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Full Fuzzy parameterized Soft Expert Set with Application to Decision Making

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Abstract

In this work, the authors consider a generalization of fuzzy parameterized soft expert set introducing the concept of full fuzzy parameterized soft expert set and study their properties. We define its basic operations and develop an algorithm to demonstrate its application in decision making. We also define the optimal choice object, developing and proving propositions relating to it. Finally, we use a concrete example to illustrate our algorithm.

Keywords: Fuzzy Set, Soft Set, Fuzzy Soft Set, Fuzzy Parameterized Soft Set, Full Fuzzy parameterized Soft Set, Basic Operations, Algebraic Properties.
MSC2010: 94D05.

1 Introduction

Zadeh in [1] introduced the concept of fuzzy set, generalizing the notion of classical set by assigning degree of membership to the elements of the set [1], [2], [3]. The assignment is a generalization of the characteristic function extending the binary codomain to the unit interval [0,1]. Zadeh's work introduced an appropriate solution to imprecision and vagueness, which dominates human thinking.

However, setting the membership function in each specific case poses a challenge, as highlighted by Molodstov in [4], an issue he addresses with the introduction and investigation of soft set theory. Described as a parameterization of fuzzy set in [5], a soft set is formulated by mapping a set of parameters to the power set of a universe, as outlined by Molodstov in [4]. This versatile concept finds applications in various domains such as game theory, operations research, economics, engineering, and physics, as extensively discussed by Molodstov in [4].

Maji and Roy [6], investigated the theory of soft and defined the notions of equal soft set, subset, superset of soft set, compliment of soft set, null soft set, and soft set operations. They offered the first practical application of soft set using the reduction of rough set as mention by Nasef *et al.* in [7] and introduced another application of soft set theory in decision making problems for real estate marketing. In [8], Cheng *et al.* proposed parametrization reduction of soft set. Also, Kong *et al.* presented the normal parameterize reduction of soft sets [10]. In 2023, ELijah and Muhammad [11] also presented a novel approach for normal parameter reduction. But soft set theory in its totality can not handle fuzzy evaluation [12].

Due to the fuzzy nature of parameters, soft set application in real life tends to be limited. As a result Maji *et al.* in [13] introduced fuzzy soft set where the universe set is fuzzified. They presented a theoretic approach of fuzzy soft set to decision making [14]. Cagman *et al.* in [15], defined fuzzy soft aggregation operator which allows for efficient decision making method. Cagman *et al.* [16] introduced the idea of fuzzifying the parameters and proposed a decision making algorithm. Also in [17], the authors fuzzified both the set of parameters and the universe set in Fuzzy parameterized Fuzzy Soft Set. Fuzzy soft set theory turn out to be an effective tool in dealing with decision making problems as presented by Majundar *et al.* ([18]) and Tripathy *et al.* [19].

However, all the previously mentioned models focused on the opinions of one expert at a time. In a different research direction, Alkhazaleh *et al.* introduced a multi-expert model in soft expert set, where opinions from multiple experts can be collectively evaluated in a single model [2]. In a related work, Alkhazaleh *et al.* generalized soft expert set to fuzzy soft expert sets, proposing a model for decision-making problems [3]. Bashir and Salleh introduced the idea of a possibility fuzzy soft expert set and explored its properties [20]. AlQudah *et al.* combined the concept of bipolar fuzzy sets into soft expert sets, presenting an algorithm based on this combination [21]. In this line of research, Bashir and Salleh introduced the concept of fuzzy parameterized soft expert set where they considered giving an important degree to each element in the set of parameters. Its properties were also studied, and its basic operations were defined. Finally, an application in decision making was provided. In 2022, Edeghagba and Muhammad introduced and investigated the concept of full fuzzy parameterized soft set and developed an algorithm based on this model, and its related properties [22].

The present work utilizes the notion of full fuzzy parameterized soft set as presented in [22] to develop a generalization of fuzzy parameterized soft expert set as introduced in [23]. By our construction we are able to investigate the notion of optimal choice object which improve our application of full fuzzy parameterized soft expert set in a decision-making problem.

2 Preliminaries

Definition 2.1. *(Molodstov, [4])* ***Soft Set:*** *Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denotes the power set of U and $A \subset E$. A pair (F, A) is called a soft set over U, where F is a mapping given by:*

$$F : A \to P(U).$$

Definition 2.2. *(Zadeh, [1])* ***Fuzzy Set:*** *Let X be a non empty set, the fuzzy set A over X is given by:*

$$A = \{(x, f_A(x)); x \in X, f_A(x) \in [0, 1]\} \text{ where } f_A : X \to [0, 1] \text{ and } f_A$$

is called a membership function

Definition 2.3. *(Alkhazakeh & Salleh, [2])* ***Soft Expert Set:*** *Let U be a universe set E be the set of parameters, X be the set of experts, O be the set of opinions $Z = E \times X \times O$, and $A \subseteq Z$ then a pair (F, A) is called a soft expert set over U, where F is a mapping given by $F : A \to P(U)$ i.e,*

$$(F, A) = \Big\{\big((e,x,o),\ F(e, x, o)\big)\ :\ e \in E, x \in X, o \in O,\ f(e, x, o) \in P(U)\Big\}$$

Definition 2.4. *(Bashir & Salleh, [23])* ***Fuzzy Parameterized Soft Expert Set:*** *Let U be the universe set, E be the set of parameters, I^E be the set of fuzzy subsets of E, X the set of experts and O the set of opinions i,e $O = \{1 = agree, 0 = disagree\}$. Let $Z = D \times X \times O$ and $A \subseteq Z$ where $D \in I^E$ then the pair$(F, A)_D$ is called fuzzy parameterized soft expert set $(FPSES)$ over U where F is a mapping given by $F_D : A \to P(U)$ and $P(U)$ is the power set of U*

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$$(F, A)_D = \{(d, x, o,), F(d, x, o) : d \in D, x \in X, o \in O, F(d, x, o) \in P(U)\}$$

Definition 2.5. *(Edeghagba & Muhammad, [22])* ***Full Fuzzy Parameterized Soft Set:*** *Let $\tilde{A} \subset \tilde{E}$. A Full Fuzzy Parameterized Soft Set (FFPS-set) $F_{\tilde{A}}$ on the universe U is given as:*

$$F_{\tilde{A}} = \{(\hat{y},\ f_{\tilde{A}}(\hat{y}))\ :\ \hat{y} \in \tilde{A},\ f_{\tilde{A}}(\hat{y}) \in P(U),\ \mu_{\hat{y}}(x) \in [0, 1],\ x \in E\}$$

where $\mu_{\hat{y}} : E \to [0, 1]$ represents a fuzzification of the set of parameters and $f_{\tilde{A}} : \tilde{E} \to P(U)$ represents the approximation function of $F_{\tilde{A}}$ such that;

$$f_{\tilde{A}}(\hat{y}) = \emptyset \text{ whenever } \mu_{\hat{y}}(x) = 0\ \forall x \in E \text{ and } \hat{y} \in \tilde{A}.$$

3 The Concept of Full Fuzzy parameterized Soft Expert Set

Definition 3.1. *Let U be the universe set,E be the set of all parameters, X be the set of experts, and O be the set of opinions $\{0 = disagre, 1 = agree\}$. Let $\tilde{E}$ be the set of all fuzzy sets of E, and $\tilde{Z}$=$\tilde{A} \times X \times O$, where $\tilde{A} \subset \tilde{E}$,and $\mathcal{A} \subseteq \tilde{Z}$.*
Now $FFPSES$ is a pair $(F, \mathcal{A})_{\tilde{A}}$ where

$$(F, \mathcal{A})_{\tilde{A}} = \{((\hat{y}, x, o),\ f_{\tilde{A}}(\hat{y}, x, o)) : \hat{y} \in \tilde{A},\ x \in X, o \in O,\ (\hat{y}, x, o) \in \mathcal{A},\ f_{\tilde{A}}(\hat{y}, x, o) \in P(U)\}$$

$$f_{\tilde{A}}\text{: } \mathcal{A} \to P(U),\ \mu_{\hat{y}} : E \to [0, 1]$$

Example 3.2. *Let $U = \{u_1, u_2, u_3, u_4, u_5, \}$, $E = \{e_1, e_2, e_3, e_4\}$, $X = \{p, q, r\}$, $O = \{0, 1\}$, $\tilde{E}$ be all possible fuzzy sets of E, which represents all possible considerations of the parameters by the experts.*
The experts consider each element of the universe in respect of all the parameters as follows:

expert p considers:

u_1 *to be* $\frac{0.3}{e_1}, \frac{0.6}{e_2}, \frac{0.7}{e_3}, \frac{0.5}{e_4}$, u_2 *to be* $\frac{0.4}{e_1}, \frac{0.5}{e_2}, \frac{0.7}{e_3}, \frac{0.3}{e_4}$,

u_3 *to be* $\frac{0.3}{e_1}, \frac{0.6}{e_2}, \frac{0.7}{e_3}, \frac{0.5}{e_4}$, u_4 *to be* $\frac{0.3}{e_1}, \frac{0.6}{e_2}, \frac{0.7}{e_3}, \frac{0.5}{e_4}$,

u_5 *to be* $\frac{0.4}{e_1}, \frac{0.5}{e_2}, \frac{0.7}{e_3}, \frac{0.3}{e_4}$.

expert q considers:

u_1 *to be* $\frac{0.5}{e_1}, \frac{0.4}{e_2}, \frac{0.8}{e_3}, \frac{0.3}{e_4}$, u_2 *to be* $\frac{0.4}{e_1}, \frac{0.5}{e_2}, \frac{0.7}{e_3}, \frac{0.3}{e_4}$,

u_3 *to be* $\frac{0.5}{e_1}, \frac{0.4}{e_2}, \frac{0.8}{e_3}, \frac{0.3}{e_4}$, u_4 *to be* $\frac{0.3}{e_1}, \frac{0.6}{e_2}, \frac{0.7}{e_3}, \frac{0.5}{e_4}$,

u_5 *to be* $\frac{0.3}{e_1}, \frac{0.6}{e_2}, \frac{0.7}{e_3}, \frac{0.5}{e_4}$.

expert r considers:

u_1 *to be* $\frac{0.4}{e_1}, \frac{0.5}{e_2}, \frac{0.7}{e_3}, \frac{0.3}{e_4}$, u_2 *to be* $\frac{0.4}{e_1}, \frac{0.5}{e_2}, \frac{0.7}{e_3}, \frac{0.3}{e_4}$,

u_3 *to be* $\frac{0.5}{e_1}, \frac{0.4}{e_2}, \frac{0.8}{e_3}, \frac{0.3}{e_4}$, u_4 *to be* $\frac{0.3}{e_1}, \frac{0.6}{e_2}, \frac{0.7}{e_3}, \frac{0.5}{e_4}$,

u_5 *to be* $\frac{0.5}{e_1}, \frac{0.8}{e_2}, \frac{0.4}{e_3}, \frac{0.3}{e_4}$.

Now it will be observed that the set of all considerstions of the elements of the universe by the experts is :

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$$\tilde{A} = \{\hat{y_1}, \hat{y_2}, \hat{y_3}\} \subset \tilde{E}.$$

where;

$$\hat{y_1} = \{\tfrac{0.3}{e_1}, \tfrac{0.6}{e_2}, \tfrac{0.7}{e_3}, \tfrac{0.5}{e_4}\},\ \hat{y_2} = \{\tfrac{0.4}{e_1}, \tfrac{0.5}{e_2}, \tfrac{0.7}{e_3}, \tfrac{0.3}{e_4}\},\ \hat{y_3} = \{\tfrac{0.5}{e_1}, \tfrac{0.4}{e_2}, \tfrac{0.8}{e_3}, \tfrac{0.3}{e_4}\},$$

Therefore $\tilde{Z}{=}\tilde{A} \times X \times O$, *be all possible opinions of all the experts, and* $\mathcal{A} \subseteq \tilde{Z}$.

From above :

$\mathcal{A} = \big\{(\hat{y_1}, p, 1),\ (\hat{y_1}, p, 0),\ (\hat{y_2}, p, 1),\ (\hat{y_2}, p, 0),\ (\hat{y_1}, q, 1),\ (\hat{y_1}, q, 0),\ (\hat{y_2}, q, 1),\ (\hat{y_2}, q, 0),\ (\hat{y_3}, q, 1),\ (\hat{y_3}, q, 0),\ (\hat{y_1}, r, 1),\ (\hat{y_1}, r, 0),\ (\hat{y_2}, r, 1),\ (\hat{y_2}, r, 0),\ (\hat{y_3}, r, 1),\ (\hat{y_3}, r, 0)\big\}$

Now, the functional values of all elements $\hat{a} \in \mathcal{A}$*:*

$f_{\tilde{A}}\ ((\frac{0.3}{e_1}, \frac{0.6}{e_2}, \frac{0.7}{e_3}, \frac{0.5}{e_4}), p, 1){=}\{u_1, u_3, u_4\},\ \ f_{\tilde{A}}\ ((\frac{0.4}{e_1}, \frac{0.5}{e_2}, \frac{0.7}{e_3}, \frac{0.3}{e_4}), p, 1){=}\{u_2, u_5\}$

$f_{\tilde{A}}\ ((\frac{0.3}{e_1}, \frac{0.6}{e_2}, \frac{0.7}{e_3}, \frac{0.5}{e_4}), q, 1){=}\{u_4, u_5\},\ \ f_{\tilde{A}}\ ((\frac{0.4}{e_1}, \frac{0.5}{e_2}, \frac{0.7}{e_3}, \frac{0.3}{e_4}), q, 1){=}\{u_2\}$

$f_{\tilde{A}}\ ((\frac{0.5}{e_1}, \frac{0.4}{e_2}, \frac{0.8}{e_3}, \frac{0.3}{e_4}), q, 1){=}\{u_1, u_3\},\ \ f_{\tilde{A}}\ ((\frac{0.3}{e_1}, \frac{0.6}{e_2}, \frac{0.7}{e_3}, \frac{0.5}{e_4}), r, 1){=}\{u_4\}$

$f_{\tilde{A}}\ ((\frac{0.4}{e_1}, \frac{0.5}{e_2}, \frac{0.7}{e_3}, \frac{0.3}{e_4}), r, 1){=}\{u_1, u_2\},\ \ f_{\tilde{A}}\ ((\frac{0.5}{e_1}, \frac{0.4}{e_2}, \frac{0.8}{e_3}, \frac{0.3}{e_4}), r, 1){=}\{u_3, u_5\}$

$f_{\tilde{A}}\ ((\frac{0.3}{e_1}, \frac{0.6}{e_2}, \frac{0.7}{e_3}, \frac{0.5}{e_4}), p, 0){=}\{u_2, u_5,\},\ \ f_{\tilde{A}}\ ((\frac{0.4}{e_1}, \frac{0.5}{e_2}, \frac{0.7}{e_3}, \frac{0.3}{e_4}), p, 0){=}\{u_1, u_3u_4,\}$

$f_{\tilde{A}}\ ((\frac{0.3}{e_1}, \frac{0.6}{e_2}, \frac{0.7}{e_3}, \frac{0.5}{e_4}), q, 0){=}\{u_1, u_2u_3,\},\ \ f_{\tilde{A}}\ ((\frac{0.4}{e_1}, \frac{0.5}{e_2}, \frac{0.7}{e_3}, \frac{0.3}{e_4}), q, 0){=}\{u_1, u_3, u_4, u_5,\}$

$f_{\tilde{A}}\ ((\frac{0.5}{e_1}, \frac{0.4}{e_2}, \frac{0.8}{e_3}, \frac{0.3}{e_4}), q, 0){=}\{u_2, u_4u_5,\},\ \ f_{\tilde{A}}\ ((\frac{0.3}{e_1}, \frac{0.6}{e_2}, \frac{0.7}{e_3}, \frac{0.5}{e_4}), r, 0){=}\{u_1, u_2u_3, u_5\}$

$f_{\tilde{A}}\ ((\frac{0.4}{e_1}, \frac{0.5}{e_2}, \frac{0.7}{e_3}, \frac{0.3}{e_4}), r, 0){=}\{u_3, u_4u_5, u_6\},\ \ f_{\tilde{A}}\ ((\frac{0.5}{e_1}, \frac{0.4}{e_2}, \frac{0.8}{e_3}, \frac{0.3}{e_4}), r, 0){=}\{u_1, u_2, u_4,\}$

Then: $(F, \mathcal{A})_{\tilde{A}}{=}\Bigg\{\Bigg[\Bigg(\Big(\frac{0.3}{e_1}, \frac{0.6}{e_2}, \frac{0.7}{e_3}\frac{0.5}{e_4}\Big), p, 1\Bigg)(u_1, u_3, u_4)\Bigg],$

$\Bigg[\Bigg(\Big(\frac{0.4}{e_1}, \frac{0.5}{e_2}, \frac{0.7}{e_3}\frac{0.3}{e_4}\Big), p, 1\Bigg)(u_2, u_5)\Bigg],\ \Bigg[\Bigg(\Big(\frac{0.3}{e_1}, \frac{0.6}{e_2}, \frac{0.7}{e_3}\frac{0.5}{e_4}), q, 1\Bigg), (u_4, u_5)\Bigg],$

$\Bigg[\Bigg(\Big(\frac{0.4}{e_1}, \frac{0.5}{e_2}, \frac{0.7}{e_3}\frac{0.3}{e_4}), q, 1\Bigg), (u_2)\Bigg],\ \Bigg[\Bigg(\Big(\frac{0.5}{e_1}, \frac{0.4}{e_2}, \frac{0.8}{e_3}\frac{0.3}{e_4}), q, 1\Bigg)(u_1, u_3)\Bigg],$

$\Bigg[\Bigg(\Big(\frac{0.3}{e_1}, \frac{0.6}{e_2}, \frac{0.7}{e_3}\frac{0.5}{e_4}), r, 1\Bigg), (u_4)\Bigg],\ \Bigg[\Bigg(\Big(\frac{0.4}{e_1}, \frac{0.5}{e_2}, \frac{0.7}{e_3}\frac{0.3}{e_4}), r, 1\Bigg), (u_1, u_2)\Bigg],$

$\Bigg[\Bigg(\Big(\frac{0.5}{e_1}, \frac{0.4}{e_2}, \frac{0.8}{e_3}\frac{0.3}{e_4}), r, 1\Bigg)(u_3, u_5)\Bigg],\ \Bigg[\Bigg(\Big(\frac{0.3}{e_1}, \frac{0.6}{e_2}, \frac{0.7}{e_3}\frac{0.5}{e_4}), p, 0\Bigg)(u_2, u_5,)\Bigg],$

$\Bigg[\Bigg(\Big(\frac{0.4}{e_1}, \frac{0.5}{e_2}, \frac{0.7}{e_3}\frac{0.3}{e_4}), p, 0\Bigg)(u_1, u_3, u_4,)\Bigg],\ \Bigg[\Bigg(\Big(\frac{0.3}{e_1}, \frac{0.6}{e_2}, \frac{0.7}{e_3}\frac{0.5}{e_4}), q, 0\Bigg), (u_1, u_2, u_3,)\Bigg],$

$\Bigg[\Bigg(\Big(\frac{0.4}{e_1}, \frac{0.5}{e_2}, \frac{0.7}{e_3}\frac{0.3}{e_4}), q, 0\Bigg), (u_1, u_3, u_4, u_5,)\Bigg],\ \Bigg[\Bigg(\Big(\frac{0.5}{e_1}, \frac{0.4}{e_2}, \frac{0.8}{e_3}\frac{0.3}{e_4}), q, 0\Bigg)(u_2, u_4, u_5,)\Bigg],$

$\Bigg[\Bigg(\Big(\frac{0.3}{e_1}, \frac{0.6}{e_2}, \frac{0.7}{e_3}\frac{0.5}{e_4}), r, 0\Bigg), (u_1, u_2, u_3, u_5,)\Bigg],\ \Bigg[\Bigg(\Big(\frac{0.4}{e_1}, \frac{0.5}{e_2}, \frac{0.7}{e_3}\frac{0.3}{e_4}), r, 0\Bigg), (u_3, u_4, u_5,)\Bigg],$

$$\Bigg[\Bigg(\Big(\frac{0.5}{e_1}, \frac{0.4}{e_2}, \frac{0.8}{e_3}\frac{0.3}{e_4}), r, 0\Bigg)(u_1, u_2, u_4,)\Bigg]\Bigg\}.$$

Definition 3.3. *Let* $(F, \mathcal{A})_{\tilde{A}}$ *and* $(G, \mathcal{B})_{\tilde{B}}$ *be two FFPSES then* $(F, \mathcal{A})_{\tilde{A}}$ *is a subset of* $(G, \mathcal{B})_{\tilde{B}}$ *written* $(F, \mathcal{A})_{\tilde{A}} \tilde{\subseteq} (G, \mathcal{B})_{\tilde{B}}$ *if*

i. $\mathcal{A} \tilde{\subseteq} \mathcal{B}$

ii. $f_{\tilde{A}}(\hat{y}_i, x_k, o) \subseteq g_{\tilde{A}}(\hat{z}_j, x_k, o)$ *for all* $(\hat{y}_i, x_k, o) \in \mathcal{A}$, *and* $\mathcal{A}, \mathcal{B} \subseteq \tilde{Z}$

Definition 3.4. *The compliment of* $FFPSES$ $(F, \mathcal{A})_{\tilde{A}}$ *denoted by* $(F, \mathcal{A})^c_{\tilde{A}}$ *is defined by*

$(F, \mathcal{A})^c_{\tilde{A}}$=$(F^c, \backsim \mathcal{A})_{\tilde{A}^c}$, *where* $f^c_{\tilde{A}^c} :\sim \mathcal{A} \to P(U)$ *is a mapping given by* $f^c_{\tilde{A}^c}(\hat{a}) = U - f_{\tilde{A}}(\hat{a})$ *for all* $\hat{a} \in \mathcal{A}$ *where* $\backsim \mathcal{A} \subseteq \{\tilde{A}^c \times X \times O\}$

Remark 3.5. *The cardinality of* $\tilde{A}$ *and* $\tilde{A}^c$ *are equal*

Proposition 3.6. *If* $(F, \mathcal{A})^c_{\tilde{A}}$ *is a* $FFPSES$ *over* U *then* $((F, \mathcal{A})^c_{\tilde{A}})^c = (F, \mathcal{A})_{\tilde{A}}$

Proof. From Definition 3.4, we have $((F, \mathcal{A})^c_{\tilde{A}})^c = ((F^c, \backsim \mathcal{A})_{\tilde{A}^c})^c$
where $(f^c)^c_{\tilde{A}^c} :\backsim (\backsim \mathcal{A}) \to P(U) = \mathcal{A} \to P(U)$ is a mapping given by $(f^c)^c_{\tilde{A}^c}(\hat{a}) = U - f^c_{\tilde{A}^c}(\hat{a}) = f_{\tilde{A}}(\hat{a})$ $\forall \hat{a} \in \backsim (\backsim \mathcal{A}) = \mathcal{A}$ where $\backsim (\backsim \mathcal{A}) \subseteq \{(\tilde{A}^c)^c \times X \times O\} = \{\tilde{A} \times X \times O\}$
therefore $((F, \mathcal{A})^c_{\tilde{A}})^c = (F, \mathcal{A})_{\tilde{A}}$ □

Definition 3.7. *The union of two* $FFPSES$ $(F, \mathcal{A})_{\tilde{A}}$ *and* $(G, \mathcal{B})_{\tilde{B}}$ *is given by:*

$$(H, \mathcal{C})_{\tilde{C}} = (F, \mathcal{A})_{\tilde{A}} \tilde{\cup} (G, \mathcal{B})_{\tilde{B}}$$

Where:

i. $\tilde{C} = \tilde{A} \Cup \tilde{B}$

ii. $h(\hat{c}) = f(\hat{a}) \cup g(\hat{b})$ *for all* $\hat{a} \in \mathcal{A}, \hat{b} \in \mathcal{B}, \hat{c} \in \mathcal{C}$,

and

$$\tilde{A} \Cup \tilde{B} = \begin{cases} \tilde{A} \tilde{\cup} \tilde{B}, & if\ |\tilde{A}| = |\tilde{B}| \\ \tilde{A} \uplus \tilde{B}, & if\ |\tilde{A}| \neq |\tilde{B}| \end{cases}$$

Where:
$\tilde{A} \uplus \tilde{B} = (\hat{A} \tilde{\cup} \tilde{B}) \cup (\tilde{A} \ \backslash \hat{A})$, *if* $|\tilde{A}| > |\tilde{B}|$, $|\hat{A}| = |\tilde{B}|$
and
$\tilde{A} \uplus \tilde{B} = (\tilde{A} \tilde{\cup} \hat{B}) \cup (\tilde{B} \ \backslash \hat{B})$, *if* $|\tilde{B}| > |\tilde{A}|$, $|\hat{B}| = |\tilde{A}|$.
For $\hat{A}$ *and* $\hat{B}$ *are arbitrary (classical) subsets of* $\tilde{A}$ *and* $\tilde{B}$ *respectively.*

Remark 3.8. *The choice of* $\hat{A}$ *(or* $\hat{B}$*) in the definition of* $\tilde{A} \Cup \tilde{B}$ *is not unique, but clearly in the classical sense the cardinality property of subsetness:*

$$|\tilde{A}|, |\tilde{B}| \leq |\tilde{A} \Cup \tilde{B}|$$

is preserved. Also in the fuzzy sense subsetness $\tilde{A}, \tilde{B} \leq \tilde{A} \Cup \tilde{B}$:

$$\forall \hat{a} \in \tilde{A} \ \ (or \ \ \hat{b} \in \tilde{B})\ \exists \hat{c} \in \tilde{A} \Cup \tilde{B} \ni \hat{a} \leq \hat{c} \ \ (or \ \ \hat{b} \leq \hat{c})$$

is preserved

Lemma 3.1. *Let* $\tilde{A}, \tilde{B}, \tilde{\subset} \tilde{E}$, *then* $\tilde{A} \Cup \tilde{B} = \tilde{B} \Cup \tilde{A}$.

Proof. By definition 3.7, assume the case $|\tilde{A}| = |\tilde{B}|$ then $\tilde{A} \Cup \tilde{B} = \tilde{A} \tilde{\cup} \tilde{B}$. Therefore since fuzzy joint union is commutative it follows $\tilde{A} \tilde{\cup} \tilde{B} = \tilde{B} \tilde{\cup} \tilde{A}$. Hence $\tilde{A} \tilde{\Cup} \tilde{B} = \tilde{B} \tilde{\Cup} \tilde{A}$. Next assume the case $|\tilde{A}| > |\tilde{B}|$ then $\tilde{A} \Cup \tilde{B} = \tilde{A} \uplus \tilde{B}$. By definition

$$\tilde{A} \Cup \tilde{B} = \tilde{A} \uplus \tilde{B} = (\hat{A} \tilde{\cup} \tilde{B}) \cup (\tilde{A} \setminus \hat{A}) = (\tilde{B} \tilde{\cup} \hat{A}) \cup (\tilde{A} \setminus \hat{A}) = \tilde{B} \uplus \tilde{A} = \tilde{B} \Cup \tilde{A}$$

(commutative property of fuzzy joint). Hence $\tilde{A} \Cup \tilde{B} = \tilde{B} \Cup \tilde{A}$.
The proof of $|\tilde{A}| > |\tilde{B}|$ in this case is given analogously. □

Lemma 3.2. *Let* $\tilde{A}, \tilde{B}, \tilde{C} \tilde{\subset} \tilde{E}$ *then* $\tilde{A} \Cup (\tilde{B} \Cup \tilde{C}) = (\tilde{A} \Cup \tilde{B}) \Cup \tilde{C}$

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Proof. From definition 3.7, we have

$$\tilde{A}\Cup(\tilde{B}\Cup\tilde{C}) = \begin{cases} \tilde{A}\tilde{\cup}(\tilde{B}\Cup\tilde{C}), & if\ |\tilde{A}| = |\tilde{B}\Cup\tilde{C}| \\ \tilde{A}\uplus(\tilde{B}\Cup\tilde{C}), & if\ |\tilde{A}| \neq |\tilde{B}\Cup\tilde{C}| \end{cases}$$

$Case\ I: |\tilde{A}| = |\tilde{B}\Cup\tilde{C}|$ By definition

$$\tilde{A}\tilde{\cup}(\tilde{B}\Cup\tilde{C}) = \begin{cases} \tilde{A}\tilde{\cup}(\tilde{B}\tilde{\cup}\tilde{C}), & if\ |\tilde{B}| = |\tilde{C}| \\ \tilde{A}\tilde{\cup}(\tilde{B}\uplus\tilde{C}), & if\ |\tilde{B}| \neq |\tilde{C}| \end{cases}$$

Again assume the case where $|\tilde{B}| = |\tilde{C}|$ then it easily follows that

$\tilde{A}\Cup(\tilde{B}\Cup\tilde{C}) = \tilde{A}\tilde{\cup}(\tilde{B}\tilde{\cup}\tilde{C}) = (\tilde{A}\tilde{\cup}\tilde{B})\tilde{\cup}\tilde{C} = (\tilde{A}\Cup\tilde{B})\Cup\tilde{C}$ (By definition and associativity property of fuzzy joint). Next assume the case where $|\tilde{B}| > |\tilde{C}|$ by definition

$\tilde{A}\tilde{\cup}(\tilde{B}\Cup\tilde{C}) = \tilde{A}\tilde{\cup}(\tilde{B}\uplus\tilde{C}) = \tilde{A}\tilde{\cup}[(\hat{B}\tilde{\cup}\tilde{C}) \cup (\tilde{B} \setminus \hat{B})]$
$= \tilde{A}\tilde{\cup}[(\hat{B} \cup (\tilde{B} \setminus \hat{B}))\tilde{\cup}\tilde{C}]$
$= \tilde{A}\tilde{\cup}[(\hat{B}\cup(\tilde{B} \cap \hat{B}^c))\ \tilde{\cup}\tilde{C}]$
$= \tilde{A}\tilde{\cup}[(\hat{B}\cup\tilde{B}) \cap (\hat{B} \cup \hat{B}^c))\ \tilde{\cup}\tilde{C}]$
$= \tilde{A}\tilde{\cup}[(\tilde{B} \cap \tilde{B})\ \tilde{\cup}\tilde{C}]$
$= \tilde{A}\tilde{\cup}(\tilde{B}\ \tilde{\cup}\tilde{C})$
$= (\tilde{A}\tilde{\cup}\tilde{B})\ \tilde{\cup}\tilde{C}$ (By definition and associativity of fuzzy joint)
$= (\tilde{A}\tilde{\cup}\tilde{B})\ \Cup\tilde{C}$ (By definition)

The proof of $|\tilde{C}| > |\tilde{B}|$ in this case is given analogously.

$Case\ II: |\tilde{A}| > |\tilde{B}\Cup\tilde{C}|$ (The proof of $|\tilde{A}| < |\tilde{B}\Cup\tilde{C}|$ in this case is given analogously). By definition it follows that

$\tilde{A}\Cup(\tilde{B}\Cup\tilde{C}) = \tilde{A}\uplus(\tilde{B}\Cup\tilde{C}) = (\hat{A}\tilde{\cup}(\tilde{B}\Cup\tilde{C})) \cup (\tilde{A} \setminus \hat{A})$

Where

$$\tilde{B}\Cup\tilde{C} = \begin{cases} \tilde{B}\tilde{\cup}\tilde{C}, & if\ |\tilde{B}| = |\tilde{C}| \\ \tilde{B}\uplus\tilde{C}, & if\ |\tilde{B}| \neq |\tilde{C}| \end{cases}$$

When $|\tilde{B}| = |\tilde{C}|$ then $\tilde{B}\Cup\tilde{C} = \tilde{B}\tilde{\cup}\tilde{C}$ and
$\tilde{A}\Cup(\tilde{B}\Cup\tilde{C}) = \tilde{A}\uplus(\tilde{B}\Cup\tilde{C}) = (\hat{A}\tilde{\cup}(\tilde{B}\tilde{\cup}\tilde{C})) \cup (\tilde{A} \setminus \hat{A})$
$= (\hat{A} \cup (\tilde{A} \setminus \hat{A}))\tilde{\cup}(\tilde{B}\tilde{\cup}\tilde{C})$
$= \tilde{A}\tilde{\cup}(\tilde{B} \cup \tilde{C}) = (\tilde{A}\tilde{\cup}\tilde{B}) \cup \tilde{C}$.(By associativity of fuzzy joint)
$= (\tilde{A}\Cup\tilde{B})\Cup\tilde{C}$ (by definition)

Next we consider $|\tilde{B}| > |\tilde{C}|$ then $\tilde{B}\Cup\tilde{C} = \tilde{B}\uplus\tilde{C}$ and
$\tilde{A}\uplus(\tilde{B}\Cup\tilde{C}) = (\hat{A}\tilde{\cup}[(\hat{B}\tilde{\uplus}\tilde{C})]) \cup (\tilde{A} \setminus \hat{A})$
$= (\hat{B}\tilde{\uplus}\tilde{C})\tilde{\cup}(\hat{A}\cup(\tilde{A} \setminus \hat{A})) = (\hat{B}\tilde{\uplus}\tilde{C})\tilde{\cup}((\hat{A}\cup\tilde{A}) \cap (\hat{A}\cup\hat{A}^c))$
$= ((\hat{B}\tilde{\cup}\tilde{C}) \cup (\tilde{B} \setminus \hat{B}))\tilde{\cup}((\hat{A}\cup\tilde{A}) \cap (\hat{A}\cup\hat{A}^c))$
$= (\tilde{C}\tilde{\cup}((\hat{B} \cup (\tilde{B} \setminus \hat{B}))\tilde{\cup}((\hat{A}\cup\tilde{A}) \cap (\hat{A}\cup\hat{A}^c))$
$= (\tilde{C}\tilde{\cup}((\hat{B}\cup\tilde{B}) \cap (\hat{B}\cup\hat{B}^c)))\tilde{\cup}((\hat{A}\cup\tilde{A}) \cap (\hat{A}\cup\hat{A}^c))$
$= \tilde{C}\tilde{\cup}(((\hat{B}\cup\tilde{B}) \cap (\hat{B}\cup\hat{B}^c))\tilde{\cup}((\hat{A}\cup\tilde{A}) \cap (\hat{A}\cup\hat{A}^c)))$ (By associativity of fuzzy joint)
$= \tilde{C}\tilde{\cup}(\tilde{B}\tilde{\cup}\tilde{A})$
$= (\tilde{A}\tilde{\cup}\tilde{B})\tilde{\cup}\tilde{C}$.(By commutativity of fuzzy joint)
$= (\tilde{A}\Cup\tilde{B})\Cup\tilde{C}$. (By definition)

Hence $\tilde{A}\Cup(\tilde{B}\Cup\tilde{C}) = (\tilde{A}\Cup\tilde{B})\Cup\tilde{C}$.
The case $|\tilde{B}| < |\tilde{C}|$ is proved analogously. □

Proposition 3.9. *If* $(F,\mathcal{A})_{\tilde{A}}$,$(G,\mathcal{B})_{\tilde{B}}$ *and* $(H,\mathcal{C})_{\tilde{C}}$ *are three* $FFPSES$ *over* U *then*

a. $(F,\mathcal{A})_{\tilde{A}}\ \tilde{\cup}\ (G,\mathcal{B})_{\tilde{B}} = (G,\mathcal{B})_{\tilde{B}}\ \tilde{\cup}\ (F,\mathcal{A})_{\tilde{A}}$.

b. $(F,\mathcal{A})_{\tilde{A}}\ \tilde{\cup}\ \left((G,\mathcal{B})_{\tilde{B}}\ \tilde{\cup}\ (H,\mathcal{C})_{\tilde{C}}\right) = \left((F,\mathcal{A})_{\tilde{A}}\ \tilde{\cup}\ (G,\mathcal{B})_{\tilde{B}}\right)\ \tilde{\cup}\ (H,\mathcal{C})_{\tilde{C}}$.

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Proof. a. From Definition 3.7, given that$(F,\mathcal{A})_{\tilde{A}} \tilde{\cup} (G,\mathcal{B})_{\tilde{B}}$ then $\tilde{A}\Cup\tilde{B}$ and $f_{\tilde{A}}(\hat{a}) \cup g_{\tilde{B}}(\hat{b})$ for all $\hat{a} \in \mathcal{A}, \hat{b} \in \mathcal{B}$ From Lemma 3.1 $\tilde{A}\Cup\tilde{B} = \tilde{B}\Cup\tilde{A}$ and since union of classical set is commutative then $f_{\tilde{A}}(\hat{a}) \cup g_{\tilde{B}}(\hat{b}) = g_{\tilde{B}}(\hat{b}) \cup f_{\tilde{A}}(\hat{a})$ Then $(F,\mathcal{A})_{\tilde{A}} \tilde{\cup} (G,\mathcal{B})_{\tilde{B}} = (G,\mathcal{B})_{\tilde{B}} \tilde{\cup} (F,\mathcal{A})_{\tilde{A}}$

b. From Definition 3.7 given that$(F,\mathcal{A})_{\tilde{A}} \tilde{\cup} \left((G,\mathcal{B})_{\tilde{B}} \tilde{\cup} (H,\mathcal{C})_{\tilde{C}}\right)$ then $\tilde{A}\Cup(\tilde{B}\Cup\tilde{C})$ and $f_{\tilde{A}}(\hat{a}) \cup (g_{\tilde{B}}(\hat{b}) \cup h_{\tilde{C}}(\hat{c}))$. From Lemma 3.2 $\tilde{A}\Cup(\tilde{B}\Cup\tilde{C}) = (\tilde{A}\Cup\tilde{B})\Cup\tilde{C}$ and union of classical set is associative

therefore $(F,\mathcal{A})_{\tilde{A}} \tilde{\cup} \left((G,\mathcal{B})_{\tilde{B}} \tilde{\cup} (H,\mathcal{C})_{\tilde{C}}\right) = \left((F,\mathcal{A})_{\tilde{A}} \tilde{\cup} (G,\mathcal{B})_{\tilde{B}}\right) \tilde{\cup} (H,\mathcal{C})_{\tilde{C}}$

□

Definition 3.10. *The intersection of two FFPSES $(F,\mathcal{A})_{\tilde{A}}$ and $(G,\mathcal{B})_{\tilde{B}}$ is given by:*

$$(H,\mathcal{C})_{\tilde{C}} = (F,\mathcal{A})_{\tilde{A}} \tilde{\cap} (G,\mathcal{B})_{\tilde{B}}$$

Where:

i. $\tilde{C} = \tilde{A} \Cap \tilde{B}$

ii. $h(\hat{c}) = f(\hat{a}) \cap g(\hat{b})$ *for all* $\hat{a} \in \mathcal{A}, \hat{b} \in \mathcal{B}, \hat{c} \in \mathcal{C}$,

and

$$\tilde{A} \Cap \tilde{B} = \begin{cases} \tilde{A}\tilde{\cap}\tilde{B}, & if\ |\tilde{A}| = |\tilde{B}| \\ \tilde{A}\sqcap\tilde{B}, & if\ |\tilde{A}| \neq |\tilde{B}| \end{cases}$$

Where:
$\tilde{A} \sqcap \tilde{B} = (\hat{A}\tilde{\cap}\tilde{B})$, *if* $|\tilde{A}| > |\tilde{B}|$, $|\hat{A}| = |\tilde{B}|$
and
$\tilde{A} \sqcap \tilde{B} = (\tilde{A}\tilde{\cap}\hat{B})$, *if* $|\tilde{B}| > |\tilde{A}|$, $|\hat{B}| = |\tilde{A}|$.
For $\hat{A}$ and $\hat{B}$ are arbitrary (classical) subsets of $\tilde{A}$ and $\tilde{B}$ respectively.

Remark 3.11. *The choice of $\hat{A}$ (or $\hat{B}$) in definition 3.10 of $\tilde{A} \Cap \tilde{B}$ is not unique, but clearly in the classical sense the cardinality property of subsetness:*

$$|\tilde{A}|, |\tilde{B}| \geq |\tilde{A} \Cap \tilde{B}|$$

is preserved. Also in the fuzzy sense subsetness $\tilde{A}, \tilde{B} \geq \tilde{A} \Cap \tilde{B}$:

$$\forall \hat{a} \in \tilde{A}\ \ (or\ \ \hat{b} \in \tilde{B})\ \exists \hat{c} \in \tilde{A} \Cap \tilde{B} \ni \hat{a} \geq \hat{c}\ \ (or\ \ \hat{b} \geq \hat{c})$$

is preserved

Lemma 3.3. *Let $\tilde{A}, \tilde{B}, \subset \tilde{E}$ then $\tilde{A}\tilde{\Cap}\tilde{B} = \tilde{B}\tilde{\Cap}\tilde{A}$*

Proof. By definition

$$\tilde{A} \Cap \tilde{B} = \begin{cases} \tilde{A}\tilde{\cap}\tilde{B}, & if\ |\tilde{A}| = |\tilde{B}| \\ \tilde{A}\sqcap\tilde{B}, & if\ |\tilde{A}| \neq |\tilde{B}| \end{cases}$$

where $\hat{A}$ is arbitrary subset of $\tilde{A}$
Assuming $|\tilde{A}| = |\tilde{B}|$, then
$\tilde{A}\Cap\tilde{B} = \tilde{A}\tilde{\cap}\tilde{B} = \tilde{B}\tilde{\cap}\tilde{A} = \tilde{B}\Cap\tilde{A}$.
Assuming $|\tilde{A}| > |\tilde{B}|$, then
$\tilde{A}\Cap\tilde{B} = \tilde{A} \sqcap \tilde{B} = \hat{A}\tilde{\cap}\tilde{B} = \tilde{B}\tilde{\cap}\hat{A} = \tilde{B} \sqcap \tilde{A} = \tilde{B}\Cap\tilde{A}$.
The proof of $|\tilde{A}| < |\tilde{B}|$ in this case is given analogously.

□

Lemma 3.4. *Let $\tilde{A}, \tilde{B}, \tilde{C} \subset \tilde{E}$ then $\tilde{A}\tilde{\Cap}(\tilde{B}\Cap\tilde{C}) = (\tilde{A}\tilde{\Cap}\tilde{B})\Cap\tilde{C}$*

Proof. By definition

$$\tilde{A}\Cap(\tilde{B}\Cap\tilde{C}) = \begin{cases} \tilde{A}\tilde{\cap}(\tilde{B}\Cap\tilde{C}) & if\ |\tilde{A}| = |\tilde{B}\Cap\tilde{C}| \\ \tilde{A}\sqcap(\tilde{B}\Cap\tilde{C}) & if\ |\tilde{A}| \neq |\tilde{B}\Cap\tilde{C}| \end{cases}$$

$Case\ I$: $|\tilde{A}| = |\tilde{B}\Cap\tilde{C}|$, if $|\tilde{B}| = |\tilde{C}|$, then by definition
$\tilde{A}\Cap(\tilde{B}\Cap\tilde{C}) = \tilde{A}\tilde{\cap}(\tilde{B}\Cap\tilde{C}) = \tilde{A}\tilde{\cap}(\tilde{B}\tilde{\cap}\tilde{C}) = (\tilde{A}\tilde{\cap}\tilde{B})\tilde{\cap}\tilde{C} = (\tilde{A}\Cap\tilde{B})\Cap\tilde{C}$.
Now if $|\tilde{B}| > |\tilde{C}|$, then
$\tilde{A}\Cap(\tilde{B}\Cap\tilde{C}) = \tilde{A}\tilde{\cap}(\tilde{B}\Cap\tilde{C}) = \tilde{A}\tilde{\cap}(\hat{B}\tilde{\cap}\tilde{C}) = (\tilde{A}\tilde{\cap}\hat{B})\tilde{\cap}\tilde{C} = (\tilde{A}\Cap\tilde{B})\Cap\tilde{C}$.
The proof of $|\tilde{B}| < |\tilde{C}|$ in this case is given analogously.
$Case\ II$: $|\tilde{A}| > |\tilde{B}\Cap\tilde{C}|$, if $|\tilde{B}| = |\tilde{C}|$, then by definition
$\tilde{A}\Cap(\tilde{B}\Cap\tilde{C}) = \hat{A}\tilde{\cap}(\tilde{B}\Cap\tilde{C}) = \hat{A}\tilde{\cap}(\tilde{B}\tilde{\cap}\tilde{C}) = (\hat{A}\tilde{\cap}\tilde{B})\tilde{\cap}\tilde{C} = (\tilde{A}\Cap\tilde{B})\Cap\tilde{C}$.
The proof of $|\tilde{A}| < |\tilde{B}\Cap\tilde{C}|$ in this case is given analogously.
Now if $|\tilde{B}| > |\tilde{C}|$, then
$\tilde{A}\Cap(\tilde{B}\Cap\tilde{C}) = \hat{A}\tilde{\cap}(\tilde{B}\Cap\tilde{C}) = \hat{A}\tilde{\cap}(\hat{B}\tilde{\cap}\tilde{C}) = (\hat{A}\tilde{\cap}\hat{B})\tilde{\cap}\tilde{C} = (\tilde{A}\Cap\tilde{B})\Cap\tilde{C}$
The proof of $|\tilde{B}| < |\tilde{C}|$ in this case is given analogously. ☐

Proposition 3.12. *If* $(F,\mathcal{A})_{\tilde{A}}$,$(G,\mathcal{B})_{\tilde{B}}$ *and* $(H,\mathcal{C})_{\tilde{C}}$ *are three* $FFPSES$ *over* u *then*

a. $(F,\mathcal{A})_{\tilde{A}}\ \tilde{\cap}\ (G,\mathcal{B})_{\tilde{B}} = (G,\mathcal{B})_{\tilde{B}}\ \tilde{\cap}\ (F,\mathcal{A})_{\tilde{A}}$

b. $(F,\mathcal{A})_{\tilde{A}}\ \tilde{\cap}\ \left((G,\mathcal{B})_{\tilde{B}}\ \tilde{\cap}\ (H,\mathcal{C})_{\tilde{C}}\right) = \left((F,\mathcal{A})_{\tilde{A}}\ \tilde{\cap}\ (G,\mathcal{B})_{\tilde{B}}\right)\ \tilde{\cap}\ (H,\mathcal{C})_{\tilde{C}}$

Proof. The proof is same as Proposition 3.9 using Lemma 3.3 and Lemma 3.4 ☐

Lemma 3.5. *Let* $\tilde{A}, \tilde{B}, \tilde{C} \subset \tilde{E}$*, then*

i. $\tilde{A}\tilde{\Cup}(\tilde{B}\Cap\tilde{C}) = (\tilde{A}\tilde{\Cup}\tilde{B}) \Cap (\tilde{A}\Cup\tilde{C})$

ii. $\tilde{A}\tilde{\Cap}(\tilde{B}\Cup\tilde{C}) = (\tilde{A}\tilde{\Cap}\tilde{B}) \Cup (\tilde{A}\Cap\tilde{C})$

Proof. By definition

$$\tilde{A}\Cup(\tilde{B}\Cap\tilde{C}) = \begin{cases} \tilde{A}\tilde{\cup}(\tilde{B}\Cap\tilde{C}), & if\ |\tilde{A}| = |\tilde{B}\Cap\tilde{C}| \\ \tilde{A}\tilde{\uplus}(\tilde{B}\Cap\tilde{C}) & if\ |\tilde{A}| \neq |\tilde{B}\Cap\tilde{C}| \end{cases}$$

$Case\ I$: Let $|\tilde{A}| = |\tilde{B}\Cap\tilde{C}|$ and $|\tilde{B}| = |\tilde{C}|$ then by definition $\tilde{A}\Cup(\tilde{B}\Cap\tilde{C}) = \tilde{A}\tilde{\cup}(\tilde{B}\tilde{\cap}\tilde{C})$. Therefore, by the distributive property of fuzzy set, it follows that $\tilde{A}\tilde{\cup}(\tilde{B}\tilde{\cap}\tilde{C}) = (\tilde{A}\tilde{\cup}\tilde{B})\tilde{\cap}(\tilde{A}\tilde{\cup}\tilde{C}) = (\tilde{A}\Cup\tilde{B})\Cap(\tilde{A}\Cup\tilde{C})$. Now for $|\tilde{B}| > |\tilde{C}|$. By definition, $\tilde{A}\Cup(\tilde{B}\Cap\tilde{C}) = \tilde{A}\tilde{\cup}(\hat{B}\tilde{\cap}\tilde{C})$, and then by distributive property of fuzzy set $\tilde{A}\tilde{\cup}(\hat{B}\tilde{\cap}\tilde{C}) = (\tilde{A}\tilde{\cup}\hat{B})\tilde{\cap}(\tilde{A}\tilde{\cup}\tilde{C}) = (\tilde{A}\Cup\tilde{B})\Cap(\tilde{A}\Cup\tilde{C})$.
The reverse order, $|\tilde{B}| < |\tilde{C}|$ in this case is prove analogously.
$Case\ II$: Let $|\tilde{A}| > |\tilde{B}\Cap\tilde{C}|$, where $|\tilde{B}| = |\tilde{C}|$, then by definition
$\tilde{A}\Cup(\tilde{B}\Cap\tilde{C}) = \tilde{A}\tilde{\uplus}(\tilde{B}\tilde{\cap}\tilde{C})) = (\hat{A}\tilde{\cup}(\tilde{B}\tilde{\cap}\tilde{C})) \cup (\tilde{A} \setminus \hat{A})$
$= ((\hat{A}\tilde{\cup}\tilde{B})\tilde{\cap}(\hat{A}\tilde{\cup}\tilde{C})) \cup (\tilde{A} \setminus \hat{A})$
$= ((\hat{A}\tilde{\cup}\tilde{B}) \cup (\tilde{A} \setminus \hat{A}))\tilde{\cap}((\hat{A}\tilde{\cup}\tilde{C}) \cup (\tilde{A} \setminus \hat{A}))$
$= (\tilde{A} \uplus \tilde{B})\tilde{\cap}(\tilde{A} \uplus \tilde{C})$ (By definition)
$= (\tilde{A}\Cup\tilde{B})\tilde{\cap}(\tilde{A}\Cup\tilde{C}) = (\tilde{A}\Cup\tilde{B})\Cap(\tilde{A}\Cup\tilde{C})$ by simple argument.
The reverse order, $|\tilde{A}| < |\tilde{B}\Cap\tilde{C}|$ in this case is prove analogously.
Next, for $|\tilde{B}| > |\tilde{C}|$
$\tilde{A}\Cup(\tilde{B}\Cap\tilde{C}) = \tilde{A}\tilde{\uplus}(\hat{B}\tilde{\cap}\tilde{C}) = (\hat{A}\tilde{\cup}(\hat{B}\tilde{\cap}\tilde{C})) \cup (\tilde{A} \setminus \hat{A})$
$= ((\hat{A}\tilde{\cup}\hat{B})\tilde{\cap}(\hat{A}\tilde{\cup}\tilde{C})) \cup (\tilde{A} \setminus \hat{A})$
$= ((\hat{A}\tilde{\cup}\hat{B}) \cup (\tilde{A} \setminus \hat{A}))\tilde{\cap}((\hat{A}\tilde{\cup}\tilde{C}) \cup (\tilde{A} \setminus \hat{A}))$
$= (\tilde{A} \uplus \hat{B})\tilde{\cap}(\tilde{A} \uplus \tilde{C})$ (By definition)
$= (\tilde{A}\Cup\hat{B})\tilde{\cap}(\tilde{A}\Cup\tilde{C}) = (\tilde{A}\Cup\tilde{B})\Cap(\tilde{A}\Cup\tilde{C})$ (By definition).
The reverse order, $|\tilde{B}| < |\tilde{C}|$ in this case is prove analogously.

☐

Proposition 3.13. *If* $(F,\mathcal{A})_{\tilde{A}}$,$(G,\mathcal{B})_{\tilde{B}}$ *and* $(H,\mathcal{C})_{\tilde{C}}$ *are three* $FFPSES$ *over* U *then*

a. $(F,\mathcal{A})_{\tilde{A}} \tilde{\cup} \left((G,\mathcal{B})_{\tilde{B}} \tilde{\cap} (H,\mathcal{C})_{\tilde{C}}\right) = \left((F,\mathcal{A})_{\tilde{A}} \tilde{\cup} (G,\mathcal{B})_{\tilde{B}}\right) \tilde{\cap}\left((F,\mathcal{A})_{\tilde{A}} \tilde{\cup} (H,\mathcal{C})_{\tilde{C}}\right)$

b. $(F,\mathcal{A})_{\tilde{A}} \tilde{\cap} \left((G,\mathcal{B})_{\tilde{B}} \tilde{\cup} (H,\mathcal{C})_{\tilde{C}}\right) = \left((F,\mathcal{A})_{\tilde{A}} \tilde{\cap} (G,\mathcal{B})_{\tilde{B}}\right) \tilde{\cup}\left((F,\mathcal{A})_{\tilde{A}} \tilde{\cap} (H,\mathcal{C})_{\tilde{C}}\right)$

Proof. From Lemma 3.5 $\tilde{A}\tilde{\Cup}(\tilde{B}\Cap\tilde{C}) = (\tilde{A}\tilde{\Cup}\tilde{B})\Cap(\tilde{A}\Cup\tilde{C})$ and base on Distributive property of classical set the proposition hold. □

Lemma 3.6. *Let* $\tilde{A}, \tilde{B} \subset \tilde{E}$ *then*

a. $(\tilde{A}\Cap\tilde{B})^c \subseteq \tilde{A}^c\Cup\tilde{B}^c$

b. $\tilde{A}^c\Cap\tilde{B}^c \subseteq (\tilde{A}\Cup\tilde{B})^c$

Proof. a. Let $|\tilde{A}| > |\tilde{B}|$. Clearly from the definitions of union and intersection of $FFPSES$, $|(\tilde{A}\Cup\tilde{B})| = |\tilde{A}|$ and $|(\tilde{A}\Cap\tilde{B})| = |\tilde{B}|$, and by remark 3.5, $|\tilde{A}| = |\tilde{A}^c|$ and $|\tilde{B}| = |\tilde{B}^c|$. Then $|(\tilde{A}\Cap\tilde{B})^c| = |\tilde{B}^c| < |\tilde{A}^c| = |\tilde{A}^c\Cup\tilde{B}^c|$. Therefore, $|(\tilde{A}\Cap\tilde{B})^c| < |\tilde{A}^c\Cup\tilde{B}^c|$. Next let $\hat{x} \in (\tilde{A}\Cap\tilde{B})^c$, then
$\hat{x} = min(\hat{y}_i, \hat{z}_i)^c$ where $\hat{y_i}^c \in \tilde{A}^c, \hat{z_j}^c \in \tilde{B}^c$
$\hat{x} = \{min(\mu_{\hat{y}}(e_i), \mu_{\hat{z}}(e_i)) : e_i \in E\}^c$
$\hat{x} = \{1 - min(\mu_{\hat{y}}(e_i), \mu_{\hat{z}}(e_i)) : e_i \in E\} \leq max\{1 - \mu_{\hat{y}}(e_i), 1 - \mu_{\hat{z}}(e_i) : e_i \in E\}$
$= max(\hat{y_i}^c, \hat{z_j}^c) = \tilde{A}^c\Cup\tilde{B}^c$.
Therefore, $(\tilde{A} \Cap \tilde{B})^c = \tilde{A}^c \Cup \tilde{B}^c$.
b. This is proved analogously. □

Remark 3.14. *Let* $\tilde{A}, \tilde{B} \subset \tilde{E}$ *then*

a. $(\tilde{A}\Cap\tilde{B})^c = \tilde{A}^c\Cup\tilde{B}^c$

b. $\tilde{A}^c\Cap\tilde{B}^c = (\tilde{A}\Cup\tilde{B})^c$
if and only if $|\tilde{A}| = |\tilde{B}|$.

Proof. a. Assume that $|\tilde{A}| = |\tilde{B}|$, then $|(\tilde{A} \Cap \tilde{B})^c| = |\tilde{A}^c| = |\tilde{A}| = |\tilde{B}| = |\tilde{B}^c| = |\tilde{A}^c \Cup \tilde{B}^c|$.
Conversely,
$|\tilde{A}| = |\tilde{A}^c| = |(\tilde{A}\Cap\tilde{B})^c| = |\tilde{A}^c\Cup\tilde{B}^c| = |\tilde{B}^c| = |\tilde{B}|$.
Therefore, $(\tilde{A} \Cap \tilde{B})^c = \tilde{A}^c \Cup \tilde{B}^c$.
b. This is proved analogously. □

Proposition 3.15. *Let* $(F,\mathcal{A})_{\tilde{A}}$ *and* $G,\mathcal{B})_{\tilde{B}}$ *be two* $FFPSES$ *then*

a. $\left((F,\mathcal{A})_{\tilde{A}} \tilde{\cap} (G,\mathcal{B})_{\tilde{B}}\right)^c \tilde{\subseteq} (F,\mathcal{A})^c_{\tilde{A}} \tilde{\cup} (G,\mathcal{B})^c_{\tilde{B}}$

b. $(F,\mathcal{A})^c_{\tilde{A}} \tilde{\cap} (G,\mathcal{B})^c_{\tilde{B}} \tilde{\subseteq} \left((F,\mathcal{A})_{\tilde{A}} \tilde{\cup} (G,\mathcal{B})_{\tilde{B}}\right)^c$

Proof. The prove follows from Lemma 3.6 and DeMorgan law of classical set. □

Definition 3.16. *An Agree* $FFPSES$ $(F,\mathcal{A})_{\tilde{A}}$ *over* U *is* $FFPSESubset$ *of* $(F,\mathcal{A})_{\tilde{A}}$ *where the opinions of all experts are Agree and is defined as follows:*

$$(F,\mathcal{A})_{\tilde{A}_1} = \left\{(\alpha, f^1_{\tilde{A}}(\alpha)) : \alpha \in \tilde{A} \times X \times \{1\}\right\}.$$

Remark 3.17. *Let* $(F,\mathcal{A})_{\tilde{A}_1}$ *be an agree* $FFPSES$ *over* U *then*
$u_i \in f(\hat{y}_p, x_j, \{1\}) \implies u_i \notin f(\hat{y}_k, x_j, \{1\})$ *for* $p \neq k$

Definition 3.18. *A Disagree* $FFPSES$ $(F,\mathcal{A})_{\tilde{A}}$ *over* U *is* $FFPSESubset$ *of* $(F,\mathcal{A})_{\tilde{A}}$ *where the opinions of all experts are Disagree and is defined as follows:*

$$(F,\mathcal{A})_{\tilde{A}_0} = \left\{(\alpha, f^0_{\tilde{A}}(\alpha)) ; \alpha \in \tilde{A} \times X \times \{0\}\right\}$$

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Definition 3.19. *If* $(F,\mathcal{A})_{\tilde{A}}$ *and* $(G,\mathcal{B})_{\tilde{B}}$ *are two* $FFPSES$ *over* U *then :*

$$(F,\mathcal{A})_{\tilde{A}}\ AND\ (G,\mathcal{B})_{\tilde{B}}\ \text{denoted}\ (F,\mathcal{A})_{\tilde{A}} \wedge (G,\mathcal{B})_{\tilde{B}}\ \text{is defined by}$$
$$(F,\mathcal{A})_{\tilde{A}} \wedge (G,\mathcal{B})_{\tilde{B}} = (H,\mathcal{A}\times\mathcal{B})_{\tilde{R}}$$

Such that: $h(\alpha,\beta)_{\tilde{R}} = f_{\tilde{A}}(\alpha) \cap g_{\tilde{B}}(\beta),\ \forall\ (\alpha,\beta) \in \mathcal{A}\times\mathcal{B}$, *where* $\tilde{R} = \tilde{A}\times\tilde{B}$

Definition 3.20. *If* $(F,\mathcal{A})_{\tilde{A}}$ *and* $(G,\mathcal{B})_{\tilde{B}}$ *are two* $FFPSES$ *over* U *then :*

$$(F,\mathcal{A})_{\tilde{A}}\ OR\ (G,\mathcal{B})_{\tilde{B}}\ \text{denoted}\ (F,\mathcal{A})_{\tilde{A}} \vee (G,\mathcal{B})_{\tilde{B}}\ \text{is defined by}$$
$$(F,\mathcal{A})_{\tilde{A}} \vee (G,\mathcal{B})_{\tilde{B}} = (H,\mathcal{A}\times\mathcal{B})_{\tilde{R}}$$

Such that: $h(\alpha,\beta)_{\tilde{R}} = f_{\tilde{A}}(\alpha) \cup g_{\tilde{B}}(\beta),\ \forall\ (\alpha,\beta) \in \mathcal{A}\times\mathcal{B}$, *where* $\tilde{R} = \tilde{A}\times\tilde{B}$

4 APPLICATION OF $FFPSES$

In this section the method of full fuzzificcation of set of parameters as presented in Edeghagba & Muhammad, [22] will be used to modify the algorithm developed by Bashir & Salleh, [23], and apply in decision making.

Definition 4.1. *Optimal Choice Object* : *Let* $F_{\tilde{A}}$ *be a* $FFPSES$ *then* $\mathcal{T}_m F_{\tilde{A}}$ *is the optimal choice Value given by*

$$\mathcal{T}_m F_{\tilde{A}} = \sum_{e_i \in E}\left(\Phi_{ij}\frac{\sum \mu_E(e_i)}{n(E)}\right) - \sum_{e_i \in E}\left(\phi_{ij}\frac{\sum \mu_E(e_i)}{n(E)}\right)\ \text{for}\ m = 1,2,3..p\ \text{and}\ n(U) = p$$

where

$$\Phi_{p_k,\hat{y}_j}(u_i) = \begin{cases} 1, & if\ u_i \in f^1(\hat{a_{kj}}) \\ 0, & if\ u_i \notin f^1(\hat{a_{kj}}) \end{cases}$$

and

$$\phi_{p_k,\hat{y}_j}(u_i) = \begin{cases} 1, & if\ u_i \in f^0(\hat{a_{kj}}) \\ 0, & if\ u_i \notin f^0(\hat{a_{kj}}) \end{cases}$$

(where Φ *and* ϕ *are entries on Agree and Disagree table respectively) and* u_m *is the Optimal Choice Object*

Definition 4.2. *Let* $\mathcal{T}_m F_{\tilde{A}}, \mathcal{T}_m G_{\tilde{B}}$ *be two optimal choice values over the same universe* U *where*

$$\mathcal{T}_m G_{\tilde{B}} > \mathcal{T}_m F_{\tilde{A}},$$

then $\mathcal{T}_m G_{\tilde{B}}$ *is called Prime Optimal Choice Value*

4.1 Algorithm

i. Take Considerations of the expert for all elements of the universe in respect of all the parameters

ii. input the $FFPSES$

iii. Deduce an $agreeFFPSES$ and $disagreeFFPSES$

iv Find $\mathcal{U}_i = \sum\left(\Phi_{ij}\frac{\sum \mu_E(e_i)}{n(E)}\right)$

v. Find $\mathcal{V}_i = \sum\left(\phi_{ij}\frac{\sum \mu_E(e_i)}{n(E)}\right)$

vi. Find $\mathcal{T}_i = \mathcal{U}_i - \mathcal{V}_i$

vii. Find m such that $\mathcal{T}_m = max\mathcal{T}_i$ then u_m is the optimal choice object

Proposition 4.3. *Let $(F, \mathcal{A})_{\tilde{A}_0}$ be a Disagree $FFPSE-Set$ over U if $u_i \in f(\hat{y_p}, x_j, \{0\}) \cap f(\hat{y_k}, x_j, \{0\})$, then $\mathcal{U}_i < \mathcal{V}_i$*

Proof. From *Remark* 3.17, let $u_i \in f(\hat{y_1}, x_j, \{1\})$, then $u_i \notin \bigcap_{p=2}^{n} f(\hat{y_p}, x_j, \{1\})$.

Therefore $\mathcal{U}_i = \sum \left(\Phi_{ij} \frac{\sum \mu_E(e_i)}{n(E)}\right) = \left(\frac{\sum \mu_E(e_i)}{n(E)}\right)$, since $\Phi_{ij} = 1$.

Likewise if $u_i \in \bigcap_{p=1}^{n} f(\hat{y_p}, x_j, \{0\})$, then

$\mathcal{V}_i = \left(\frac{(\sum \mu_E(e_i)}{n(E)}\right) + \left(\frac{\sum \mu_E(e_i)}{n(E)}\right) + ... + \left(\frac{\sum \mu_E(e_i)}{n(E)}\right) > \left(\frac{\sum \mu_E(e_i)}{n(E)}\right) = \mathcal{U}_i$.

Therefore $\mathcal{U}_i < \mathcal{V}_i$

□

Example 4.4. *An organisation announced that a number of grants to be awarded for researchers in the areas of Science, Technology and Innovation. The grants are to be given out based on the quality of the concept to be submitted by an applicant. With regards to this the organisation have hired experts in those areas for the screening of the applications.The experts are to consider the following parameters;e_1 = Originality of work, e_2 = Relevance of work,e_3 = Design of the work, e_4 = Crossboarder relevance of the work. Each of these criteria(parameters) are to be considered graded for each of the applicants to enable the organisation know how best to award the grant. Assume the organisation has grants 20million, 25million, 35million, and 50 million Naira to be awarded to four applicants $\{u_1, u_2, u_3, u_4\}$.*

Now the considerations of the articles by three experts, p_1, p_2, p_3 are as follows,

$$\begin{aligned} p_1 \Rightarrow u_1 &\rightarrow (\tfrac{0.6}{e_1}, \tfrac{0.7}{e_2}, \tfrac{0.8}{e_3}, \tfrac{0.4}{e_4}) \\ ,u_2 &\rightarrow (\tfrac{0.4}{e_1}, \tfrac{0.6}{e_2}, \tfrac{0.4}{e_3}, \tfrac{0.5}{e_4}), \\ u_3 &\rightarrow (\tfrac{0.4}{e_1}, \tfrac{0.7}{e_2}, \tfrac{0.6}{e_3}, \tfrac{0.5}{e_4}), \\ u_4 &\rightarrow (\tfrac{0.4}{e_1}, \tfrac{0.7}{e_2}, \tfrac{0.6}{e_3}, \tfrac{0.5}{e_4}), \end{aligned}$$

$$\begin{aligned} p_2 \Rightarrow u_1 &\rightarrow (\tfrac{0.4}{e_1}, \tfrac{0.6}{e_2}, \tfrac{0.4}{e_3}, \tfrac{0.5}{e_4}), \\ u_2 &\rightarrow (\tfrac{0.4}{e_1}, \tfrac{0.7}{e_2}, \tfrac{0.6}{e_3}, \tfrac{0.5}{e_4}), \\ u_3 &\rightarrow (\tfrac{0.4}{e_1}, \tfrac{0.6}{e_2}, \tfrac{0.4}{e_3}, \tfrac{0.5}{e_4}), \\ u_4 &\rightarrow (\tfrac{0.6}{e_1}, \tfrac{0.7}{e_2}, \tfrac{0.8}{e_3}, \tfrac{0.4}{e_4}), \end{aligned}$$

$$\begin{aligned} p_3 \Rightarrow u_1 &\rightarrow (\tfrac{0.6}{e_1}, \tfrac{0.7}{e_2}, \tfrac{0.8}{e_3}, \tfrac{0.4}{e_4}), \\ u_2 &\rightarrow (\tfrac{0.4}{e_1}, \tfrac{0.7}{e_2}, \tfrac{0.6}{e_3}, \tfrac{0.5}{e_4}), \\ u_3 &\rightarrow (\tfrac{0.6}{e_1}, \tfrac{0.7}{e_2}, \tfrac{0.8}{e_3}, \tfrac{0.4}{e_4}), \\ u_4 &\rightarrow (\tfrac{0.4}{e_1}, \tfrac{0.7}{e_2}, \tfrac{0.6}{e_3}, \tfrac{0.5}{e_4}). \end{aligned}$$

From the considerations above it follows that $\tilde{A} = \{\hat{y_1}, \hat{y_2}, \hat{y_3}\} \subset \tilde{E}$.where

$$\hat{y_1} = \{\tfrac{0.4}{e_1}, \tfrac{0.7}{e_2}, \tfrac{0.6}{e_3}, \tfrac{0.5}{e_4}\}$$

$$\hat{y_2} = \{\tfrac{0.6}{e_1}, \tfrac{0.7}{e_2}, \tfrac{0.8}{e_3}, \tfrac{0.4}{e_4}\}$$

$$\hat{y_3} = \{\tfrac{0.4}{e_1}, \tfrac{0.6}{e_2}, \tfrac{0.4}{e_3}, \tfrac{0.5}{e_4}\}.$$

Now the functional values are,

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$$f_{\tilde{A}}\ ((\frac{0.4}{e_1},\frac{0.7}{e_2},\frac{0.6}{e_3},\frac{0.5}{e_4}),p_1,1)=\{u_3,u_4\}$$

$$f_{\tilde{A}}\ ((\frac{0.6}{e_1},\frac{0.7}{e_2},\frac{0.8}{e_3},\frac{0.4}{e_4}),p_1,1)=\{u_1,\}$$

$$f_{\tilde{A}}\ ((\frac{0.4}{e_1},\frac{0.6}{e_2},\frac{0.4}{e_3},\frac{0.5}{e_4}),p_1,1)=\{u_2\}$$

$$f_{\tilde{A}}\ ((\frac{0.4}{e_1},\frac{0.7}{e_2},\frac{0.6}{e_3},\frac{0.5}{e_4}),p_2,1)=\{u_2\}$$

$$f_{\tilde{A}}\ ((\frac{0.6}{e_1},\frac{0.7}{e_2},\frac{0.8}{e_3},\frac{0.4}{e_4}),p_2,1)=\{u_4\}$$

$$f_{\tilde{A}}\ ((\frac{0.4}{e_1},\frac{0.6}{e_2},\frac{0.4}{e_3},\frac{0.5}{e_4}),p_2,1)=\{u_1,u_3\}$$

$$f_{\tilde{A}}\ ((\frac{0.4}{e_1},\frac{0.7}{e_2},\frac{0.6}{e_3},\frac{0.5}{e_4}),p_3,1)=\{u_2,u_4\}$$

$$f_{\tilde{A}}\ ((\frac{0.6}{e_1},\frac{0.7}{e_2},\frac{0.8}{e_3},\frac{0.4}{e_4}),p_3,1)=\{u_1,u_3\}$$

$$f_{\tilde{A}}\ ((\frac{0.4}{e_1},\frac{0.7}{e_2},\frac{0.6}{e_3},\frac{0.5}{e_4}),p_1,0)=\{u_1,u_2\}$$

$$f_{\tilde{A}}\ ((\frac{0.6}{e_1},\frac{0.7}{e_2},\frac{0.8}{e_3},\frac{0.4}{e_4}),p_1,0)=\{u_2,u_3,u_4\}$$

$$f_{\tilde{A}}\ ((\frac{0.4}{e_1},\frac{0.6}{e_2},\frac{0.4}{e_3},\frac{0.5}{e_4}),p_1,0)=\{u_1,u_3,u_4\}$$

$$f_{\tilde{A}}\ ((\frac{0.4}{e_1},\frac{0.7}{e_2},\frac{0.6}{e_3},\frac{0.5}{e_4}),p_2,0)=\{u_1,u_3,u_4\}$$

$$f_{\tilde{A}}\ ((\frac{0.6}{e_1},\frac{0.7}{e_2},\frac{0.8}{e_3},\frac{0.4}{e_4}),p_2,0)=\{u_1,u_2,u_3\}$$

$$f_{\tilde{A}}\ ((\frac{0.4}{e_1},\frac{0.6}{e_2},\frac{0.4}{e_3},\frac{0.5}{e_4}),p_2,0)=\{u_2,u_4\}$$

$$f_{\tilde{A}}\ ((\frac{0.4}{e_1},\frac{0.7}{e_2},\frac{0.6}{e_3},\frac{0.5}{e_4}),p_3,0)=\{u_1,u_3\}$$

$$f_{\tilde{A}}\ ((\frac{0.6}{e_1},\frac{0.7}{e_2},\frac{0.8}{e_3},\frac{0.4}{e_4}),p_3,0)=\{u_2,u_4\}.$$

The FFPSES is

$$(F,\mathcal{A})_{\tilde{A}}=\Bigg\{\Big[((\frac{0.4}{e_1},\frac{0.7}{e_2},\frac{0.6}{e_3}\frac{0.5}{e_4}),p_1,1)(u_3,u_4)\Big]\Big[((\frac{0.6}{e_1},\frac{0.7}{e_2},\frac{0.8}{e_3}\frac{0.4}{e_4}),p_1,1)(u_1)\Big]$$

$$\Big[((\frac{0.4}{e_1},\frac{0.6}{e_2},\frac{0.4}{e_3}\frac{0.5}{e_4}),p_1,1),(u_2)\Big]\Big[((\frac{0.4}{e_1},\frac{0.7}{e_2},\frac{0.6}{e_3}\frac{0.5}{e_4}),p_2,1),(u_2)\Big]$$

$$\Big[((\frac{0.6}{e_1},\frac{0.7}{e_2},\frac{0.8}{e_3}\frac{0.4}{e_4}),p_2,1)(u_4)\Big]\Big[((\frac{0.4}{e_1},\frac{0.6}{e_2},\frac{0.4}{e_3}\frac{0.5}{e_4}),p_2,1)(u_1,u_3)\Big]$$

$$\Big[((\frac{0.4}{e_1},\frac{0.7}{e_2},\frac{0.6}{e_3}\frac{0.5}{e_4}),p_3,1)(u_2,u_4)\Big]\Big[((\frac{0.6}{e_1},\frac{0.7}{e_2},\frac{0.8}{e_3}\frac{0.4}{e_4}),p_3,1)(u_1,u_3)\Big]$$

$$\Big[((\frac{0.4}{e_1},\frac{0.7}{e_2},\frac{0.6}{e_3}\frac{0.5}{e_4}),p_1,0)(u_1,u_2)\Big]\Big[((\frac{0.6}{e_1},\frac{0.7}{e_2},\frac{0.8}{e_3}\frac{0.4}{e_4}),p_1,0)(u_2,u_3,u_4)\Big]$$

$$\Big[((\frac{0.4}{e_1},\frac{0.6}{e_2},\frac{0.4}{e_3}\frac{0.5}{e_4}),p_1,0),(u_1,u_3,u_4)\Big]\Big[((\frac{0.4}{e_1},\frac{0.7}{e_2},\frac{0.6}{e_3}\frac{0.5}{e_4}),p_2,0),(u_1,u_3,u_4)\Big]$$

$$\Big[((\frac{0.6}{e_1},\frac{0.7}{e_2},\frac{0.8}{e_3}\frac{0.4}{e_4}),p_2,0)(u_1,u_2,u_3)\Big]\Big[((\frac{0.4}{e_1},\frac{0.6}{e_2},\frac{0.4}{e_3}\frac{0.5}{e_4}),p_2,0)(u_2,u_4)\Big]$$

$$\Big[((\frac{0.4}{e_1},\frac{0.7}{e_2},\frac{0.6}{e_3}\frac{0.5}{e_4}),p_3,0)(u_1,u_3)\Big]\Big[((\frac{0.6}{e_1},\frac{0.7}{e_2},\frac{0.8}{e_3}\frac{0.4}{e_4}),p_3,0)(u_2,u_4)\Big]\Bigg\}.$$

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From the $FFPSES$ above, agree and disagree $FFPSES$ will be deduced as below

$$(F,\mathcal{A})_{\tilde{A}_1}=\Bigg\{\Big[((\tfrac{0.4}{e_1},\tfrac{0.7}{e_2},\tfrac{0.6}{e_3}\tfrac{0.5}{e_4}),p_1,1)(u_3,u_4)\Big]\Big[((\tfrac{0.6}{e_1},\tfrac{0.7}{e_2},\tfrac{0.8}{e_3}\tfrac{0.4}{e_4}),p_1,1)(u_1)\Big]$$

$$\Big[((\tfrac{0.4}{e_1},\tfrac{0.6}{e_2},\tfrac{0.4}{e_3}\tfrac{0.5}{e_4}),p_1,1),(u_2)\Big]\Big[((\tfrac{0.4}{e_1},\tfrac{0.7}{e_2},\tfrac{0.6}{e_3}\tfrac{0.5}{e_4}),p_2,1),(u_2)\Big]$$

$$\Big[((\tfrac{0.6}{e_1},\tfrac{0.7}{e_2},\tfrac{0.8}{e_3}\tfrac{0.4}{e_4}),p_2,1)(u_4)\Big]\Big[((\tfrac{0.4}{e_1},\tfrac{0.6}{e_2},\tfrac{0.4}{e_3}\tfrac{0.5}{e_4}),p_2,1)(u_1,u_3)\Big]$$

$$\Big[((\tfrac{0.4}{e_1},\tfrac{0.7}{e_2},\tfrac{0.6}{e_3}\tfrac{0.5}{e_4}),p_3,1)(u_2,u_4)\Big]\Big[((\tfrac{0.6}{e_1},\tfrac{0.7}{e_2},\tfrac{0.8}{e_3}\tfrac{0.4}{e_4}),p_3,1)(u_1,u_3)\Big]\Bigg\},$$

and

$$(F,\mathcal{A})_{\tilde{A}_0}=\Big[((\tfrac{0.4}{e_1},\tfrac{0.7}{e_2},\tfrac{0.6}{e_3}\tfrac{0.5}{e_4}),p_1,0)(u_1,u_2)\Big]\Big[((\tfrac{0.6}{e_1},\tfrac{0.7}{e_2},\tfrac{0.8}{e_3}\tfrac{0.4}{e_4}),p_1,0)(u_2,u_3,u_4)\Big]$$

$$\Big[((\tfrac{0.4}{e_1},\tfrac{0.6}{e_2},\tfrac{0.4}{e_3}\tfrac{0.5}{e_4}),p_1,0),(u_1,u_3,u_4)\Big]\Big[((\tfrac{0.4}{e_1},\tfrac{0.7}{e_2},\tfrac{0.6}{e_3}\tfrac{0.5}{e_4}),p_2,0),(u_1,u_3,u_4)\Big]$$

$$\Big[((\tfrac{0.6}{e_1},\tfrac{0.7}{e_2},\tfrac{0.8}{e_3}\tfrac{0.4}{e_4}),p_2,0)(u_1,u_2,u_3)\Big]\Big[((\tfrac{0.4}{e_1},\tfrac{0.6}{e_2},\tfrac{0.4}{e_3}\tfrac{0.5}{e_4}),p_2,0)(u_2,u_4)\Big]$$

$$\Big[((\tfrac{0.4}{e_1},\tfrac{0.7}{e_2},\tfrac{0.6}{e_3}\tfrac{0.5}{e_4}),p_3,0)(u_1,u_3)\Big]\Big[((\tfrac{0.6}{e_1},\tfrac{0.7}{e_2},\tfrac{0.8}{e_3}\tfrac{0.4}{e_4}),p_3,0)(u_2,u_4)\Big]\Bigg\}.$$

Next from both sets above we obtain the tables below

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Table 1: $AgreeFFPSES$-**set Table**

$\hat{a}_{i_j}$/U	$\frac{\sum \mu_E(e_i)}{n(E)}$	u_1	u_2	u_3	u_4
$\left((\frac{0.4}{e_1}, \frac{0.7}{e_2}, \frac{0.6}{e_3}, \frac{0.5}{e_4}), p_1\right)$	0.55	0	0	1	1
$((\frac{0.6}{e_1}, \frac{0.7}{e_2}, \frac{0.8}{e_3} \frac{0.4}{e_4}), p_1)$	0.625	1	0	0	0
$((\frac{0.4}{e_1}, \frac{0.6}{e_2}, \frac{0.4}{e_3} \frac{0.5}{e_4}), p_1)$	0.475	0	1	0	0
$((\frac{0.4}{e_1}, \frac{0.7}{e_2}, \frac{0.6}{e_3}, \frac{0.5}{e_4}), p_2)$	0.55	0	1	0	0
$((\frac{0.6}{e_1}, \frac{0.7}{e_2}, \frac{0.8}{e_3} \frac{0.4}{e_4}), p_2)$	0.625	0	0	0	1
$((\frac{0.4}{e_1}, \frac{0.6}{e_2}, \frac{0.4}{e_3} \frac{0.5}{e_4}), p_2)$	0.475	1	0	1	0
$((\frac{0.4}{e_1}, \frac{0.7}{e_2}, \frac{0.6}{e_3}, \frac{0.5}{e_4}), p_3)$	0.55	0	1	0	1
$((\frac{0.6}{e_1}, \frac{0.7}{e_2}, \frac{0.8}{e_3} \frac{0.4}{e_4}), p_3)$	0.625	1	0	1	0
$\mathcal{U}_i = \sum_{e_i \in \hat{y}_j} \left(\mu_{ij} \frac{\sum \mu_E(e_i)}{n(E)}\right)$	0	$\mathcal{U}_1 = 1.725$	$\mathcal{U}_2 = 1.5$	$\mathcal{U}_3 = 1.175$	$\mathcal{U}_4 = 1.65$

Table 2: $DisagreeFFPSES$-**set Table**

$\hat{a}_{i_j}$/U	$\frac{\sum \mu_E(e_i)}{n(E)}$	u_1	u_2	u_3	u_4
$((\frac{0.4}{e_1}, \frac{0.7}{e_2}, \frac{0.6}{e_3}, \frac{0.5}{e_4}), p_1,)$	0.55	1	1	0	0
$((\frac{0.6}{e_1}, \frac{0.7}{e_2}, \frac{0.8}{e_3} \frac{0.4}{e_4}), p_1,)$	0.625	0	1	1	1
$((\frac{0.4}{e_1}, \frac{0.6}{e_2}, \frac{0.4}{e_3} \frac{0.5}{e_4}), p_1)$	0.475	1	0	1	1
$((\frac{0.4}{e_1}, \frac{0.7}{e_2}, \frac{0.6}{e_3}, \frac{0.5}{e_4}), p_2,)$	0.55	1	0	1	1
$((\frac{0.6}{e_1}, \frac{0.7}{e_2}, \frac{0.8}{e_3} \frac{0.4}{e_4}), p_2,)$	0.625	1	1	1	0
$((\frac{0.4}{e_1}, \frac{0.6}{e_2}, \frac{0.4}{e_3} \frac{0.5}{e_4}), p_2)$	0.475	0	1	0	1
$((\frac{0.4}{e_1}, \frac{0.7}{e_2}, \frac{0.6}{e_3}, \frac{0.5}{e_4}), p_3,)$	0.55	1	0	1	0
$((\frac{0.6}{e_1}, \frac{0.7}{e_2}, \frac{0.8}{e_3} \frac{0.4}{e_4}), p_3)$	0.625	0	1	0	1
$\mathcal{V}_i = \sum_{e_i \in \hat{y}_j} \left(\mu_{ij} \frac{\sum \mu_E(e_i)}{n(E)}\right)$	0	$\mathcal{V}_1 = 3.375$	$\mathcal{V}_2 = 3.3$	$\mathcal{V}_3 = 3.45$	$\mathcal{V}_4 = 2.05$

$\mathcal{U}_i$	$\mathcal{V}_i$	$\mathcal{T}_i$=$\mathcal{U}_i - \mathcal{V}_i$
$\mathcal{U}_1 = 1.725$	$\mathcal{V}_1 = 3.375$	$\mathcal{T}_1 = -1.65$
$\mathcal{U}_2 = 1.5$	$\mathcal{V}_2 = 3.3$	$\mathcal{T}_2 = -1.8$
$\mathcal{U}_3 = 1.175$	$\mathcal{V}_3 = 3.45$	$\mathcal{T}_3 = -2.275$
$\mathcal{U}_4 = 1.65$	$\mathcal{V}_4 = 2.05$	$\mathcal{T}_4 = -0.885$

Now $m = 4$ since $\mathcal{T}_4 = -0.885 = max\ \mathcal{T}_i$
then u_4 is the *optimal choice*
And the articles will be considered for grant as follows
$u_4 \rightarrow 50 million\ naira,$
$u_1 \rightarrow 35 million\ naira$
, $u_2 \rightarrow 25 million\ naira,$
$u_3 \rightarrow 20 million\ naira$

Proposition 4.5. *Let $F_{\tilde{A}}$ and $G_{\tilde{B}}$ be two FFPSES such that $F_{\tilde{A}} \subseteq G_{\tilde{B}}$ then*

i. $\mathcal{T}_m G_{\tilde{B}}$ is the Prime Optimal Choice Object $\mathcal{T}_{\tilde{m}}$

ii. $\mathcal{T}_m G_{\tilde{B}} = \mathcal{T}_m(F_{\tilde{A}} \tilde{\cup} G_{\tilde{B}})$

iii. $\mathcal{T}_m F_{\tilde{A}} = \mathcal{T}_m(F_{\tilde{A}} \tilde{\cap} G_{\tilde{B}})$

Proof. i. Let $F_{\tilde{A}} \tilde{\subseteq} G_{\tilde{B}}$ then $\tilde{A} \tilde{\subseteq} \tilde{B}$ (where $\hat{y}_i \in \tilde{A}, \hat{z}_j \in \tilde{B}$ and $\hat{y} = \mu_{\hat{y}}(e_i), \hat{z} = \mu_{\hat{z}}(e_i) \forall e_i \in E$). Therefore $\mu_{\hat{y}}(e_i) \leq \mu_{\hat{z}}(e_i)$ then $\frac{\sum \mu_{\hat{y}}(e_i)}{n(E)} \leq \frac{\sum \mu_{\hat{z}}(e_i)}{n(E)}$. By definition 3.3 it follows that if $u_i \in (F, \mathcal{A})_{\tilde{A}_1}$ then $u_i \in (G, \mathcal{B})_{\tilde{B}_1}$, thus

$$\sum_{e_i \in E} \left(\Phi_{ij} \frac{\sum \mu_{\hat{y}}(e_i)}{n(E)} \right) \leq \sum_{e_i \in E} \left(\Phi_{ij} \frac{\sum \mu_{\hat{z}}(e_i)}{n(E)} \right) \tag{4.1}$$

$$\sum_{e_i \in E} \left(\phi_{ij} \frac{\sum \mu_{\hat{y}}(e_i)}{n(E)} \right) \leq \sum_{e_i \in E} \left(\phi_{ij} \frac{\sum \mu_{\hat{z}}(e_i)}{n(E)} \right). \tag{4.2}$$

Subtract equation 4.2 from 4.1

$$\left(\sum_{e_i \in E} \left(\Phi_{ij} \frac{\sum \mu_{\hat{y}}(e_i)}{n(E)} \right) - \sum_{e_i \in E} \left(\phi_{ij} \frac{\sum \mu_{\hat{y}}(e_i)}{n(E)} \right) \right) \leq \left(\sum_{e_i \in E} \left(\Phi_{ij} \frac{\sum \mu_{\hat{z}}(e_i)}{n(E)} \right) - \sum_{e_i \in \hat{y}_j} \left(\phi_{ij} \frac{\sum \mu_{\hat{z}}(e_i)}{n(E)} \right) \right)$$

$\mathcal{T}_m F_{\tilde{A}} \leq \mathcal{T}_m G_{\tilde{B}}$

ii. $\mathcal{T}_m(F_{\tilde{A}} \tilde{\cup} G_{\tilde{B}}) = \sum_{e_i \in \hat{E}} \left(\Phi_{ij} \frac{\sum \mu_{\hat{y} \tilde{\cup} \hat{z}}(e_i)}{n(E)} \right) - \sum_{e_i \in \hat{E}} \left(\phi_{ij} \frac{\sum \mu_{\hat{y} \tilde{\cup} \hat{z}}(e_i)}{n(E)} \right)$

Since $F_{\tilde{A}} \tilde{\subseteq} G_{\tilde{B}}$, it follows that $\tilde{A} \tilde{\subseteq} \tilde{B}$ and $\hat{y}_i \tilde{\leq} \hat{z}_j$, thus $\mu_{\hat{z}}(e_i) \geq \mu_{\hat{y}}(e_i)$, then $\mu_{\hat{z}}(e_i) \in \hat{z}_j \in \tilde{A} \Cup \tilde{B} = \tilde{B}$, therefore $f_{\tilde{A}}(\hat{y}) \subseteq g_{\tilde{B}}(\hat{z}) = h_{\tilde{C}}(\hat{c})$ where $\tilde{C} = \tilde{A} \Cup \tilde{B}$. Now

$$\sum_{e_i \in E} \left(\Phi_{ij} \frac{\sum \mu_{\hat{z}}(e_i)}{n(E)} \right) = \sum_{e_i \in E} \left(\Phi_{ij} \frac{\sum \mu_{\hat{y} \tilde{\cup} \hat{z}}(e_i)}{n(E)} \right) \tag{4.3}$$

$$\sum_{e_i \in E} \left(\phi_{ij} \frac{\sum \mu_{\hat{z}}(e_i)}{n(E)} \right) = \sum_{e_i \in E} \left(\phi_{ij} \frac{\sum \mu_{\hat{y} \tilde{\cup} \hat{z}}(e_i)}{n(E)} \right). \tag{4.4}$$

Subtract equation 4.4 from equation 4.3

$$\sum_{e_i \in E} \left(\Phi_{ij} \frac{\sum \mu_{\hat{z}}(e_i)}{n(E)} \right) - \sum_{e_i \in E} \left(\phi_{ij} \frac{\sum \mu_{\hat{z}}(e_i)}{n(E)} \right) = \sum_{e_i \in E} \left(\Phi_{ij} \frac{\sum \mu_{\hat{y} \tilde{\cup} \hat{z}}(e_i)}{n(E)} \right) - \sum_{e_i \in E} \left(\phi_{ij} \frac{\sum \mu_{\hat{y} \tilde{\cup} \hat{z}}(e_i)}{n(E)} \right)$$

therefore $\mathcal{T}_m G_{\tilde{B}} = \mathcal{T}_m(F_{\tilde{A}} \tilde{\cup} G_{\tilde{B}})$

iii. The prove follows analogously.

□

5 Conclusion

In this research the authors focused on the development of $FFPSE - Set$, which is a novel model combining $FFPS - Set$ and $FPSE - Set$. Some of its set theoretic and algebraic properties were investigated. Additionally, we have describe our model with some numerical examples. Furthermore, it is applied to decision making problem, in which case an algorithm is developed.

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