International Journal of Mathematical Sciences and Optimization: Theory and Applications
10(4), 2024, Pages 70 - 80
https://doi.org/10.5281/zenodo.14710364

On Vector Lyapunov Functions and Uniform Eventual Stability of Nonlinear Impulsive Differential Equations

J. E. Ante [1*], J. O. Achuobi [2], U. P. Akai [1], E. J. Oduobuk [1], A. B. Inyang [1]

1. Department of Mathematics, Topfaith University, Mkpatak.
2. Department of Mathematics, University of Calabar, Calabar, Nigeria.
* Corresponding author: jackson.ante@topfaith.edu.ng*, jonasachuobi@unical.edu.ng, up.akai@topfaith.edu.ng, ej.oduobuk@topfaith.edu.ng, ab.inyang@topfaith.edu.ng

Article Info
Received: 07 July 2024 Revised: 11 November 2024
Accepted: 29 November 2024 Available online: 15 December 2024

Abstract

In this paper, the uniform eventual stability of nonlinear impulsive differential equations with fixed moments of impulse is examined using the vector Lyapunov functions which is generalized by a class of piecewise continuous functions. Together with comparison results, sufficient conditions for the uniform eventual stability are presented. Results obtained extends the more restrictive scalar case in the literature to a more comprehensive framework for uniform eventual stability analysis.

Keywords: Uniform Eventual Stability, Impulsive Differential Equation, Lyapunov Method.
MSC2010: 34A12; 34A37.

1 Introduction

Many evolution processes display characteristics where abrupt state changes occur at specific instances. These processes often experience short-lived disturbances, whose duration are minuscule compared to the entire evolution of the system. As a result, it is reasonable to model such disturbances as instantaneous occurrences, commonly represented by impulses. This impulsive nature is evident in various applications, such as biological events that involve threshold dynamics, bursting rhythm models in medical and biological studies, optimal control models in economic theory, pharmacokinetic models, and frequency-modulated systems [1].

Impulsive differential equations have become central in the study of dynamic systems that are influenced by sudden, transient changes at discrete points in time. These abrupt shifts, such as those seen in mechanical shocks, biological cycles, or economic fluctuations, introduce complexities in stability analysis that traditional methods cannot fully address. In this framework, Lyapunov stability theory has proven to be an essential tool for evaluating system stability, as documented in [2–13]. However, classical approaches often fall short of capturing the unique and complex stability behaviors inherent in nonlinear impulsive differential equations. Therefore, impulsive differential equations offer a robust and realistic framework for modeling complex real-world phenomena that cannot be accurately described by ordinary differential equations. This framework provides an invaluable perspective on the stability and dynamics of systems where impulses significantly influence the system's behavior over time.

As discussed in [14], many perturbation and adaptive control problems involve scenarios where the focus is not on an equilibrium or invariant point but rather on eventually stable sets that are asymptotically invariant. This perspective allows us to view Lyapunov stability as a particular instance within the broader category of eventual stabilities. For decades, there has been significant interest among researchers in examining the qualitative properties of impulsive differential equations, as evidenced by studies in [1, 15–20].

Regarding the stability of perturbed differential equations, results in [26] address the eventual stability of impulsive differential systems under bounded perturbations. In particular, [14] developed sufficient conditions for the preservation of uniform eventual stability in impulsive differential systems with non-fixed impulse moments, utilizing vanishing perturbations and employing piecewise continuous auxiliary functions that generalize traditional Lyapunov functions. Additional advancements include the work in [21], which provided results on the uniform eventual stability for impulsive differential systems with bounded perturbations and non-fixed impulse moments. Further, [27] investigated eventual stability and boundedness for impulsive systems with supremum norms by using a class of piecewise continuous functions analogous to Lyapunov functions and employing the Razumikhin technique. These developments underscore the critical role of eventual stability in impulsive systems, where adaptive control and perturbation factors necessitate a broader analytical approach beyond standard Lyapunov stability.

In this paper, we explore the uniform eventual stability of impulsive differential systems by employing vector Lyapunov functions, generalized through a class of piecewise continuous auxiliary functions. By utilizing these generalized Lyapunov functions alongside comparison results, we establish sufficient conditions for the uniform eventual stability of solutions within these systems. Furthermore, an illustrative example is provided to demonstrate the practical application and effectiveness of the proposed stability criteria.

Our approach introduces a more versatile stability analysis framework that accounts for impulsive effects and non-fixed moments within dynamic systems. This method extends traditional Lyapunov stability theory, offering broader applicability for systems that exhibit complex perturbation behaviors and transient impulses. The results obtained contribute to the theoretical understanding of eventual stability and offer practical insights for control and stability assessment in real-world impulsive systems.

2 Preliminaries, Notations and Definitions

Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space with norm $\|.\|$, and let Ω represent a domain within $\mathbb{R}^n$ that contains the origin. Define the sets $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R} = (-\infty, \infty)$, with an initial time $t_0 \in \mathbb{R}_+$ and $t > 0$.

Consider a subset $J \subset \mathbb{R}_+$. We define the class of functions $PC[J, \Omega]$ as those functions $\alpha : J \to \Omega$ such that $\alpha(t)$ is piecewise continuous with potential points of discontinuity $t_k \in J$, at which $\alpha(t)$ is well-defined.

Now, consider the following impulsive differential system:

$$\begin{aligned} \mu' = \Xi(t, \mu), \quad & t \geq t_0,\ t \neq t_k,\ k = 1, 2, \dots, \\ \Delta\mu = I_k(\mu(t_k)), \quad & k \in \mathbb{N},\ t = t_k, \\ \mu(t_0) = \mu_0, & \end{aligned} \tag{2.1}$$

where $\mu, \mu_0 \in \mathbb{R}^N$, $\Xi : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$, and $t_0 \in \mathbb{R}_+$. Here, $I_k : \mathbb{R}^N \to \mathbb{R}^N$ denotes the impulsive effect at moments t_k.

We analyze this system under the following assumptions:

IJMSO

International Journal of Mathematical Sciences and Optimization: Theory and Applications
10(4), 2024, Pages 70 - 80
https://doi.org/10.5281/zenodo.14710364

(i) (A_0) The impulse moments satisfy $0 < t_1 < t_2 < \cdots < t_k < \ldots$ and $t_k \to \infty$ as $k \to \infty$.

(ii) The function $\Xi : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ is continuous on each interval $(t_{k-1}, t_k]$. For every $\mu \in \mathbb{R}^N$ and $k = 1, 2, \ldots$, the limit $\lim_{(t,y)\to(t_k^+,\mu)} \Xi(t,y) = \Xi(t_k^+, \mu)$ exists.

(iii) Each impulsive function $I_k : \mathbb{R}^N \to \mathbb{R}^N$ is well-defined and maps states at impulse moments.

In this paper, we assume that the function Ξ is Lipschitz continuous with respect to its second argument. Furthermore, we specify that $\Xi(t, 0) \equiv 0$ and $I_k(0) = 0$ for all k, ensuring that the trivial solution exists for system (2.1), see [22, 23]. The impulse times t_k, where $k = 1, 2, \ldots$, are fixed and satisfy $t_0 < t_1 < t_2 < \ldots$ with the property $\lim_{k\to\infty} t_k = \infty$.

The system (2.1), together with the initial condition $\mu(t_0) = \mu_0$, is assumed to have a solution $\mu(t; t_0, \mu_0) \in PC([t_0, \infty), \mathbb{R}^N)$. It is worth noting that sufficient conditions for the existence and uniqueness of global solutions to system (2.1) have been established in [16], [24], [25], [18], and [28].

The second equation in (2.1) is called the impulsive condition, and the function $I_k(\mu(t_k))$ gives the amount of jump of the solution at the point t_k. Let $V : \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}_+^N$. Then V is said to belong to class L if,

(i) V is continuous in $(t_{k-1}, t_k] \times \mathbb{R}^N$ and for each $\mu \in \mathbb{R}^N$, $k = 1, 2, \ldots$ and $\lim_{(t,y)\to(t_k^+,\mu)} V(t,y) = V(t_k^+, \mu)$ exists;

(ii) V is locally Lipschitz with respect to its second argument μ. For $(t, \mu) \in (t_{k-1}, t_k] \times \mathbb{R}^N$, we define the upper right Dini derivative of V with respect to (2.1) as,

$$D^+V(t,\mu) = \limsup_{h\to 0^+} \frac{1}{h}\{V(t+h, \mu + h\Xi(t,\mu)) - V(t,\mu)\} \tag{2.2}$$

Definition 2.1. *A function $\Upsilon \in C[\mathbb{R}^n, \mathbb{R}^n]$ is said to be quasi-monotone non-decreasing in μ, if $\mu \leq y$ and $\mu_i = y_i$ for $1 \leq i \leq n$ implies $\Upsilon_i(\mu) \leq \Upsilon_i(y), \forall i$.*

Definition 2.2. *The set $\mu(t) \equiv 0$ in system (2.1) is defined to be:*

- *$(S1)$ Eventually stable: For every $\epsilon > 0$, there exists a time $T = T(\epsilon) > 0$ and a corresponding $\delta = \delta(t_0, \epsilon)$ for each initial time $t_0 \in \mathbb{R}_+$ and each $\mu_0 \in \mathbb{R}^N$ such that $\|\mu_0\| < \delta$ implies $\|\mu(t; t_0, \mu_0)\| < \epsilon$ for all $t \geq t_0$.*
- *$(S2)$ Eventually uniformly stable: The stability condition in $(S1)$ holds with δ independent of the initial time t_0.*

Definition 2.3. *A function $a(r)$ is defined to belong to the class $\mathcal{K}$ if it satisfies the following conditions: $a \in C[\mathbb{R}_+, \mathbb{R}_+]$, $a(0) = 0$, and $a(r)$ is strictly increasing with respect to r.*

In this paper, we define the following sets:

$$\begin{aligned} \bar{S}_\psi &= \{\mu \in \mathbb{R}^N : \|\mu\| \leq \psi\} \\ S_\psi &= \{\mu \in \mathbb{R}^N : \|\mu\| < \psi\} \end{aligned}$$

Suffice to say here that the inequalities between vectors are understood to be component-wise inequalities.

Alongside (2.1) we shall consider a comparison system of the form

$$\begin{aligned} u' = \Upsilon(t,u), t \geq t_0, t \neq t_k, k = 1, 2, \ldots \\ \Delta u = \psi_k(u(t_k)), k \in N, t = t_k \\ u(t_0^+) = u_0, \end{aligned} \tag{2.3}$$

IJMSO

International Journal of Mathematical Sciences and Optimization: Theory and Applications
10(4), 2024, Pages 70 - 80
https://doi.org/10.5281/zenodo.14710364

existing for $t \geq t_0$, where $u \in \mathbb{R}^n$, $\mathbb{R}_+ = [t_0, \infty)$, $\Upsilon : \mathbb{R}_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n$ continuous in $(t_{k-1}, t_k]$, $\psi_k : \mathbb{R}^n \to \mathbb{R}^n$ and $\Upsilon(t, u)$ is quasimonotone nondecreasing in u, and $\Upsilon(t, 0) \equiv 0$, where Υ is the continuous mapping of $\mathbb{R}_+ \times \mathbb{R}^n_+$ into $\mathbb{R}^n$. The function $\Upsilon \in C[\mathbb{R}_+ \times \mathbb{R}^n_+, \mathbb{R}^n]$ is such that for any initial data $(t_0, u_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, the system (2.3) with initial condition $u(t_0) = u_0$ is assumed to have a solution $u(t; t_0, u_0) \in PC([t_0, \infty), \mathbb{R}^n)$. Note that some sufficient conditions for the existence of solution of (2.3) has been examined in [16], [18], [20] and [28].

Lemma 2.1. *Assume that conditions $A_0(i)$, (ii), (iii) hold, and that $\Xi(t, 0) = 0$ and that $I_k(0) = 0$. Then the interval J can be extended to the maximal interval of existence $[t_0, \infty)$.*

Proof. Given that conditions $A_0(i)$, (ii), and (iii) are satisfied, along with $\Xi(t, 0) = 0$ and $I_k(0) = 0$, it follows from the existence theorem for the impulsive differential equation $\mu' = \Xi(t, \mu(t))$ [18] that the solution $\mu(t) = \mu(t, t_0, \mu_0)$ of the initial value problem (2.1) is well-defined on each interval $(t_{k-1}, t_k]$, where $k = 1, 2, \ldots$.

Furthermore, since $t_0 < t_1 < \ldots$ and $\lim_{k\to\infty} t_k = \infty$, we conclude that the interval J can be extended to $[t_0, \infty)$ for $t \geq t_0$.

□

3 Main Results

In this section, we begin by proving the comparison results, then proceed to establish the necessary conditions for the uniform eventual stability of the set $x(t) \equiv 0$ of impulsive differential systems with fixed moments of impulse.

(Comparison results) Assume that

$\Upsilon \in C[\mathbb{R}_+ \times \mathbb{R}^n_+, \mathbb{R}^n]$ and $\Upsilon(t, u)$ is quasimonotone nondecreasing in u for each $u \in \mathbb{R}^n$ and $\lim_{(t,y)\to(t_k^+,u)} \Upsilon(t, y) = \Upsilon(t_k^+, u)$ exists;

$V \in C[\mathbb{R}_+ \times \mathbb{R}^N, \mathbb{R}^N_+]$ and $V \in L$ such that
$D^+V(t, \mu) \leq \Upsilon(t, V(t, \mu)), t \neq t_k, (t, \mu) \in \mathbb{R}_+ \times \mathbb{R}^N$ and $V(t_k^+, \mu + I_k(\mu(t_k))) \leq \rho_k(V(t, \mu)), t = t_k, \mu \in S_\psi$ and the function $\rho_k : \mathbb{R}^{\mathrm{N}}_+ \to \mathbb{R}^N_+$ is nondecreasing for $k = 1, 2, ...$

Let $r(t) = r(t; t_0, u_0) \in PC([t_0, T], \mathbb{R}^n)$ be the maximal solution of the initial value problem (IVP) for the impulsive differential equation (IDE) (2.3) existing on $[t_0, \infty)$.
Then,

$$V(t, \mu(t)) \leq r(t), t \geq t_0 \tag{3.1}$$

where $\mu(t) = \mu(t; t_0, \mu_0) \in PC([t_0, T], \mathbb{R}^N)$ is any solution of (2.1) existing on $[t_0, \infty)$, provided that

$$V(t_0^+, \mu_0) \leq u_0. \tag{3.2}$$

Proof. Let $\mu(t) = \mu(t, t_0, \mu_0)$ be any solution of (2.1) existing on $t \geq t_0$, such that $V(t_0^+, \mu_0) \leq u_0$.

Set $m(t) = V(t, \mu(t))$ for $t \neq t_k$ so that for small $h > 0$, we have

$$m(t + h) - m(t) = V(t + h, \mu(t + h)) - V(t, \mu)$$

$$m(t+h)-m(t) = V(t+h, \mu(t+h))-V(t+h, \mu(t)+h\Xi(t, \mu(t)))+V(t+h, \mu(t)+h\Xi(t, \mu(t)))-V(t, \mu)$$

Since $V(t, \mu)$ is locally Lipschitzian in μ for $t \in (t_k, t_{k+1}]$, we have

$$m(t + h) - m(t) \leq k\|\mu(t + h) - (\mu(t) + h\Xi(t, \mu(t)))\|e + V(t + h, \mu(t) + h\Xi(t, \mu(t))) - V(t, \mu)$$

International Journal of Mathematical Sciences and Optimization: Theory and Applications
10(4), 2024, Pages 70 - 80
https://doi.org/10.5281/zenodo.14710364

Dividing by $h > 0$ and taking the limsup as $h \to 0^+$ we have

$$\limsup_{h\to 0^+} \frac{1}{h}[m(t+h) - m(t)] \le \limsup_{h\to 0^+} \frac{1}{h}[k\|\mu(t+h) - \mu(t) - h\Xi(t,\mu(t))\|]e + \limsup_{h\to 0^+} \frac{1}{h}[V(t+h,\mu(t)+h\Xi(t,\mu(t))) - V(t,\mu)]$$

where k is the local Lipschitz constant and $e = (1, 1, ...1)^T$

$$D^+m(t) = D^+V(t,\mu(t)) \le \Upsilon(t, V(t,\mu(t)))$$

Using condition (ii) of Theorem 3.1 we arrive at

$$V(t,\mu(t)) \le r(t), t \ne t_k \qquad (3.3)$$
$$V(t_0^+,\mu_0) \le u_0$$

Also,

$$m(t_k^+) = V(t_k^+, \mu(t_k) + I_k(\mu(t_k^+))) \le \psi_k(m(t_k^+)) \qquad (3.4)$$

Hence, by Cor. 1.7.1 in [17], we obtain the desired estimate of 3.1. □

Corollary 3.1. *Assume that*

(i) $\Upsilon \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$ *and* $\Upsilon(t,u)$ *is quasimonotone nondecreasing in* u *for each* $u \in \mathbb{R}^n$ *and* $\lim_{(t,y)\to(t_k^+,u)} \Upsilon(t,y) = \Upsilon(t_k^+,u)$ *exists;*

(ii) $V \in C[\mathbb{R}_+ \times \mathbb{R}^N, \mathbb{R}_+^N]$ *and* $V \in L$ *such that* $D^+V(t,\mu) \ge \Upsilon(t, V(t,\mu)), t \ne t_k, (t,\mu) \in \mathbb{R}_+ \times \mathbb{R}^N$ *and* $V(t, \mu + I_k(\mu(t_k))) \ge \rho_k(V(t,\mu)), t = t_k, \mu \in S_\psi$ *and the function* $\rho_k : \mathbb{R}_+^{\mathrm{N}} \to \mathbb{R}_+^N$ *is nondecreasing for* $k = 1, 2, ...$

(iii) Let $p(t) = p(t; t_0, u_0) \in PC([t_0, T], \mathbb{R}_+^N)$ *be the minimal solution of the IVP for the IDE* (2.3) *existing on* $[t_0, \infty)$.

Then,

$$V(t,\mu(t)) \ge p(t), t \ge t_0 \qquad (3.5)$$

where $\mu(t) = \mu(t; t_0, \mu_0) \in PC([t_0, T], \mathbb{R}^N)$ *is any solution of* (2.1) *existing on* $[t_0, \infty)$*, provided that*

$$V(t_0^+, \mu_0) \ge u_0. \qquad (3.6)$$

Proof. Let $\mu(t) = \mu(t, t_0, \mu_0)$ be any solution of (2.1) existing on $t \ge t_0$, such that $V(t_0^+, \mu_0) \ge u_0$.

Set $m(t) = V(t, \mu(t))$ for $t \ne t_k$ so that for small $h > 0$, we have

$$m(t+h) - m(t) = V(t+h, \mu(t+h)) - V(t,\mu)$$

$$m(t+h) - m(t) = V(t+h, \mu(t+h)) - V(t+h, \mu(t)+h\Xi(t,\mu(t))) + V(t+h, \mu(t)+h\Xi(t,\mu(t)))) - V(t,\mu)$$

Since $V(t,\mu)$ is locally Lipschitzian in μ for $t \in (t_k, t_{k+1}]$, we have

$$m(t+h) - m(t) \ge k\|\mu(t+h) - (\mu(t) + h\Xi(t,\mu(t)))\|e + V(t+h, \mu(t)+h\Xi(t,\mu(t))) - V(t,\mu)$$

Dividing by $h > 0$ and taking the limsup as $h \to 0^+$ we have

$$\limsup_{h\to 0^+} \frac{1}{h}[m(t+h) - m(t)] \ge \limsup_{h\to 0^+} \frac{1}{h}[k\|\mu(t+h) - \mu(t) - h\Xi(t,\mu(t))\|]e + \limsup_{h\to 0^+} \frac{1}{h}[V(t+h,\mu(t)+h\Xi(t,\mu(t))) - V(t,\mu)]$$

where k is the local Lipschitz constant and $e = (1, 1, ...1)^T$

$$D^+m(t) = D^+V(t, \mu(t))$$

Using condition (ii) of Cor 3.2 we arrive at

$$D^+V(t, \mu) \geq \Upsilon(t, m(t)), t \neq t_k m(t_0^+) \geq u_0 \tag{3.7}$$

Also,

$$m(t_k^+) = V(t_k^+, \mu(t_k) + I_k(\mu(t_k^+))) \leq \psi_k(m(t_k^+)) \tag{3.8}$$

Hence, by Cor. 1.7.1 in [17], we obtain the desired estimate of 3.5. □

[Uniform Stability] Assume the following

(i) $\Upsilon \in C[\mathbb{R}_+ \times \mathbb{R}_+^n, \mathbb{R}^n]$ satisfies $(A_0)(ii)$ and $\Upsilon(t, u)$ is quasi-monotone non-decreasing in u with $\Upsilon(t, 0) \equiv 0$.

(ii) $V : \mathbb{R}_+ \times S_\psi \to \mathbb{R}_+^N$, $V \in L$ is locally Lipschitzian in μ with $V(t, 0) \equiv 0$ such that

$$D^+V(t, \mu) \leq \Upsilon(t, V(t, \mu)), t \neq t_k, (t, \mu) \in \mathbb{R}_+ \times S_\psi \tag{3.9}$$

holds for all $(t, \mu) \in \mathbb{R}_+ \times S_\psi$.

(iii) there exists a $\psi_0 > 0$ such that $\mu_0 \in S_\psi$ implies that

$$\mu + I_k(\mu) \in S_\psi \textit{ and } V(t_k^+, \mu + I_k(\mu(t_k))) \leq \psi_k(V(t, \mu)), t = t_k, \mu \in S_\psi$$

and the function $\psi_k : \mathbb{R}_+^N \to \mathbb{R}_+^N$ is nondecreasing for $k = 1, 2, ...$

(iv) $b(\|\mu\|) \leq V_0(t, \mu) \leq a(\|\mu\|)$, where $a, b \in \mathcal{K}$ and $V_0(t, \mu) = \sum_{i=1}^N V_i(t, \mu)$

Then the uniform eventual stability of the set $u(t) \equiv 0$ of the IDE (2.3) implies the uniform eventual stability of the set $\mu(t) \equiv 0$ of (2.1).

Proof. Let $0 < \epsilon < \psi$ and $t_0 \in \mathbb{R}_+$ be given.

Assume that the solution $u = 0$ of (3.5) is uniformly stable. Then, for each $b(\epsilon) > 0$ and $t_0 \in \mathbb{R}_+$, there exists a positive function $\delta_1 = \delta_1(\epsilon) > 0$ such that, whenever

$$u_0 = \sum_{i=1}^n u_{i0} \leq \delta, \text{ we have } \sum_{i=1}^n u_i(t; t_0, u_0) < b(\epsilon), \quad t \geq t_0, \tag{3.10}$$

where $u(t; t_0, u_0)$ represents any solution of (3.5).
Let us choose $V(t_0^+, \mu_0) \leq u_0$ and

$$\sum_{i=1}^n u_{i0} = a(t_0, \|\mu_0\|)$$

Since $a(t, \mathcal{K})$ and $a \in C[\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+]$ we can find a positive function $\delta = \delta(t_0, \epsilon) > 0$ such that

$$a(t_0, \|\mu_0\|) < \delta_1 \textit{ and } \|\mu_0\| < \delta \tag{3.11}$$

hold simultaneously. We claim that if

$$\|\mu_0\| \leq \delta, \textit{ then } \|\mu(t, t_0, \mu_0)\| \leq \epsilon, t \geq t_0.$$

Assume, for contradiction, that this claim is false. Then, there would exist a point $t_1 > t_0$ and a solution $\mu(t)$ with $\|\mu_0\| < \delta$ such that

International Journal of Mathematical Sciences and Optimization: Theory and Applications
10(4), 2024, Pages 70 - 80
https://doi.org/10.5281/zenodo.14710364

$$\|\mu(t_1)\| = \epsilon \; and \; \|\mu(t)\| < \epsilon, \; for \; t \in [t_0, t_1). \tag{3.12}$$

This implies that $\mu(t) + I_k(\mu(t_k)) \in S_\psi$ for $t \in [t_0, t_1)$.
From equation (3.1) we have that

$$V_0(t, \mu(t)) \leq r_0(t, t_0, u_0) \; for \; t \in [t_0, t_1). \tag{3.13}$$

Combining condition (iv) and (3.13) we have

$$b(\epsilon) \leq \sum_{i=1}^{n} V_i(t_1, \mu(t_1)) \leq \sum_{i=1}^{n} r_i(t; t_0, u_0) \tag{3.14}$$

Using equations (3.14), (3.13) and (3.10) we have,

$$b(\epsilon) \leq \sum_{i=1}^{n} V_i(t_1, \mu(t_1)) \leq \sum_{i=1}^{n} r_i(t; t_0, u_0) < b(\epsilon)$$

which leads to an absurdity that $b(\epsilon) < b(\epsilon)$.
Hence, the uniform eventual stability of the set $u(t) = 0$ of (2.3) implies the uniform eventual stability of $\mu(t) = 0$ of (2.1). □

4 Application

Let the points $t_k, t_k < t_{k+1}$, $lim_{k\to\infty} t_k \to \infty$ be fixed. Consider the vector impulsive differential equations

$$\begin{aligned} x_1' &= -4x_1 + x_2 sinx_1 + x_1 secx_2, t \neq t_k \\ x_2' &= x_1 cosx_2 - 2x_2 secx_1 - x_2 sinx_1, t \neq t_k \\ \Delta x_1 &= c_k, \Delta x_2 = d_k, k = 1, 2, .. \end{aligned} \tag{4.1}$$

for $t \geq t_0$, with initial conditions

$$x_1(t_0^+) = x_0 \quad \text{and} \quad x_2(t_0^+) = x_0$$

Consider a vector $V = (V_1, V_2)^T$, where
$V_1(t, x_1, x_2) = |x_1|$ and $V_2(t, x_1, x_2) = |x_2|$, with $x = (x_1, x_2) \in \mathbb{R}^2$, and its associated norm defined by $\|x\| = |x_1| + |x_2|$.
Now

$$V_0(t, x) = \sum_{i=1}^{2} V_i(t, x_1, x_2) = |x_1| + |x_2| \tag{4.2}$$

and so $b(\|x\|) \leq V_0(t, x) \leq a(\|x\|)$ with $b(r) = r$ and $a(r) = r^2$, implying that $a, b \in \mathcal{K}$. From (2.2) we compute the Dini derivative for $V_1 = |x_1|$ for $t > 0, t \neq t_k$ as follows:

$$D^+V_1(t; x_1, x_2) = \limsup_{h\to 0^+} \frac{1}{h}\big\{V(t + h, x + hf(t, x)) - V(t, x)\big\}$$

$$D^+V_1 \quad = \quad \limsup_{h\to 0^+} \frac{1}{h}\big\{|x_1 + hf_1(t, x_1)| - |x_1|\big\} \tag{4.3}$$

IJMSO

International Journal of Mathematical Sciences and Optimization: Theory and Applications
10(4), 2024, Pages 70 - 80
https://doi.org/10.5281/zenodo.14710364

$$
\begin{aligned}
&= \limsup_{h\to 0^+} \frac{1}{h}\{hf_1(t,x_1)\} \\
&\le f_1(t,x_1) \\
&= -4x_1 + x_2 sinx_1 + x_1 secx_2 \\
&= x_1(-4 + secx_2) + x_2 sinx_1 \\
&\le |x_1|(-4 + \frac{1}{|cosx_2|}) + |x_2|(|sinx_1|)
\end{aligned}
$$

where trigonometric functions are bounded by 1

$$
\le \quad |x_1|(-4+1) + |x_2|(1)
$$

$$
D^+V_1 \quad \le \quad -3V_1 + V_2 \tag{4.4}
$$

Also, for $x_0 \in S_\psi$, for $t = t_k, k = 1, 2, ..,$ we have

$$
V(t, x(t) + c_k) = |c_k + x(t)| \le V(t, x(t))
$$

Again, for $V_2 = |x_2|, wehave$

$$
\begin{aligned}
D^+V_2 &\le f_2(t,x_2) \\
&= x_1 cosx_2 - 2x_2 secx_1 - x_2 sinx_1 \\
&= x_1 cosx_2 - x_2(2secx_1 - sinx_1) \\
&\le |x_1|(|cosx_2|) - |x_2|(\frac{2}{|cosx_1|} + |sinx_1|) \\
&\le |x_1| - |x_2|(2+1)
\end{aligned}
$$

$$
D^+V_2 \quad \le \quad V_1 - 3V_2 \tag{4.5}
$$

Also, for $x_0 \in S_\psi$, for $t = t_k, k = 1, 2, ..,$ we have

$$
V(t, x(t) + d_k) = |d_k + x(t)| \le V(t, x(t))
$$

Combining (4.5) and (4.4) we have

$$
D^+V \le \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = g(t, V) \tag{4.6}
$$

Now consider the comparison system

$$
u' = g(t, u) = Au \tag{4.7}
$$

where $A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$.
The vectorial inequality (4.7) and all other conditions of Theorem (3) are satisfied since the eigenvalues of A are all negative real parts. Hence, the system (4.1) is uniformly eventually stable . Therefore, the set $x(t) \equiv 0$ is uniformly eventually stable.

5 Conclusion

In this paper, the uniform eventual stability of impulsive differential system is examined by employing the vector Lyapunov functions which is generalized by a class of piecewise continuous auxilliary functions. Together with comparison results, sufficient conditions for the uniform eventual stability solution is established with illustrative example.

6 Conflicts of interest

The authors declare that there is no conflicts of interest.

References

[1] Lakshmikantham, V., Bainov, D. D., P. S. Simeonov. (1989). The Theory of Impulsive Differential Equations. *Series in Modern Applied Mathematics*, (6).

[2] Achuobi, J.O., Akpan, E.P., George, R., & Ofem, A.E. (2024). Stability analysis of Caputo fractional time-dependent systems with delay using vector lyapunov functions, *A*IMS Mathematics. **9**:28079-28099 . doi: 10.3934/math.20241362.

[3] Adeyanju, A. A., Oni, S. A., Adams, D. O., Oni, S. A., & Olutimo, A. L. (2023). On the stability, boundedness and asymptotic behaviour of solutions of differential equations of third-order. International Journal of Mathematical Sciences and Optimization: Theory and Applications, **9**(2):121 - 132. https://doi.org/10.6084/zenodo.10384294

[4] Ante, J.E., Itam, O.O., Atsu, J.U., Essang, S.O., Abraham, E.E. & Ineh, M. P. (2024). On the novel auxiliary Lyapunov function and uniform asymptotic practical stability of nonlinear impulsive Caputo fractional differential equations via new modelled generalized Dini derivative, *A*frican Journal of Mathematics and Statistics Studies. **7**:11-33.

[5] Ante, J.E., Atsu, J.U., Abraham, E.E., Itam, O.O., Oduobuk, E.J. & Inyang, A. B. (2024). On a class of piecewise continuous Lyapunov functions and asymptotic practical stability of nonlinear impulsive Caputo fractional differential equations via new modelled generalized Dini derivative, *I*EEE-SEM Journal Publication. **12**:2320-9151.

[6] Ante, J. E., Atsu, J. U., Maharaj, A., Abraham, E. E., and Narain, O. K. (2024). On a class of piecewise continuous Lyapunov functions and uniform eventual stability of nonlinear impulsive Caputo fractional differential equations via new modelled generalized Dini derivative, *Asia Pac. J. Math*, **11**(99):1-20. https://apjm.apacific.org/PDFs/11-99.pdf.

[7] Ante, J.E., Inyang, A.B., Oduobuk, E.J., and Akai, U.P. (2024). On the vector Lyapunov functions and eventual stability of nonlinear impulsive differential equations, *International Journal of Mathematical Analysis and Modelling*, **7**(2):185–199.

[8] Atsu, J. U., Ante, J. E., Inyang, A . B., & Akpan, U. D. (2024). A survey on the vector Lyapunov functions and practical stability of nonlinear impulsive Caputo fractional differential equations via new modelled generalized Dini derivative. *IOSR Journal of Mathematics* **20**:28-42.

[9] Bishop, S. A., & Nnubia, A. C. (2021). Stability of Nonlocal Stochastic Volterra Equations. International Journal of Mathematical Sciences and Optimization: Theory and Applications, **7**(2):48-55. https://doi.org/10.52968/28302767

[10] Igobi, D. K., Ndiyo, E., & Ineh, M. P. (2023). Variational stability results of dynamic equations on time-scales using generalized ordinary differential equations. *World Journal of Applied Science & Technology*, **15**(2):245-254.

[11] Ineh, M.P., Achuobi, J.O., Akpan, E.P., & Ante, J.E. (2024). CDq On the uniform stability of Caputo fractional differential equations using vector Lyapunov functions. *The Journals of the Nigerian Association of Mathematical Physics*, **68**:51-64.

[12] Ineh, M.P., & Akpan, E.P. (2024). Lyapunov uniform asymptotic stability of Caputo fractional dynamic equations on time scale using a generalized derivative, *The Transactions of the Nigerian Association of Mathematical Physics*, **20**:117-132.

[13] Michael Precious Ineh, Edet Peter Akpan,& Hossam A. Nabwey (2024). A novel approach to Lyapunov stability of Caputo fractional dynamic equations on time scale using a new generalized derivative. *AIMS Mathematics*, **9** (12):34406-34434. Doi: 10.3934/math.20241639

[14] Sood, A. & Srvastava, S. K. (2016). On retention of Eventual Stability of Perturbed Impulsive Differential Systems. *J. Math. fund. Sci*, **48**(1):1-11.

[15] Akpan, E. P., & Akinyele, O. (1992). On the ϕ_0-stability of comparison differential systems. *Journal of mathematical analysis and applications,* **164**(2):307-324.

[16] Igobi, D. K., Ante, J. E. (2018). Results on Existence and Uniqueness of Solution of Impulsive Neutral Integro-Differential System. *Journal of Mathematics Research*, **10**(4). Doi: 10.5539/jmr.v10n4p165

[17] Lakshmikantham, V., and Leela, S. (1969). *Differential and Integral Inequalities. Theory and Application.* **1**.

[18] Lipcsey, Z., Ugboh, J. A., Esuabana, I. M. & Isaac, I. O. (2020). Existence Theorem for Impulsive Differential Equations with Measurable Right Side for Handling Delay Problems. *Hindawi Journal of Mathematics*, **17**.

[19] Srivastava, S. K. and Amanpreet, K. A. (2009). New Approach to Stability of Impulsive Differential Equations. *Int. Journal of Math. Analysis*, **3**(4):179-185.

[20] Ugboh, J. A. and Esuabana, I. M. (2018). Existence and Uniqueness Result for a Class of Impulsive Delay Differential Equations. *International Journal of Chemistry, Mathematics and Physics*, **2**(4).

[21] Kulev, G. K. & Bainov, D. D. (1988). Stability of Systems with Impulses by the Direct Method of Lyapunov, *Bull Austral. Math. Soc.*, **38**:113-123.

[22] Ante, J. E., Essang, S. O., Itam, O. O. & John, E. I. (2024). On the Existence of Maximal Solution and Lyapunov Practical Stability of Nonlinear Impulsive Caputo Fractional Derivative via Comparison Principle. *Advanced Journal of Science, Technology and Engineering*, **4**(4):92-110.

[23] Ante, J.E., Abraham, E. E., Ebere, U.E., Udogworen, W.K., and Akpan, C.S. (2024). On the Global Existence of Solution and Lyapunov Asymptotic Practical stability for Nonlinear Impulsive Caputo Fractional Derivative via Comparison Principle, *Sch J Phys Math Stat*, **11**:160–172.

[24] Lakshmikantham, V., Leela, S., & Vasundhara Devi, J. (2009). Theory of Fractional Dynamic Systems. *Cambridge Scientific Publishers.*

[25] Lakshmikantham, V., Vatsala, A. S. (2007). General Uniqueness and Monotone Iterative Technique for Fractional Differential Equationss. *Applied mathematics Letters* **21**:828-834.

[26] Soliman, A. A. (2003). Stability Criteria of Impulsive Differential Systems. *Applied Mathematics and Computation*, **134**(2-3):445-457.

[27] Stamova, I. (2011). Eventual Stability and Eventual Boundedness for Impulsive Differential Equations with "Supremum". *Mathematical Modelling and Analysis,* **16**(2):304-314.

[28] Wu, C. (2020). A General Comparison Principle for Caputo Fractional-order Ordinary Differential Equations. *Fractals*, **28**(4):2050070.