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Note On Hyers-Ulam Stability Criteria for Third Order Nonlinear Differential Equations with Forcing Term

I.Fakunle [1*], P.O. Arawomo [2], A.B.Ipinlaye [1], B.N.Akintewe [1]

1. Department of Mathematics, Adeyemi Federal University of Education, Ondo, 35005, Nigeria.
2. Department of Mathematics, University of Ibadan, Ibadan,200005, Nigeria.
* Corresponding author: fakunlesanmi@gmail.com*, ipinlayeab@aceondo.edu.ng, akintewebn@ace.edu.ng, womopeter@gmail.com

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Abstract

The stability of the ordinary differential equations has been investigated and the investigation is ongoing. In this paper we are concerned with note on Hyers-Ulam stability(HUs) criteria for third order nonlinear differential equations with forcing term. The third order nonlinear differential equations invesgated were transformed to integral equation, then, applied Bihari inequality and Gronwall-Bellman-Bihari(GBB) type inequality to arrive at our results. New criteria were established to prove HUs of nonlinear third order differential equations. Finally, examples are given to illustrate correctness our results.

Keywords: Forcing Term, Integral Inequality, New Criteria, Hyers-Ulam Stability, Nonlinear Differential Equation.
MSC2010: 65R20, 47N10.

1 Introduction

The equations of interest in this paper,are the following third order nonlinear differential equations:

$$\begin{aligned} &u'''(t) + R_1(t, u(t), u'(t))u''(t) + R_2(t, u(t), u'(t))u'(t) \\ &\quad +p(t)\gamma(u(t)) + Q(t, u(t)) = P(t, u(t), u'(t)) \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} &u'''(t) + \beta(t)f(u(t))u''(t) + \alpha(t)g(u(t))u'(t) \\ &\quad +p(t)\gamma(u(t)) = P(t, u(t), u'(t)) \end{aligned} \tag{1.2}$$

on setting initial conditions as

$$u(t_0) = u'(t_0) = u''(t_0) = 0, \tag{1.3}$$

where $R_1(t_0, 0, 0) = 0$, $R_2(t_0, 0, 0) = 0$, $Q(t_0, 0) = 0$, $P(t_0, 0, 0) = 0$, R_1, $R_2, P \in C(\mathbf{I} \times \mathbf{R^2}, \mathbf{R})$, $Q \in C(\mathbf{I} \times \mathbf{R}, \mathbf{R})$, $g, f, \gamma \in C(\mathbf{R}_+, \mathbf{R}_+)$, $\mathbf{I} = (0, \infty), \mathbf{R}_+ = [0, \infty), \mathbf{R} = (-\infty, \infty)$. Several assumptions are given as follow:

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i $P(t, u(t), u'(t)) = \phi(t)\kappa(u(t))h(u'(t)^4)$,

ii $R_1(t, u(t), u'(t)) = \alpha(t)\nu(u(t))u'(t)^n$, where $n \in \mathbf{N}$,

iii $R_2(t, u(t), u'(t)) = a(t)u'(t)^2 + b(t)\psi(u'(t))$,

iv $Q(t, u(t)) = v(t)r(t)\varpi(u(t))$, where $\phi, \alpha, a, b, v, r \in C(\mathbf{R}_+)$, $\kappa, \gamma, \psi, h, \varpi \in C(\mathbf{R}_+)$.

Ulam [32] in 1940, gave a wide range of talk before the Mathematics Club of the University of Winsconsin in which he discussed a number of important unsolved problems. A year later, the solution to this question was given by Hyers [14] for additive functions defined on Banach space in 1941. Later, the result of Hyers [14] generalised by Rassias [26] in 1978.

Alsina and Ger [4] in 1988 were the first authors who investigated the HUs of the first order linear differential equation

$$u'(t) = u(t). \tag{1.4}$$

This result of Alsina and Ger has been generalised by Takahasi*et al* [29]. Takahasi*et al* [29] investigated that the HUs holds for the first order differential equation

$$u(t) = \lambda u(t). \tag{1.5}$$

Miura *et al* [23] proved the HUs of linear differential equation of the form

$$u'(t) + g(t)u(t) = 0. \tag{1.6}$$

Jung [17] obtained the HUs of linear differential equations of the form

$$\varphi(t)u'(t) = u(t). \tag{1.7}$$

Jung [15] investigated the HUs of the nonhomogenous linear differential equation of fist order

$$u'(t) + p(t)u(t) + q(t) = 0. \tag{1.8}$$

From this work, the author improved the result of Jung [17] and Miura [23]. Jung [16] proved the HUs of the differential equations of the form

$$tu'(t) + \alpha u(t) + \beta t^r x_0 = 0. \tag{1.9}$$

and

$$t^2u''(t) + \alpha tu'(t) + \alpha u(t) + \beta u(t) = 0. \tag{1.10}$$

Wang*et at* [33] investigated the HUs of the first order nonhomogenous linear differential equation

$$p(t)u'(t) + q(t)u(t) + r(t) = 0. \tag{1.11}$$

Li [19] proved the HUs to the differential equation of the form

$$u''(t) + \lambda^2 u(t). \tag{1.12}$$

Li and Shen [18] proved the stability of the homogenenous linear differential equation of second order

$$u''(t) + \alpha u(t) + \beta u(t) = 0 \tag{1.13}$$

and

$$u''(t) + \alpha u(t) + \beta u(t) = f(t) \tag{1.14}$$

in the sense of Hyers-Ulam.

Furthermore, the following authors investigated the HUs of the third order linear differential equations. These include: Abdollahpou*et al* [1], Murali and PonmanaSelva [21], [22], Tunc and Bicer [30] and Tripathy and Satapathy [31]

The following authors went further in their discussion on HUs of nonlinear differential equations these authors include Rus [27], [28], Qarawani [24], [25], Algfiary and Jung [3], Fakunle [8], [9], [10], [11], [12], [13]. However,author such as Adeyanju *et.al,* [2] approached the proof of stability of differential equations of third order by constructing a complete Lyapunov function, while Bishop and Nnubia [6] investigated the stability of nonlocal stochastic Volterra equations through the sense of Hyers-Ulam-Rassias. Bishop and Nnubia employed Gronwall lemma to established their result.

Being motivated by the works of Fakunle and Arawomo [8],Bishop and Nnubia [6] and other papers listed in the literature, we now study the HUs of equation (1.1) and (1.2) using Gronwall-Bellman-Bihari type inequality.

2 preliminaries

The following definitions, lemmas and theorems are necessary for our results

Definition 2.1. Equation (1.1) has the HUs with the initial condition (1.3)if there exists a positive constant $K > 0$ with the following properties: For every $\epsilon > 0$, $u(t) \in C^2(\mathbf{R}_+)$ where t is sufficiently large in $\mathbf{I}$

$$\begin{aligned} |u'''(t) + R_1(t,u(t),u'(t))u'' + R_2(t,u(t),u'(t))u'(t) + q(t)\gamma(u(t)) \\ +Q(t,u(t)) - P(t,u(t),u'(t))| \leq \epsilon, \end{aligned} \tag{2.1}$$

then, there exists some solutions $u_0(t) \in C^2(\mathbf{R}_+)$ of equation (1.1) such that

$$|u(t) - u_0(t)| \leq K\epsilon$$

and satisfies the initial conditions (1.3).

Definition 2.2. The differential equation (1.2) has the HUs with initial condition (1.3), if there exists a positive constant $K > 0$ with the following property: for every $u(t) \in C^2(\mathbf{R}_+)$, which satisfies

$$\begin{aligned} |u'''(t) + \beta(t)f(u(t))u''(t) + \alpha(t)g(u(t))u'(t) \\ +p(t)\gamma(u(t)) - P(t,u(t),u'(t))| \leq \epsilon, \end{aligned} \tag{2.2}$$

then there exists a function $u_0(t) \in C^2(\mathbf{R}_+)$ satisfies (1.2) with initial conditions (1.3) such that

$$|u(t) - u_0(t)| \leq K\epsilon,$$

we call such K a Hyers-Ulam constant(HUc) for the differential equation.

Definition 2.3. A function $\omega : [0,\infty) \to [0,\infty)$ is said to belong to a class Ψ if

i $\omega(u)$ is nondecreasing and continuous for $u \geq 0$

ii $(\frac{1}{v})\omega(u) \leq \omega(\frac{u}{v})$ for all u and $v \geq 1$.

iii there exists a function ϕ, continuous on $[0,\infty)$ with $\omega(\alpha u) \leq \phi(\alpha)\omega(u)$ for $\alpha \geq 0$

Lemma 2.4. [5] Let $u(t)$, $f(t)$ be positive continuous functions defined on $t_0 \leq t \leq b, (\leq \infty)$ and $K > 0$, $M \geq 0$, further let $\omega(u)$ be a nonnegative, nondecreasing continuous function for $u \geq 0$, then the inequality

$$u(t) \leq K + M\int_{t_0}^{t} f(s)\omega(u(s))ds, \ \ t_0 \leq t < b, \tag{2.3}$$

implies the inequality

$$u(t) \leq \Omega^{-1}\left(\Omega(k) + M\int_{t_0}^{t} f(s)ds\right), \ \ t_0 \leq t \leq b' \leq b, \tag{2.4}$$

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where

$$\Omega(u) = \int_{u_0}^{u} \frac{dt}{\omega(t)}, \quad 0 < u_0 < u. \tag{2.5}$$

In the case $\omega(0) > 0$ or $\Omega(0+)$ is finite, one may take $u_0 = 0$ and Ω^{-1} is the inverse function of Ω and t must be in the subinterval $[t_0, b']$ of $[t_0, b]$ such that

$$\Omega(k) + M \int_{t_0}^{t} f(s)ds \in Dom(\Omega^{-1}).$$

Theorem 2.5. [20] If $f(t)$ and $g(t)$ are continuous in $[t_0, t] \subseteq \mathbf{I}$ and $f(t)$ does not change sign in the interval, then there is a point $\xi \in [t_0, t]$ such that $\int_{t_0}^{t} g(s)f(s)ds = g(\xi)\int_{t_0}^{t} f(s)ds$

Theorem 2.6. [8, 10] Suppose $u(t), r(t), h(t) \in C(\mathbf{I}, \mathbf{R}_+)$ and $\varpi(u), \beta(u) \in \Psi$ are nonnegative, monotonic, nondecreasing, continuous and $\omega(u)$ be submultiplicative for $u > 0$. Let

$$u(t) \leq K + T\int_{t_0}^{t} r(s)\beta(u(s))ds + L\int_{t_0}^{t} h(s)\varpi(u(s))ds \tag{2.6}$$

for K, T and L positive constants, then

$$\begin{aligned} u(t) \leq \Omega^{-1}\left(\Omega(K) + L\int_{t_0}^{t} h(s)\varpi\left(F^{-1}\left(F(1) + T\int_{t_0}^{s} r(\alpha)d\alpha\right)\right)ds\right) \\ F^{-1}\left(F(1) + T\int_{t_0}^{t} r(s)ds\right) \end{aligned} \tag{2.7}$$

where $\beta(u) \neq \varpi(u)$, Ω is defined in equation (2.5) and $F(u)$ is defined as

$$F(u) = \int_{u_0}^{u} \frac{ds}{\beta(s)}, \quad 0 < u_0 \leq u, \tag{2.8}$$

F^{-1}, Ω^{-1} are the inverses of F, Ω respectively and t is in the subinterval $(0, b) \in \mathbf{I}$ so that

$$F(1) + T\int_{t_0}^{t} r(s)ds \in Dom(F^{-1})$$

and

$$\Omega(K) + L\int_{t_0}^{t} h(s)\varpi\left(F^{-1}\left(F(1) + T\int_{t_0}^{t} r(\alpha)d\alpha\right)\right)ds \in Dom(\Omega^{-1})$$

Corollary 2.7. [8,10] Suppose $\rho(t)$ is a nonnegative, monotonic, nondecreasing continuous function on $\mathbf{R}_+$. Let

$$u(t) \leq \rho(t) + T\int_{t_0}^{t} r(s)\beta(u(s))ds + L\int_{t_0}^{t} h(s)\varpi(u(s))ds, \tag{2.9}$$

for T and L be positive constants, then

$$\begin{aligned} u(t) \leq \rho(t)\Omega^{-1}\left(\Omega(1) + L\int_{t_0}^{t} h(s)\varpi\left(F^{-1}\left(F(1) + T\int_{t_0}^{t} r(\alpha)d\alpha\right)\right)ds\right) \\ F^{-1}\left(F(1) + T\int_{t_0}^{t} r(s)ds\right), \quad t \in \mathbf{I}, \end{aligned} \tag{2.10}$$

where $\Omega(u)$ and $F(u)$ are defined as in (2.5) and (2.8) respectively.

Theorem 2.8. [8,10] If $u(t), r(t), h(t), \rho(t), g(t) \in C(\mathbf{R}_+)$ and $\omega, f, \gamma \in \Psi$ be nonnegative, monotonic, nondecreasing continuous functions. Let γ be submultiplicative. If

$$u(t) \leq \rho(t) + A\int_{t_0}^{t} r(s)\beta(u(s))ds + B\int_{t_0}^{t} h(s)\varpi(u(s))ds + L\int_{t_0}^{t} g(s)\gamma(u(s))ds \tag{2.11}$$

for $K, A, B, L > 0$, then

$$u(t) \leq \rho(t)\Upsilon^{-1} \left[\Upsilon(1) + L\int_{t_0}^{t} g(s)\gamma\left[\Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{s} h(\alpha)\varpi\left(T(\alpha)\right)d\alpha\right)T(s)\right]ds\right] \Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{t} h(s)\varpi\left(T(s)\right)ds\right)T(t) \tag{2.12}$$

where $T(t)$ is given as

$$T(t) = F^{-1}\left(F(1) + A\int_{t_0}^{t} r(s)ds\right) \tag{2.13}$$

and

$$\Upsilon(r) = \int_{r_0}^{r} \frac{ds}{\gamma(s)}, \quad 0 < r_0 \leq r, \tag{2.14}$$

and F^{-1}, Ω^{-1} and Υ^{-1} are the inverses of F, Ω, Υ respectively $t \in (0, b) \subset (I)$. So that

$$\Upsilon(1) + L\int_{t_0}^{t} g(s)\gamma\left[\Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{s} h(\alpha)\varpi\left(T(\alpha)\right)d\alpha\right)T(s)\right]ds \in Dom(\Upsilon^{-1}$$

Remark 2.9. Lemma 2.4 is known as Bihari inequality while Theorem 2.6, Corollary 2.7 and Theorem 2.8 are called GBB type inequalities. Theorem 2.6, Corollary 2.7 and Theorem 2.8 are extensions of Lemma 2.4.They are used based on nonlinear terms that exist in the integral equations which our nonlinear third order ordinary differential equations are transformed.

3 Main results

In addition to the assumptions imposed on functions R_1, R_2, Q and P appearing in (1.1) and (1.2),the following hypothesis are required:

i Let $\int_{t_0}^{\infty} |u'(\rho)|d\rho \leq L$, where L is a positive constant.

ii $\lim_{t_0 \to \infty} \int_{t_0}^{t} \phi(s)ds \leq n_1 < \infty$, where $n_1 > 0$,

iii $\lim_{t_0 \to \infty} \int_{t_0}^{t} b(s)ds \leq n_2 < \infty$, where $n_2 > 0$,

iv $\lim_{t_0 \to \infty} \int_{t_0}^{t} v(s)r(s)ds \leq n_3 < \infty$, where $n_3 > 0$,

v $\lim_{t_0 \to \infty} \int_{t_0}^{t} a(s)ds \leq n_4 < \infty$, where $n_4 > 0$,

vi $\lim_{t_0 \to \infty} \int_{t_0}^{t} \alpha(s)ds \leq n_5 < \infty$, where $n_5 > 0$,

vii $\lim_{t_0 \to \infty} \int_{t_0}^{t} \beta(s)ds \leq n_6 < \infty$ where $n_6 > 0$,

viii $|\Phi(u(t))| \geq |u(t)|$,

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ix $|u'(t)| \leq \lambda$ where $\lambda > 0$

where $\phi, p, v, m, \alpha, q, b, \beta \in C(\mathbf{R}_+)$. We also investigate (1.1) and (1.2) when the forcing term $P(t, u(t), u'(t)) = 0$.

Theorem 3.1. If the assumptions (i)-(iv) are satisfied together with hypothesis (i)-(vi),(viii) then the equation (1.1) is Hyers-Ulam stable with HUc given as

$$\begin{aligned} K_1 = &\left(\frac{L}{\sigma} + n_3 \frac{\lambda\psi}{\sigma} + n_4 \frac{\lambda^4}{\sigma}\right) \\ \Upsilon^{-1} &\left[\Upsilon(1) + n_1\kappa \left[\Omega^{-1}\left(\Omega(1) + n_2 \frac{\lambda^2}{\sigma}\psi\left(T_1^*\right)\right) T_1^*\right]\right] \\ &\Omega^{-1}\left(\Omega(1) + n_2 \frac{\lambda^2}{\sigma}\psi\left(T_1^*\right)\right) T_1^*, \end{aligned} \tag{3.1}$$

where

$$T_1^* = F^{-1}\left(F(1) + n_5 \frac{\delta\lambda^{n+1}}{\sigma}\right).$$

Proof. Inequality (2.1) and assumptions (i),(ii), (iii) and (vi) are used to obtain

$$\begin{aligned} &u'''(t) + \alpha(t)\nu(u(t))u'(t)^n u''(t) + \left(a(t)u'(t)^2 + b(t)\psi(u'(t))\right) u'(t) \\ &+q(t)\gamma(u(t)) + v(t)r(t)\varpi(u(t)) - \phi(t)\kappa(u(t))h(u'(t)^4) \leq \epsilon. \end{aligned} \tag{3.2}$$

Let $u''(t)$ be differentiable function on $\mathbf{R}_+$, if $u'''(t) \geq 0 \ \forall \ t \in \mathbf{R}_+$, then $u''(t)$ is nondecreasing on $\mathbf{R}_+$ and $u''(t) \geq \delta$ where $\delta > 0$. (3.2) when multiplying by $u'(t)$ becomes

$$\begin{aligned} &\alpha(t)\nu(u(t))u'(t)^{n+1}\delta + \left(a(t)u'(t)^2 + b(t)\psi(u'(t))\right) u'(t)^2 + q(t)\gamma(u(t))u'(t) \\ &+v(t)r(t)\varpi(u(t))u'(t) - \phi(t)\kappa(u(t))h(u'(t)^4)u'(t) \leq u'(t)\epsilon. \end{aligned}$$

With the application of Theorem 2.6 implies that there exists $\xi, \tau, \pi, \chi, \tau \in [t_0, t]$ such that

$$\begin{aligned} &\delta u'(\xi)^{n+1} \int_{t_0}^t \alpha(s)\nu(u(s))ds + u'(\tau)^4 \int_{t_0}^t a(s)ds + u'(\pi)^2 \int_{t_0}^t b(s)\psi(u(s))ds \\ &+ \int_{t_0}^t q(s)\gamma(u(s))u'(s)ds + u'(\tau)\varpi(u(\tau)) \int_{t_0}^t v(s)r(s)ds \\ &-h(u'(\chi)^4)u'(\chi) \int_{t_0}^t \phi(s)\kappa(u(s))ds \leq \epsilon \int_{t_0}^t u'(s)ds. \end{aligned}$$

Setting

$$\Phi(u(t)) = \int_{u_0}^{u(t)} \gamma(u(s))ds. \tag{3.3}$$

Applying equation (3.3), if $q'(t) \geq 0$, let $q(t)$ be nondcreasing function on $\mathbf{R}_+$, then $q(t) \geq \sigma$ where $\sigma > 0$. We arrive at

$$\begin{aligned} \sigma\Phi(|u(t)|) \leq \epsilon &\int_{t_0}^t |u'(s)|ds + \delta|u'(\xi)|^{n+1} \int_{t_0}^t \alpha(s)\nu(|u(s)|)ds \\ &+|u'(\tau)|^4 \int_{t_0}^t a(s)ds + |u'(\pi)|^2 \int_{t_0}^t b(s)\psi(|u(s)|)ds \\ &+|u'(\tau)||\varpi(|u(\tau)|) \int_{t_0}^t v(s)r(s)ds \\ &+|h(u'(\chi)^4)||u'(\chi)| \int_{t_0}^t \phi(s)\kappa(|u(s)|)ds, \end{aligned}$$

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Let the function ϖ be bounded function on $\mathbf{R}_+$, then there exists positive constant ψ such that $\varpi(|u(\tau)|) \leq \psi$ and using hypotheses (i),(vii),(viii) and (ix) to obtain

$$|u(t)| \leq \epsilon \left(\frac{L}{\sigma} + \frac{\lambda\psi}{\sigma} \int_{t_0}^{t} v(s)r(s)ds + \frac{\lambda^4}{\sigma} \int_{t_0}^{t} a(s)ds \right) + \frac{\delta\lambda^{n+1}}{\sigma} \int_{t_0}^{t} \alpha(s)\nu(|u(s)|)ds + \frac{\lambda^2}{\sigma} \int_{t_0}^{t} b(s)\psi(|u(s)|)ds + \frac{|h(\lambda^4)|\lambda}{\sigma} \int_{t_0}^{t} \phi(s)\kappa(|u(s)|)ds.$$

By applying Theorem 2.8, we obtain

$$|u(t)| \leq \epsilon \left(\frac{L}{\sigma} + \frac{\lambda\psi}{\sigma} \int_{t_0}^{t} v(s)r(s)ds + \frac{\lambda^4}{\sigma} \int_{t_0}^{t} a(s)ds \right) \Upsilon^{-1} \left[\Upsilon(1) + \int_{t_0}^{t} \phi(s)\kappa \left[\Omega^{-1} \left(\Omega(1) + \frac{\lambda^2}{\sigma} \int_{t_0}^{s} b(\alpha)\psi\left(T_1(\alpha)\right) d\alpha \right) T_1(s) \right] ds \right] \Omega^{-1} \left(\Omega(1) + \frac{\lambda^2}{\sigma} \int_{t_0}^{t} b(s)\psi\left(T_1(s)\right) ds \right) T_1(t), \tag{3.4}$$

where

$$T_1(t) = F^{-1} \left(F(1) + \frac{\delta\lambda^{n+1}}{\sigma} \int_{t_0}^{t} \alpha(s)ds \right).$$

Employing the hypotheses (ii)- (vi), we arrive at

$$|u(t)| \leq \epsilon \left(\frac{L}{\sigma} + n_3 \frac{\lambda\psi}{\sigma} + n_4 \frac{\lambda^4}{\sigma} \right) \Upsilon^{-1} \left[\Upsilon(1) + n_1\kappa \left[\Omega^{-1} \left(\Omega(1) + n_2 \frac{\lambda^2}{\sigma} \psi\left(T_1^*\right) \right) T_1^* \right] \right] \Omega^{-1} \left(\Omega(1) + n_2 \frac{\lambda^2}{\sigma} \psi\left(T_1^*\right) \right) T_1^*,$$

where

$$T_1^* = F^{-1} \left(F(1) + n_5 \frac{\delta\lambda^{n+1}}{\sigma} \right).$$

Hence,

$$|u(t) - u(t_0)| \leq |u(t)| \leq K_1\epsilon.$$

Therefore, we arrive at the result. □

Remark 3.2. *The result of Theorem 3.1 is an extension of the result of Theorem 6 in Fakunle and Arawomo [8]. GBB type inequality of Theorem 2.8 is used to arrive at our result.*

Theorem 3.3. Suppose that the assumptions (i),(ix) is satisfied. Let $u''(t)$ be differentiable function on $\mathbf{R}_+$, if $u'''(t) \geq 0 \; \forall \; t \in \mathbf{R}_+$, then $u''(t)$ is nondecreasing on $\mathbf{R}_+$, furthermore, $u'' \geq \delta$ where δ a positive constant. Equation (1.2) is Hyers-Ulam stable with HUc determined as K_2.

Proof. From inequality (2.2), we use the assumption (i) and hypotheses of Theorem 3.3 to obtain

$$\beta(t)f(u(t))u''(t) + \alpha(t)g(u(t))u'(t) + p(t)\gamma(u(t)) - \phi(t)\kappa(u(t))h(u'(t)^4) \leq \epsilon,$$

It is clear from Theorem 2.6, there exists $\xi, \varrho, \rho \in [t_0, t]$ such that

$$\delta u'(\xi) \int_{t_0}^{t} \beta(s)f(u(s))ds + u'(\varrho)^2 \int_{t_0}^{t} \alpha(s)g(u(s))ds + \int_{t_0}^{t} p(s)\gamma(u(s))u'(s)ds - u'(\rho)h(u'(\rho)^4) \int_{t_0}^{t} \phi(s)\kappa(u(s))ds \leq \epsilon \int_{t_0}^{t} u'(s)ds$$

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and apply equation (3.3) to obtain

$$\sigma\Phi(|u(t)|) \leq \epsilon \int_{t_0}^{t} |u'(s)|ds + \delta|u'(\xi)| \int_{t_0}^{t} \beta(s)f(|u(s)|)ds$$
$$+|u'(\varrho)^2| \int_{t_0}^{t} \alpha(s)g(|u(s)|)ds + |u'(\rho|)|h(u'(\rho)^4)| \int_{t_0}^{t} \phi(s)\kappa(|u(s)|)ds.$$

From hypotheses (i), (vii) and (ix) we have

$$|u(t)| \leq \frac{L\epsilon}{\sigma} + \frac{\lambda\delta}{\sigma} \int_{t_0}^{t} \beta(s)f(|u(s)|)ds$$
$$+\frac{\lambda^2}{\sigma} \int_{t_0}^{t} \alpha(s)g(|u(s)|)ds + \frac{\lambda h(\lambda^4)}{\sigma} \int_{t_0}^{t} \phi(s)\kappa(|u(s)|)ds.$$

Applying Theorem 2.8, we obtain

$$\begin{aligned} |u(t)| \leq \frac{L\epsilon}{\sigma}\Upsilon^{-1} \Bigg[\Upsilon(1) + \frac{\lambda h(\lambda^4)}{\sigma} \\ \int_{t_0}^{t} \phi(s)\kappa \left[\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\sigma}\int_{t_0}^{s} \alpha(\tau)g\left(T_2(\tau)\right)d\tau\right)T_2(s)\right] ds\Bigg] \\ \Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\sigma}\int_{t_0}^{t} \alpha(s)g\left(T_2(s)\right)ds\right)T_2(t), \end{aligned} \tag{3.5}$$

where $T_2(t)$ is given as

$$T_2(t) = F^{-1}\left(F(1) + \frac{\lambda\delta}{\sigma}\int_{t_0}^{t} \beta(s)ds\right).$$

We used hypotheses (v)- (x)to arrive at

$$|u(t)| \leq \frac{L\epsilon}{\sigma}\Upsilon^{-1}\left[\Upsilon(1) + \frac{\lambda h(\lambda^4)}{\sigma}n_1\kappa\left[\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\sigma}n_5 g\left(T_2^*\right)\right)T_2^*\right]\right]$$
$$\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\sigma}n_5 g\left(T_2^*\right)\right)T_2^*,$$

where T_2^* is given as

$$T_2^* = F^{-1}\left(F(1) + n_6\frac{\lambda\delta}{\sigma}\right).$$

Hence,

$$|u(t) - u(t_0)| \leq |u(t)| \leq K_2\epsilon.$$

Therefore,

$$K_2 = \frac{L}{\sigma}\Upsilon^{-1}\left[\Upsilon(1) + \frac{\lambda h(\lambda^4)}{\sigma}n_1\kappa\left[\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\sigma}n_5 g\left(T_2^*\right)\right)T_2^*\right]\right]$$
$$\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\sigma}n_5 g\left(T_2^*\right)\right)T_2^*,$$

□

Remark 3.4. *The result of Theorem 3.3 extended the result of Theorem 6 in Fakunle and Arawomo [8]. GBB type inequality of Theorem2.8 is applied to obtain our result.*

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Theorem 3.5. Supposed the assumptions (ii)-(vi) are satisfied . If $P(t, u(t), u'(t)) = 0$ in equation (1.1) is assumed. Then

$$\begin{aligned} u'''(t) + R_1(t, u(t), u'(t))u''(t) + R_2(t, u(t), u'(t))u'(t) \\ +q(t)\gamma(u(t)) + Q(t, u(t)) = 0, \end{aligned} \tag{3.6}$$

equation (3.4) has HUs with HUc which is determined as

$$\begin{aligned} K_3 = \left(\frac{L}{\sigma} + n_4\frac{\lambda^4}{\sigma}\right) \\ \Upsilon^{-1}\left[\Upsilon(1) + n_3\frac{\lambda}{\sigma}\varpi\left[\Omega^{-1}\left(\Omega(1) + n_2\frac{\lambda^2}{\sigma}\psi\left(T_3^*\right)\right)T_3^*\right]\right] \\ \Omega^{-1}\left(\Omega(1) + n_2\frac{\lambda^2}{\sigma}\psi\left(T_3^*\right)\right)T_3^*, \end{aligned} \tag{3.7}$$

where

$$T_3^* = F^{-1}\left(F(1) + n_5\frac{\delta\lambda^{n+1}}{\sigma}\right).$$

Proof. Inequality (2.1) with assumption on function P i.e $P(t, u(t), u'(t)) = 0$ and multiplication by $u'(t)$ to have

$$\begin{aligned} u'''(t)u'(t) + R_1(t, u(t), u'(t))u''(t)u'(t) \\ +R_2(t, u(t), u'(t))u'(t)^2 + q(t)\gamma(u(t))u'(t) + Q(t, u(t))u'(t) \le \epsilon u'(t). \end{aligned} \tag{3.8}$$

By the hypothesis of Theorem 3.3 to get

$$\begin{aligned} \delta R_1(t, u(t), u'(t))u'(t) + R_2(t, u(t), u'(t))u'(t)^2 \\ +q(t)\gamma(u(t))u'(t) + Q(t, u(t))u'(t) \le \epsilon u'(t). \end{aligned}$$

Due to the assumptions (ii)-(iv) we obtain

$$\begin{aligned} \delta\alpha(t)\nu(u(t))u'(t)^{n+1} + \left(a(t)u'(t)^2 + b(t)\psi(u(t))\right)u'(t)^2 \\ +q(t)\gamma(u(t))u'(t) + v(t)r(t)\varpi(u(t))u'(t) \le \epsilon. \end{aligned}$$

The application of Theorem 2.6 implies that there exists $\xi, \rho, \tau, \pi, \chi \in [t_0, t]$ such that

$$\begin{aligned} \delta u'(\xi)\int_{t_0}^{t}\alpha(s)\nu(u(s))u'(s)^{n+1}ds + u'(\rho)^4\int_{t_0}^{t}a(s)ds \\ +u'(\tau)^2\int_{t_0}^{t}b(s)\psi(u(s))ds + \int_{t_0}^{t}q(s)\gamma(u(s))u'(s)ds \\ +u'(\pi)\int_{t_0}^{t}v(s)r(s)\varpi(u(s))ds \le \epsilon\int_{t_0}^{t}u'(s)ds. \end{aligned}$$

Applying (3.3) and let $q(t)$ be nondcreasing function on $\mathbf{R}_+$ then $q'(t) \ge 0$, $q(t) \ge \sigma$ for $\sigma > 0$

$$\begin{aligned} \sigma\Phi(|u(t)|) \le \epsilon\int_{t_0}^{t}|u'(s)|ds + \delta|u'(\xi)|^{n+1}\int_{t_0}^{t}\alpha(s)\nu(|u(s)|)ds \\ +|u'(\tau)|^4\int_{t_0}^{t}a(s)ds + |u'(\pi)|^2\int_{t_0}^{t}b(s)\psi(|u(s)|)ds \\ +|u'(\tau)|\int_{t_0}^{t}v(s)r(s)\varpi(|u(s)|)ds, \end{aligned}$$

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Using the hypotheses (i),(vii) and (ix) to obtain

$$|u(t)| \leq \epsilon \left(\frac{L}{\sigma} + \frac{\lambda^4}{\sigma} \int_{t_0}^{t} a(s)ds \right) + \frac{\delta \lambda^{n+1}}{\sigma}$$
$$\int_{t_0}^{t} \alpha(s)\nu(|u(s)|)ds + \frac{\lambda^2}{\sigma} \int_{t_0}^{t} b(s)\psi(|u(s)|)ds + \frac{\lambda}{\sigma} \int_{t_0}^{t} v(s)r(s)\varpi(|u(s)|)ds.$$

We use Theorem 2.8 to obtain

$$|u(t)| \leq \epsilon \left(\frac{L}{\sigma} + \frac{\lambda^4}{\sigma} \int_{t_0}^{t} a(s)ds \right)$$
$$\Upsilon^{-1} \left[\Upsilon(1) + \frac{\lambda}{\sigma} \int_{t_0}^{t} v(s)r(s)\varpi \left[\Omega^{-1} \left(\Omega(1) + \frac{\lambda^2}{\sigma} \int_{t_0}^{s} b(\alpha)\psi\left(T_3(\alpha)\right) d\alpha \right) T_3(s) \right] ds \right]$$
$$\Omega^{-1} \left(\Omega(1) + \frac{\lambda^2}{\sigma} \int_{t_0}^{t} b(s)\psi\left(T_3(s)\right) ds \right) T_3(t),$$

where $T_3(t)$ is given as

$$T_3(t) = F^{-1} \left(F(1) + \frac{\delta \lambda^{n+1}}{\sigma} \int_{t_0}^{t} \alpha(s)ds \right).$$

By using the hypotheses (iii),(iv),(v) and (vi) we obtain

$$|u(t)| \leq \epsilon \left(\frac{L}{\sigma} + n_4 \frac{\lambda^4}{\sigma} \right)$$
$$\Upsilon^{-1} \left[\Upsilon(1) + n_3 \frac{\lambda}{\sigma} \varpi \left[\Omega^{-1} \left(\Omega(1) + n_2 \frac{\lambda^2}{\sigma} \psi\left(T_3^*\right) \right) T_3^* \right] \right]$$
$$\Omega^{-1} \left(\Omega(1) + n_2 \frac{\lambda^2}{\sigma} \psi\left(T_3^*\right) \right) T_3^*,$$

where T_3^* is given as

$$T_3^* = F^{-1} \left(F(1) + n_5 \frac{\delta \lambda^{n+1}}{\sigma} \right).$$

Hence,

$$|u(t) - u(t_0)| \leq |u(t)| \leq K_3 \epsilon,$$

where,

$$K_3 = \left(\frac{L}{\sigma} + n_4 \frac{\lambda^4}{\sigma} \right)$$
$$\Upsilon^{-1} \left[\Upsilon(1) + n_3 \frac{\lambda}{\sigma} \varpi \left[\Omega^{-1} \left(\Omega(1) + n_2 \frac{\lambda^2}{\sigma} \psi\left(T_3^*\right) \right) T_3^* \right] \right]$$
$$\Omega^{-1} \left(\Omega(1) + n_2 \frac{\lambda^2}{\sigma} \psi\left(T_3^*\right) ds \right) T_3^*$$

□

Remark 3.6. *Again GBB stated in Theorem 2.8is used to arrive at the result of Theorem 3.5.This result is compared with the result of Theorem 3.1 it seems as if there is no difference in HUc.*

Theorem 3.7. Let $P(t, u(t), u'(t)) = 0$ then (1.2) becomes

$$\begin{aligned} u'''(t) + \beta(t)f(u(t))u''(t) + \alpha(t)g(u(t))u'(t) + p(t) \\ \gamma(u(t)) = 0 \end{aligned} \tag{3.9}$$

is Hyers-Ulam stable with HUc given as

$$K_4 = \frac{L}{\sigma}\Omega^{-1}\left(\Omega(1) + n_5\frac{\lambda^2}{\sigma}g\left(F^{-1}\left(F(1) + n_6\frac{\lambda\delta}{\sigma}\right)\right)\right) F^{-1}\left(F(1) + n_6\frac{\lambda\delta}{\sigma}\right). \tag{3.10}$$

Proof. Using inequality (2.2) in the form

$$u'''(t) + \beta(t)f(u(t))u''(t) + \alpha(t)g(u(t))u'(t) + p(t)\gamma(u(t)) \leq \epsilon.$$

By hypothesis of Theorem 3.3 we obtain

$$\beta(t)f(u(t))u''(t) + \alpha(t)g(u(t))u'(t) + p(t)\gamma(u(t)) \leq \epsilon. \tag{3.11}$$

Multiplying inequality (3.11) by $u'(t)$ and by the application of Theorem 2.6 implies there exists $\xi, \varrho, \rho \in [t_0, t]$ such that

$$\delta u'(\xi)\int_{t_0}^{t}\beta(s)f(u(s))ds + u'(\varrho)^2\int_{t_0}^{t}\alpha(s)g(u(s))ds + \int_{t_0}^{t}p(s)\gamma(u(s))u'(s)ds \leq \epsilon\int_{t_0}^{t}u'(s)ds, \tag{3.12}$$

Using equation (3.3) to obtain

$$\sigma\Phi(|u(t)|) \leq \epsilon\int_{t_0}^{t}|u'(s)|ds + \delta|u'(\xi)|\int_{t_0}^{t}\beta(s)f(|u(s)|)ds + |u'(\varrho)^2|\int_{t_0}^{t}\alpha(s)g(|u(s)|)ds.$$

From hypotheses (i),(vii),(ix),we have

$$|u(t)| \leq \frac{L\epsilon}{\sigma} + \frac{\lambda\delta}{\sigma}\int_{t_0}^{t}\beta(s)f(|u(s)|)ds + \frac{\lambda^2}{\sigma}\int_{t_0}^{t}\alpha(s)g(|u(s)|)ds.$$

By Theorem 2.7 we arrive at

$$|u(t)| \leq \frac{L\epsilon}{\sigma}\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\sigma}\int_{t_0}^{t}\alpha(s)g\left(F^{-1}\left(F(1) + \frac{\lambda\delta}{\sigma}\int_{t_0}^{t}\beta(\mu)d\mu\right)\right)ds\right) F^{-1}\left(F(1) + \frac{\lambda\delta}{\sigma}\int_{t_0}^{t}\beta(s)ds\right), \quad t \in \mathbf{I},$$

Using hypotheses (vi)- (vii) to get

$$|u(t)| \leq \frac{L\epsilon}{\sigma}\Omega^{-1}\left(\Omega(1) + n_5\frac{\lambda^2}{\sigma}g\left(F^{-1}\left(F(1) + n_6\frac{\lambda\delta}{\sigma}\right)\right)\right) F^{-1}\left(F(1) + n_6\frac{\lambda\delta}{\sigma}\right).$$

Hence,

$$|u(t) - u(t_0)| \leq |u(t)| \leq K_4\epsilon.$$

Therefore,

$$K_4 = \frac{L}{\sigma}\Omega^{-1}\left(\Omega(1) + n_5\frac{\lambda^2}{\sigma}g\left(F^{-1}\left(F(1) + n_6\frac{\lambda\delta}{\sigma}\right)\right)\right) F^{-1}\left(F(1) + n_6\frac{\lambda\delta}{\sigma}\right).$$

□

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Remark 3.8. *GBB type inequality in Corollary 2.7 is used to arrive at the result.*

Example 3.9. Consider the following equation

$$u'''(t) + \frac{1}{t^2}u^4(t)(u'(t))^6 u''(t) + \frac{1}{t^4}u^2(t)(u'(t))^6 + t^4 u^2(t) + \frac{1}{t^4}u^2(t) = \frac{1}{t^4}u^2(t)(u'(t))^4, \quad t > 0,$$

by using the appropriate hypotheses (ii)-(vii) and allowing the following: $\phi(t) = \frac{1}{t^4}, b(t) = \frac{1}{t^5}, \alpha(t) = \frac{1}{t^4}, a(t) = \frac{1}{t^2}$ and $v(t)r(t) = \frac{1}{t^4}$,. Substituting the above parameters into the inequality (3.4) in the proof of Theorem 3.1 to have

$$|u(t)| \leq \epsilon \left(\frac{L}{\sigma} + \frac{\lambda\psi}{\sigma} \int_{t_0}^{t} \frac{1}{s^4} ds + \frac{\lambda^4}{\sigma} \int_{t_0}^{t} \frac{1}{s^2} ds \right)$$
$$\Upsilon^{-1} \left[\Upsilon(1) + \int_{t_0}^{t} \frac{1}{t^4} \kappa \left[\Omega^{-1} \left(\Omega(1) + \frac{\lambda^2}{\sigma} \int_{t_0}^{s} \frac{1}{\alpha^5} \psi\left(T_1(\alpha)\right) d\alpha \right) T_1(s) \right] ds \right]$$
$$\Omega^{-1} \left(\Omega(1) + \frac{\lambda^2}{\sigma} \int_{t_0}^{t} \frac{1}{s^5} \psi\left(T_1(s)\right) ds \right) T_1(t),$$

where

$$T_1(t) = F^{-1} \left(F(1) + \frac{\delta\lambda^{n+1}}{\sigma} \int_{t_0}^{t} \frac{1}{s^4} ds \right).$$

By making use of these following estimations:

i $\lim_{t_0 \to \infty} \int_{t_0}^{t} \frac{1}{s^4} ds \leq n_3$

ii $\lim_{t_0 \to \infty} \int_{t_0}^{t} \frac{1}{s^2} ds \leq n_4$

iii $\lim_{t_0 \to \infty} \int_{t_0}^{t} \frac{1}{s^5} ds \leq n_1$

iv $\lim_{t_0 \to \infty} \int_{t_0}^{t} \frac{1}{s^5} ds \leq n_2$

v $\lim_{t_0 \to \infty} \int_{t_0}^{t} \frac{1}{s^4} ds \leq n_5$

The result is given as

$$|u(t)| \leq \epsilon \left(\frac{L}{\sigma} + n_3\frac{\lambda\psi}{\sigma} + n_4\frac{\lambda^4}{\sigma} \right)$$
$$\Upsilon^{-1} \left[\Upsilon(1) + n_1\kappa \left[\Omega^{-1} \left(\Omega(1) + n_2\frac{\lambda^2}{\sigma} \psi\left(T_1^*\right) \right) T_1^* \right] \right]$$
$$\Omega^{-1} \left(\Omega(1) + n_2\frac{\lambda^2}{\sigma} \psi\left(T_1^*\right) \right) T_1^*,$$

where

$$T_1^* = F^{-1} \left(F(1) + n_5\frac{\delta\lambda^{n+1}}{\sigma} \right).$$

then, the HSc is given as

$$K = \left(\frac{L}{\sigma} + n_3\frac{\lambda\psi}{\sigma} + n_4\frac{\lambda^4}{\sigma} \right)$$
$$\Upsilon^{-1} \left[\Upsilon(1) + n_1\kappa \left[\Omega^{-1} \left(\Omega(1) + n_2\frac{\lambda^2}{\sigma} \psi\left(T_1^*\right) \right) T_1^* \right] \right]$$
$$\Omega^{-1} \left(\Omega(1) + n_2\frac{\lambda^2}{\sigma} \psi\left(T_1^*\right) \right) T_1^*.$$

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Example 3.10. Consider the following equation

$$u'''(t) + \frac{1}{t^2}u^2(t)u''(t) + \frac{1}{t^5}u^4(t)(u'(t)) + \frac{1}{t^4}u^2(t) = \frac{1}{t^4}u^2(t)(u'(t))^4, \quad t > 0.$$

Let $\beta(t) = \frac{1}{t^2}, \alpha(t) = \frac{1}{t^5}, \phi(t) = \frac{1}{t^4}$, by using the appropriate hypotheses (ii)-(vii), we substitute to (3.5) to arrive at

$$|u(t)| \leq \frac{L\epsilon}{\sigma}\Upsilon^{-1}\Bigg[\Upsilon(1) + \frac{\lambda h(\lambda^4)}{\sigma} \int_{t_0}^{t} \frac{1}{s^4}\kappa\left[\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\sigma}\int_{t_0}^{s}\frac{1}{\tau^5}g\left(T_2(\tau)\right)d\tau\right)T_2(s)\right]ds\Bigg] \Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\sigma}\int_{t_0}^{t}\frac{1}{s^5}g\left(T_2(s)\right)ds\right)T_2(t),$$

where $T_2(t)$ is given as

$$T_2(t) = F^{-1}\left(F(1) + \frac{\lambda\delta}{\sigma}\int_{t_0}^{t}\frac{1}{s^2}ds\right).$$

Furthermore, we have

$$|u(t)| \leq \frac{L\epsilon}{\sigma}\Upsilon^{-1}\left[\Upsilon(1) + \frac{\lambda h(\lambda^4)}{\sigma}n_1\kappa\left[\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\sigma}n_5 g\left(T_2^*\right)\right)T_2^*\right]\right] \Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\sigma}n_5 g\left(T_2^*\right)\right)T_2^*,$$

where T_2^* is given as

$$T_2^* = F^{-1}\left(F(1) + n_6\frac{\lambda\delta}{\sigma}\right).$$

where

i $\lim_{t_0\to\infty}\int_{t_0}^{t}\frac{1}{s^4}ds \leq n_1$

ii $\lim_{t_0\to\infty}\int_{t_0}^{t}\frac{1}{s^5}ds \leq n_5$

iii $\lim_{t_0\to\infty}\int_{t_0}^{t}\frac{1}{s^2}ds \leq n_6$

then, HSc is calculated as

$$K = \frac{L}{\sigma}\Upsilon^{-1}\left[\Upsilon(1) + \frac{\lambda h(\lambda^4)}{\sigma}n_1\kappa\left[\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\sigma}n_5 g\left(T_2^*\right)\right)T_2^*\right]\right] \Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\sigma}n_5 g\left(T_2^*\right)\right)T_2^*.$$

4 Conclusion

In this study, GBB has been used to investigate the HUs of non-autonomous nonlinear third order ordinary differential equations. The results obtained extended some of the results in the literature, we also considered third order differential equations without the forcing term, the equations are also Hyers-Ulam stable and Hyers-Ulam constants are obtained. However, the results obtained are slightly different in terms of HUc from those that are with forcing term.

5 Conflicts of Interest

The authors declared no conflicts interest.

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