 \subset \Omega$ and $\Omega_1 \neq \Omega_2$. Let u_1 and u_2 be the eigenfunctions corresponding to $\lambda_{1,p}(\Omega_1)$ and $\lambda_{1,p}(\Omega_2)$ satisfying $\nabla_X p(x) \nabla_X u_1 = 0$ and $\nabla_X p(x) \nabla_X u_2 = 0$. Then*

$$\lambda_{1,p}(\Omega_1) > \lambda_{1,p}(\Omega_2)$$

if they both exist.

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Proof. Let u_1 and u_2 be positive eigenfunctions corresponding to $\lambda_{1,p}(\Omega_1)$ and $\lambda_{1,p}(\Omega_2)$, respectively. Clearly with $\phi \in C_0^\infty(\Omega)$, we have by Picone identity that

$$0 \leq \int_\Omega L(\phi, u_2)dx = \int_\Omega R(\phi, u_2)dx.$$

Replacing ϕ by u_1 and applying Proposition 3.4 we have

$$\lambda_{1,p}(\Omega_1) - \lambda_{1,p}(\Omega_2) \geq 0.$$

If we have $\lambda_{1,p}(\Omega_1) = \lambda_{1,p}(\Omega_2)$, then $L(u_1, u_2) = 0$ a.e. in Ω and thus $u_1 = \alpha u_2$ for some constant $\alpha > 0$. However, this is impossible when $\Omega_1 \subset \Omega_2$ and $\Omega_1 \neq \Omega_2$. □

Next is the uniqueness and simplicity results.

3.3 Uniqueness and simplicity of first eigenvalue

Proposition 3.6. *Let there exists $\lambda > 0$ and a strictly positive solution $v \in W_0^{1,p(x)}(\Omega)$ of (3.1) such that $\nabla_X p(x) \nabla_X v = 0$. Then we have*

$$\lambda_{1,p}(\Omega) = \lambda.$$

Moreover, let u_1 be the corresponding eigenfunction to $\lambda_{1,p}(\Omega)$. Then any other $u \in W_0^{1,p(x)}(\Omega)$ corresponding to $\lambda_{1,p}(\Omega)$ is a constant multiple of u_1.

Proof. Let $u_1 \in W_0^{1,p(x)}(\Omega)$ be the eigenfunction corresponding to $\lambda_{1,p}(\Omega)$ and u be a positive solution of (3.1). Applying Picone identity by choosing $\epsilon > 0$ (small) as follows:

$$\begin{aligned} 0 &\leq \int_\Omega L(u, u_1 + \epsilon)dx \\ &= \int_\Omega |\nabla_X u|^{p(x)} dx + \int_\Omega \frac{u^{p(x)}}{f(u_1+\epsilon)} \nabla_X^* (|\nabla_X u_1|^{p(x)-2} \nabla_X) u_1 dx \\ &= \lambda \int_\Omega g(x)|u|^{p(x)} dx - \lambda_{1,p}(\Omega) \int_\Omega g(x) \frac{u^{p(x)}}{(u_1+\epsilon)^{p(x)-1}} |u_1|^{p(x)-2} u_1 dx, \end{aligned}$$

where we have set $f(u_1 + \epsilon) = (u_1 + \epsilon)^{p(x)-1}$. Taking the limit as $\epsilon \to 0^+$, applying Fatou's Lemma and Lebesgue dominated convergence theorem, then

$$\lambda_{1,p}(\Omega) \leq \lambda.$$

On the other hand by Proposition 3.4, we have

$$\lambda_{1,p}(\Omega) \geq \lambda.$$

This therefore implies that $\lambda_{1,p}(\Omega) = \lambda$. By this we have proved the uniqueness part.

Now by the hypothesis of the theorem we have for $\phi, \psi \in C_0^\infty(\Omega)$ that

$$\int_\Omega |\nabla_X u|^{p(x)-2} \langle \nabla_X u, \nabla_X \phi \rangle dx = \lambda_{1,p} \int_\Omega g(x)|u|^{p(x)-2} u\phi dx, \tag{3.12}$$

$$\int_\Omega |\nabla_X u_1|^{p(x)-2} \langle \nabla_X u_1, \nabla_X \psi \rangle dx = \lambda_{1,p} \int_\Omega g(x)|u_1|^{p(x)-2} u_1 \psi dx. \tag{3.13}$$

Taking $\phi = u$ and $\psi = \frac{|u|^p}{(u_1+\epsilon)^{p-1}}$ into (3.12) and (3.13), respectively, and sending $\epsilon \to 0^+$, we arrive at

$$\begin{aligned} \int_\Omega |\nabla_X u|^{p(x)} dx &= \lambda_{1,p} \int_\Omega g(x)|u|^{p(x)} dx \\ &= \int_\Omega |\nabla_X u_1|^{p(x)-2} \nabla_X u_1 \nabla_X \Big(\frac{|u|^{p(x)}}{u_1^{p(x)-1}} \Big) dx, \end{aligned}$$

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which implies (by choosing $f(u_1) = u_1^{p(x)-2}$)

$$\int_\Omega R(u,u_1)dx = \int_\Omega L(u,u_1)dx = 0$$

and consequently, $\nabla_X(u/v) = 0$, i.e., $u = \alpha u_1$ for some positive constant $\alpha > 0$.

□

The next proposition gives the sign changing nature of any other eigenfunction associated to an eigenvalue other than $\lambda_{1,p}(\Omega)$.

Proposition 3.7. *Any eigenfunction v corresponding to an eigenvalue $\lambda \neq \lambda_{1,p}(\Omega)$ such that $\nabla_X p(x)\nabla_X v = 0$ changes sign.*

Proof. By contradiction we suppose $v > 0$ does not change sign (the case $v \leq 0$ can be handled similarly). Let $\phi > 0$ be an eigenfunction corresponding to $\lambda_{1,p}(\Omega)$. Choosing any $\epsilon > 0$ as before, applying Picone identity, we have

$$\begin{aligned} 0 &\leq \int_\Omega L(\phi, v+\epsilon)dx \\ &= \int_\Omega \left[|\nabla_X\phi|^{p(x)} - \nabla_X\Big(\frac{\phi^{p(x)}}{f(v+\epsilon)}\Big)|\nabla_X v|^{p(x)-2}\nabla_X v \right] dx \\ &= \int_\Omega |\nabla_X\phi|^{p(x)}dx + \int_\Omega \frac{\phi^{p(x)}}{f(v+\epsilon)}\mathscr{L}_p v dx. \end{aligned}$$

Since $\frac{\phi^{p(x)}}{(v+\epsilon)^{p(x)-1}}$ is admissible in the weak formulation of (3.1) satisfied by (ϕ, λ), we arrive at

$$0 \leq \lambda_{1,p}(\Omega)\int_\Omega g(x)|\phi|^{p(x)}dx - \lambda\int_\Omega \frac{\phi^{p(x)}}{f(v+\epsilon)}g(x)|v|^{p(x)-2}v dx.$$

Setting $f(v+\epsilon) = (v+\epsilon)^{p(x)-1}$ and letting $\epsilon \to 0^+$ in the last inequality as usual we obtain

$$0 \leq (\lambda_{1,p} - \lambda)\int_\Omega g(x)\phi^{p(x)}dx,$$

which is a contradiction since $\int_\Omega g(x)\phi^{p(x)}dx = 1$. Thus v must change sign.

□

4 Variable exponent Caccioppoli estimates for general vector fields

Picone identity is applied to prove some variable exponent Caccioppoli estimates for general vector fields in this section. Recall that

$$1 < p^- := ess \inf_{x\in\bar\Omega} p(x) \leq p(x) \leq p^+ := ess \sup_{x\in\bar\Omega} p(x) < \infty.$$

Without giving rise to confusion but for simplicity sake we write $p := p(x)$ and $q =: q(x)$. We also denote $q^- := ess \inf_{x\in\bar\Omega} q(x)$ and $q^+ := ess \sup_{x\in\bar\Omega} q(x)$.

Theorem 4.1. *Let v be a positive sub-solution of (3.1) in $\Omega \subset M$. Then for every fixed $q(x) > p(x) - 1$, $p(x) > 1$, $\nabla_X v\nabla_X p(x) = 0$, $\nabla_X v\nabla_X q(x) = 0$ and $\lambda \in \mathbb{R}$, we have*

$$\int_\Omega v^{q-p}\phi^p|\nabla_X v|^p dx \leq C_{p,q}^{p^+}\int_\Omega v^q|\nabla_X\phi|^p dx + C_{\lambda,p,q}\int_\Omega g(x)v^q\phi^p dx \tag{4.1}$$

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for every nonnegative functions $\phi \in C_0^\infty(\Omega)$, *where*

$$C_{p,q}^{p^+} := \left(\frac{p^+}{q^- - p^+ + 1}\right)^{p^+} \quad \text{and} \quad C_{\lambda,p,q} := \left(\frac{\lambda p^+}{q^- - p^+ + 1}\right).$$

Proof. Let $u = v^{q/p}\phi$, where ϕ is a nonnegative test function and v is a sub-solution of (3.1), we compute

$$\begin{aligned}\nabla_X\left(v^{q/p}\phi\right) &= \phi\nabla_X(v^{q/p}) + v^{q/p}\nabla_X\phi \\ &= \phi v^{q/p}\ln v\left(\frac{\nabla_X q}{p} - \frac{q\nabla_X p}{p^2}\right) + \frac{q}{p}v^{\frac{q-p}{p}}\phi\nabla_X v + v^{q/p}\nabla_X\phi\end{aligned}$$

so that

$$\begin{aligned}\langle\nabla_X v, \nabla_X\left(v^{q/p}\phi\right)\rangle &= \phi v^{q/p}\ln v\left(\frac{\nabla_X q}{p} - \frac{q\nabla_X p}{p^2}\right)\nabla_X v \\ &\quad + \frac{q}{p}v^{\frac{q-p}{p}}\phi|\nabla_X v|^2 + v^{q/p}\langle\nabla_X\phi, \nabla_X v\rangle.\end{aligned}$$

Now using the the fact that v is a sub-solution of (3.1) and the condition that $\nabla_X v\nabla_X p(x) \equiv 0$ and $\nabla_X v\nabla_X q(x) \equiv 0$ in the Picone identity $L(u,v) \geq 0$, we have

$$\begin{aligned}0 &\leq \int_\Omega L(v^{q/p}\phi, v) \\ &= \int_\Omega |\nabla_X\left(v^{q/p}\phi\right)|^p dx + \int_\Omega \frac{f'(v)}{(f(v))^2}|v^{q/p}|^p|\phi\nabla_X v|^p dx \\ &\quad - \int_\Omega q\frac{|v^{q/p}\phi|^{p-1}}{f(v)}\phi v^{\frac{q-p}{p}}|\nabla_X v|^p dx \\ &\quad - \int_\Omega p\frac{|v^{q/p}\phi|^{p-1}}{f(v)}v^{q/p}|\nabla_X v|^{p-2}\langle\nabla_X\phi, \nabla_X v\rangle dx.\end{aligned} \tag{4.2}$$

Considering the condition $f'(v) \geq (p(x)-1)\left[f(v)^{\frac{p(x)-2}{p(x)-1}}\right]$, we can then choose $f(v) = v^{p(x)-1}$. Then (4.2) reads

$$\begin{aligned}0 \leq \int_\Omega |\nabla_X\left(v^{q/p}\phi\right)|^p dx + \int_\Omega (p-1)v^{q-p}|\phi\nabla_X v|^p dx - \int_\Omega qv^{q-p}|\phi\nabla_X v|^p dx \\ - \int_\Omega p|v^{\frac{q-p}{p}}\phi|^{p-1}v^{q/p}|\nabla_X v|^{p-2}\langle\nabla_X\phi, \nabla_X v\rangle dx.\end{aligned} \tag{4.3}$$

Using the $\varepsilon(x)$-modified version of the Young's inequality in Lemma 2.2 with $\Phi = v^{q/p}|\nabla_X\phi|$ and $\Psi = v^{\frac{q-p}{p}}\phi|\nabla_X v|$, we can estimate the last term of (4.3) as follows

$$\begin{aligned}-\int_\Omega p|v^{\frac{q-p}{p}}\phi|^{p-1}v^{q/p}|\nabla_X v|^{p-2}\langle\nabla_X\phi, \nabla_X v\rangle dx \\ \leq \int_\Omega p|v^{\frac{q-p}{p}}\phi|^{p-1}|\nabla_X v|^{p-1}v^{q/p}\nabla_X\phi dx \\ \leq \int_\Omega \varepsilon^{1-p}v^q|\nabla_X\phi|^p dx + \int_\Omega \varepsilon(p-1)v^{q-p}|\phi\nabla_X v|^p dx,\end{aligned} \tag{4.4}$$

where $\varepsilon(x)$ is a continuous bounded function on Ω, which will be chosen later. Substituting (4.4)

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into (4.3) we get

$$
\begin{aligned}
0 \leq & \int_\Omega |\nabla_X \left(v^{q/p}\phi\right)|^p dx - \int_\Omega [q-p+1-\varepsilon(p-1)] v^{q-p} |\phi \nabla_X v|^p dx \\
& + \int_\Omega \varepsilon^{1-p} v^q |\nabla_X \phi|^p dx \\
\leq & \lambda \int_\Omega g(x) |v^{q/p}\phi|^p dx - \mathcal{C}^1_{\epsilon,p,q} \int_\Omega v^{q-p} |\phi \nabla_X v|^p dx + \mathcal{C}^2_{\epsilon,p} \int_\Omega v^q |\nabla_X \phi|^p dx,
\end{aligned}
$$

where we have used $\int_\Omega |\nabla_X u|^{p(x)} dx \leq \lambda \int_\Omega g(x)|u|^{p(x)} dx$ for the sub-solution of (3.1). Here

$$
\mathcal{C}^1_{\epsilon,p,q} := q^- - p^+ + 1 - \bar{\varepsilon}(p^+ - 1) \quad \text{and} \quad \mathcal{C}^2_{\epsilon,p} := \bar{\varepsilon}^{1-p^+},
$$

where $\bar{\varepsilon} := \sup_\Omega \varepsilon(x)$.
Rearranging the last inequality we arrive at

$$
\int_\Omega v^{q-p} |\phi \nabla_X v|^p dx \leq \frac{\mathcal{C}^2_{\epsilon,p}}{\mathcal{C}^1_{\epsilon,p,q}} \int_\Omega v^q |\nabla_X \phi|^p dx + \frac{\lambda}{\mathcal{C}^1_{\epsilon,p,q}} \int_\Omega g(x) |v^{q/p}\phi|^p dx.
$$

We can now choose a suitable number $\bar{\varepsilon}$ as $\bar{\varepsilon} := \frac{q^- - p^+ + 1}{p^+}$ and then compute

$$
\begin{aligned}
\frac{1}{\mathcal{C}^1_{\epsilon,p,q}} &:= \frac{1}{q^- - p^+ + 1 - \bar{\varepsilon}(p^+ - 1)} = \frac{p^+}{q^- - p^+ + 1}, \\
\frac{\mathcal{C}^2_{\epsilon,p}}{\mathcal{C}^1_{\epsilon,p,q}} &:= \frac{\bar{\varepsilon}^{1-p^+}}{q^- - p^+ + 1 - \bar{\varepsilon}(p^+ - 1)} = \left(\frac{p^+}{q^- - p^+ + 1}\right)^{p^+}.
\end{aligned}
$$

The proof is therefore complete. □

The following two corollaries can be deduced from Theorem 4.1 using the same assumptions.

Corollary 4.2. *Let v be a positive sub-solution of (3.1) in Ω satisfying $\nabla_X v \nabla_X p(x) = 0$. If $g(x) \equiv 0$ and $p(x) = q(x)$ in Ω. Then we have*

$$
\int_\Omega \phi^{p(x)} |\nabla_X v|^{p(x)} dx \leq (p^+)^{p^+} \int_\Omega v^{p(x)} |\nabla_X \phi|^{p(x)} dx
$$

for every nonnegative function $\phi \in C_0^\infty(\Omega)$.

Corollary 4.3. *Let v be a positive sub-solution of (3.1) in Ω satisfying $\nabla_X v \nabla_X p(x) = 0$. Letting $\lambda = 1$ and $p(x) = q(x)$ in Ω. Then we have*

$$
\int_\Omega \phi^{p(x)} |\nabla_X v|^{p(x)} dx \leq (p^+)^{p^+} \int_\Omega v^{p(x)} |\nabla_X \phi|^{p(x)} dx + p^+ \int_\Omega g(x) v^{q(x)} \phi^{p(x)} dx
$$

for every nonnegative function $\phi \in C_0^\infty(\Omega)$.

Remark 4.4. *Suppose $M = \mathbb{R}^n$, $p(x) = p$ (constant) and $q(x) = q$ (constant):*

1. *Corollary 4.2 reduces to [23, Corollary 3.1] and [36, Equation 5.27].*
2. *Corollary 4.3 reduces to [37, Corollary A.6].*

We remark also that analogous result to Theorem 4.1 holds for positive sup-solutions of (3.1) with $q(x) < p(x) - 1$.

Theorem 4.5. *Let v be a positive sup-solution of (3.1) in $\Omega \subset M$. Then for every fixed $q(x) < p(x) - 1$, $p(x) > 1$, $\nabla_X v \nabla_X p(x) = 0$, $\nabla_X v \nabla_X q(x) = 0$ and $\lambda \in \mathbb{R}$, we have*

$$\int_\Omega v^{q-p}\phi^p |\nabla_X v|^p dx \le C_{p,q}^{p^-} \int_\Omega v^q |\nabla_X \phi|^p dx + C_{\lambda,p,q} \int_\Omega g(x) v^q \phi^p dx \tag{4.5}$$

for every nonnegative functions $\phi \in C_0^\infty(\Omega)$, where

$$C_{p,q}^{p^-} := \left(\frac{p^+}{p^- - q^+ - 1}\right)^{p^+} \quad \text{and} \quad C_{\lambda,p,q}^- := -\left(\frac{\lambda p^+}{p^- - q^+ - 1}\right).$$

Remark 4.6. *Setting $q = 0$ in (4.5) we obtain a particular case whose right hand side is independent of the nonnegative function v. That is*

$$\int_\Omega |\phi \nabla_X \log v|^p dx \le \left(\frac{p^+}{p^- - 1}\right)^{p^+} \int_\Omega |\nabla_X \phi|^p dx - \left(\frac{\lambda p^+}{p^- - 1}\right) \int_\Omega g(x)\phi^p dx. \tag{4.6}$$

This is the variable exponent logarithmic Caccioppolli inequality. Precisely, If $g(x) \equiv 0$, then (4.6) reduces to a new version of the well known logarithmic Caccioppolli inequality for positive $p(x)$-superharmonic functions

$$\int_\Omega |\phi \nabla_X \log v|^{p(x)} dx \le \left(\frac{p^+}{p^- - 1}\right)^{p^+} \int_\Omega |\nabla_X \phi|^{p(x)} dx,$$

where $1 < p^- < p^+ < \infty$. Note that $v \in W_{loc}^{1,p(x)}$ is said to be $p(x)$-superharmonic if it satisfies $\int_\Omega |\nabla_X u|^{p(x)-2} \langle \nabla_X u, \nabla_X \phi \rangle dx \ge 0$. Interested reader is hereby referred to [38] and [39] for $p(=$ constant)-superharmonic case.

Conflict of Interest

The authors declare that they have no conflict of interests.

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