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Green functions of Fractional Partial Differential Equation With a Modified Elliptic Potential By a Weighted Bessel Function

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Abstract

In this paper, the Green function of a fractional partial differential equation $[(-\Delta)^{1+\alpha} + V(\mathbf{x})]\Psi = \delta(\mathbf{x} - \mathbf{a}), \alpha \in (0, 1)$ is obtained where the Laplacian Δ, the potential $V(\mathbf{x})$ and the Dirac delta function $\delta(\mathbf{x})$ are defined over a closed ball $\overline{B(\mathbf{0}, r)}$ of radius $r > 0$ in an Euclidean space $\mathbb{R}^n$ and $V(\mathbf{x})$ is a modified vector-valued Weierstrass sigma elliptic potential weighted by a Bessel function. A combination of Fourier and Hankel transform techniques are employed in obtaining the main result.

Keywords: Elliptic functions, Hankel transform, Reisz derivative, weighted Fourier transform.
MSC2010: 26A33, 34B27, 33E05, 42A38.

1 Introduction

The evaluation of Green's function is potent technique for obtaining solutions of differential equations in physics and engineering. Such solutions which are unique are used to model some complex phenomena like heat distribution, fluid dynamics, electromagnetic fields, nano-particles motion, wave reaction diffusion and so on. On the other hand, the study of fractional partial differential equations (FPDEs) and integral equations have in recent time attracted much interest because of its multifaceted applications in science and mathematics (see [1, 3–5, 10, 11, 19]). Recently, Youyu et. al. [20], studied Green function for Caputo fractional sturm-Louiville boundary conditions. The aforementioned authors have studied Laplacian of fractional order without potential but in this paper, we solve a fractional partial differential equation with a modified elliptic potential by a Bessel function weight to obtain its associated Green function. The choice of this potential is deliberate due to the fact that literature are scarce on the study of FPDEs with elliptic potential and the Bessel function $\mathcal{J}_\tau(|\mathbf{a}||\mathbf{x}|)$ is a good weight because after reaching its amplitude it decays like $\frac{1}{\sqrt{|\mathbf{a}||\mathbf{x}|}}$ (see [6]). The method employed in this work is similar to Green's function integral equation method (GFIM) [16] and differs from other methods employed in solving other PDEs (see [12, 13]).

The search for Green functions for FPDE with potential functions forms the motivation for this study and as such have implications in the study of fractional spectral shift functions. FPDE with potential functions are very useful in modelling systems which entails short-range memory and non-local hereditary phenomena (see [8]).

The outline of the paper is stated in what follows. Section two considers preliminaries on the formulation of the problem and basic theorems required to resolve the problem. Section three consists of the Main result in the paper in which the Green function is obtained. Section four, consists of the conclusion and summary of result.

2 Preliminaries

Consider a Laplacian Δ on $\mathbb{R}^n$, which is essentially self adjoint on $\mathscr{C}_c^\infty(\mathbb{R}^n)$. We define a partial differential operator with a fractional order which is close in spirit with the definition of fractional Laplace operator obtainable in [9], given by

$$L_{n,\alpha} := (-\Delta)^{1+\alpha}, \qquad n \in \mathbb{Z}^+, 0 < \alpha < 1. \tag{2.1}$$

Consider spherically symmetric function $\varphi(\mathbf{x})$ with $\mathbf{x} \in \mathbb{R}^n$ being a radial coordinate (cf: [15], §1.7.2, 8^o) such that $\mathbf{x} \in \overline{B(0,r)} \subset \mathbb{R}^n$. It is well-known (cf: [15], §1.7.2, 8^o, Eq.(8)) that

$$-\Delta = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{n-1}{r}\frac{\mathrm{d}}{\mathrm{d}r}, \tag{2.2}$$

$\mathbf{x} = (\xi_i)_{i=1}^n$ and $r = |\mathbf{x}| = \sqrt{\sum_{i=1}^n \xi_i^2}$.

Thus, one applies (2.4) with the operator factorisation to (2.1) to obtain

$$L_{n,\alpha} = (-1)^{1+\alpha}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{n-1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\left(\frac{\mathrm{d}}{\mathrm{d}r} + \frac{n-1}{r}\right)^\alpha \frac{\mathrm{d}^\alpha}{\mathrm{d}r^\alpha} \tag{2.3}$$

Given an elliptic perturbation, the Weierstrass elliptic σ-function given by (cf. [2], §18.5.6-8, pp.635-636)

$$\sigma(\mathbf{x}) = \sum_{m,n=0}^{+\infty} 2^{n-m} a_{m,n} g_2^m g_3^n \frac{\mathbf{x}^{4m+6n+1}}{(4m+6n+1)!},$$

with $g_2 = 60\sum'_{u,v}(2u\omega_1 + 2v\omega_2)^{-4}, g_3 = 140\sum'_{u,v}(2u\omega_1 + 2v\omega_2)^{-6}$ are invariant constants depending on periods of the Weierstrass $\wp$-function. Here, $\sum'_{u,v}$ is sum to infinity excluding the origin $(0,0), a_{0,0} = 1$, and

$$a_{m,n} = 3(m+1)a_{m+1,n-1} + \frac{16}{3}(n+1)a_{m-2,n+1} - \frac{1}{3}(2m+3n-1)(4m+6n+1)a_{m-1,n}.$$

Let a Bessel function of the first kind with order τ be defined in terms of the confluent hypergeometric function of the first kind by the integral

$$\mathcal{J}_\tau(s) := \frac{2^{-\tau}s^\tau}{\Gamma(1+\tau)}\, {}_0F_1(1+\tau; -\frac{1}{4}s^2)$$

(cf: [18], Eq.(37), p.200). Then, for a fixed $\mathbf{a} \in \overline{B(\mathbf{0},r)}$,

$$[L_{n,\alpha} + \sigma(\mathbf{x})\mathcal{J}_\tau(|\mathbf{a}||\mathbf{x}|)]\varphi(\mathbf{x}) = \delta(\mathbf{x}-\mathbf{a}), \tag{2.4}$$

where $\delta(\cdot)$ is the Dirac delta function and $\tau \in \mathbb{Z}^+$.

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Definition 2.1. *(cf: [15], p.153, Weighted Fourier Transform). Let $\Omega \subset \mathbb{R}^n$. The weighted Fourier transform $\mathscr{F}_\omega : L^2(\Omega) \xrightarrow{\cong} L^2(\Omega)$ is defined as*

$$\mathscr{F}_\omega[f] = \langle f(x), \chi_s \rangle_\omega = \int_\Omega f(x) e^{-j\langle x,y \rangle} \mathrm{d}x \equiv S_{n-1} \int_0^\infty \int_0^{2\pi} f(r)\chi_{-s}(r,\theta)\omega(r)\mathrm{d}r\mathrm{d}\theta.$$

Here, $s = |y| = \sqrt{\sum_{i=1}^n \eta_i^2}$, the character in $\mathbb{R}^n$ is $\chi_s(r,\theta) = e^{jrs\cos\theta}\sin^{n-2}\theta, j = \sqrt{-1}$, the weight function for $L_{n,\alpha}$ is $\omega(r) = r^{1-n} > 0$, $rs\cos\theta = \langle x, y\rangle = \sum_{k=1}^n \xi_i \eta_i$, and S_{n-1} is the surface area of the hyper-sphere $\mathbb{S}^{n-1}$ given by $S_{n-1} = 2\frac{\sqrt{\pi^n}}{\Gamma(\frac{n}{2})}$ (cf: [15], Eq.(8), p.75).

Further simplification of $\mathscr{F}_\omega[f]$ yields an expression in terms of Bessel function of the first kind with order μ given by $\mathcal{J}_\mu(\cdot), \mu \in \mathbb{Q}$.

$$\begin{aligned} \mathscr{F}_\omega[f] &= 2\pi^{\frac{n}{2}}\left(\frac{s}{2}\right)^{1-\frac{n}{2}} \int_0^\infty r^{\frac{n}{2}} f(r)\mathcal{J}_{\frac{n}{2}-1}(rs)\mathrm{d}r \\ &= (2\pi)^{\frac{n}{2}} s^{1-\frac{n}{2}} \int_0^\infty f(r) r^{\frac{n}{2}} \mathcal{J}_{\frac{n}{2}-1}(rs)\mathrm{d}r. \end{aligned} \tag{2.5}$$

It is now necessary to establish some propositions that will be needed in this work.

Proposition 2.2. *Let $a = |\mathbf{a}|$ then*

$$\mathscr{F}_\omega[\delta(\mathbf{x}-\mathbf{a})] = \frac{1}{2}(2\pi)^{\frac{n}{2}} s^{1-\frac{n}{2}} a^{\frac{n}{2}} \mathcal{J}_{\frac{n}{2}-1}(as) \tag{2.6}$$

Proof. Setting $\psi(r) = r^{\frac{n}{2}}\mathcal{J}_{1-\frac{n}{2}}(rs)$ and since $\delta(r-a)$ is an even function, it follows that

$$\begin{aligned} \mathscr{F}_\omega[\delta(\mathbf{x}-\mathbf{a})] &= (2\pi)^{\frac{n}{2}} s^{1-\frac{n}{2}} \int_0^\infty \psi(r)\delta(r-a)\mathrm{d}r \\ &= \frac{1}{2}(2\pi)^{\frac{n}{2}} s^{1-\frac{n}{2}}\psi(a). \end{aligned}$$

Hence the result. □

Lemma 2.3. *Let $L_{n,\alpha}$ be as obtainable in equation (2.3). Then, by Reisz derivative*

$$\mathscr{F}_\omega[L_{n,\alpha}\varphi] = (-1)^{1+\alpha}\frac{n(n+1)}{2} s^{2(1+\alpha)}\widehat{\varphi}^\omega(s), s > 0. \tag{2.7}$$

Proof.

$$\begin{aligned} \mathscr{F}_\omega[L_{n,\alpha}\varphi] &= (-1)^{1+\alpha}\left[s^2 + (n-1)\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^{-1}(s)\right]\left[s + (n-1)\left(\frac{\mathrm{d}}{\mathrm{d}s}\right)^{-1}(1)\right]^\alpha s^\alpha \widehat{\varphi}^\omega(s) \\ &= (-1)^{1+\alpha}\left[s^2 + (n-1)\frac{s^2}{2}\right][s + (n-1)s]^\alpha s^\alpha \widehat{\varphi}^\omega(s) \\ &= (-1)^{1+\alpha}\frac{n(n+1)}{2} s^{2(1+\alpha)}\widehat{\varphi}^\omega(s), \quad s > 0. \end{aligned}$$

□

Definition 2.4 ([14], §9-2, Definition 9.2)**.** *Let $f : \mathbb{R}^0_+ \to \mathbb{R}$ be a function. Then, the Hankel transform*

$$G_\nu(s) \equiv \mathscr{H}_\nu[g(r)] = \int_0^\infty rg(r)\mathcal{J}_\nu(sr)\mathrm{d}r, \quad s > 0$$

and the inverse Hankel transform is given by

$$g(r) = \mathscr{H}_\nu^{-1}[G_\nu(s)] = \int_0^\infty sG_\nu(s)\mathcal{J}_\nu(sr)\mathrm{d}s, \quad \nu > -\frac{1}{2}.$$

Next, we consider the Fourier transform of the potential part.

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Lemma 2.5. *Let*

$$c_{m,n} = \frac{2^{n-m} a_{m,n} g_2^m g_3^n}{(4m+6n+1)!}$$

where $a_{m,n}, g_2, g_3$ *are as earlier defined and* $c_{0,0} = 1$. *Then,*

$$\mathscr{F}_\omega[\sigma(\mathbf{x})\mathcal{J}_\tau(|\mathbf{a}||\mathbf{x}|)\varphi(\mathbf{x})] = (2\pi)^{\frac{n}{2}} s^{1-\frac{n}{2}} \sum_{m,n}^{\infty}{}' c_{m,n} \frac{2^{4m+\frac{13n}{2}+2} a^{4m+7n+2} (s^2-a^2)^{-(4m+\frac{13n}{2}+2)}}{s^{\frac{n}{2}-1}\Gamma(-(4m+\frac{13n}{2}+1))}$$

Proof. Let

$$c_{m,n} = \frac{2^{n-m} a_{m,n} g_2^m g_3^n}{(4m+6n+1)!}$$

where $a_{m,n}, g_2, g_3$ are as earlier defined and $c_{0,0} = 1$. Then,

$$\begin{aligned}
\mathscr{F}_\omega[\sigma(\mathbf{x})\mathcal{J}_\tau(|\mathbf{a}||\mathbf{x}|)\varphi(\mathbf{x})] &= (2\pi)^{\frac{n}{2}} s^{1-\frac{n}{2}} \int_0^\infty \sigma(r)\mathcal{J}_\tau(ar)\varphi(r) r^{\frac{n}{2}} \mathcal{J}_{\frac{n}{2}-1}(rs)\mathrm{d}r \\
&= (2\pi)^{\frac{n}{2}} s^{1-\frac{n}{2}} \sum_{m,n}^{\infty}{}' c_{m,n} \int_0^\infty r^{4m+6n+1} r^{\frac{n}{2}} \mathcal{J}_\tau(ar)\mathcal{J}_{\frac{n}{2}-1}(rs)\mathrm{d}r \widehat{\varphi}^\omega(s) \\
&= (2\pi)^{\frac{n}{2}} s^{1-\frac{n}{2}} \sum_{m,n}^{\infty}{}' c_{m,n} \mathscr{H}_{\frac{n}{2}-1}\left[r^{4m+\frac{13n}{2}+1}\mathcal{J}_\tau(ar)\right]\widehat{\varphi}^\omega(s).
\end{aligned}$$

Setting $\nu = \frac{n}{2}+1$ such that $\tau - \nu = 4m + \frac{13n}{2} + 1$ yields $\tau = 4m+7n+2$. By using the Hankel transform (cf: [14], §9-14, Table 9.2 (15))

$$\mathscr{H}[r^{\tau-\nu}\mathcal{J}_\tau(ar)] = \frac{2^{\tau-\nu+1} a^\nu (s^2-a^2)^{\nu-\tau-1}}{s^\nu \Gamma(\nu-\tau)}.$$

Hence,

$$\mathscr{F}_\omega[\sigma(\mathbf{x})\mathcal{J}_\tau(|\mathbf{a}||\mathbf{x}|)\varphi(\mathbf{x})] = (2\pi)^{\frac{n}{2}} s^{1-\frac{n}{2}} \sum_{m,n}^{\infty}{}' c_{m,n} \frac{2^{4m+\frac{13n}{2}+2} a^{4m+7n+2} (s^2-a^2)^{-(4m+\frac{13n}{2}+2)}}{s^{\frac{n}{2}-1}\Gamma(-(4m+\frac{13n}{2}+1))}\widehat{\varphi}^\omega(s).$$

□

3 The Main Result

The solution of equation (2.4) is the crux of this paper. In the theorem that follows, the solution of the resulting fractional partial differential equation with modified Weierstrass elliptic potential weighted by a Bessel function is obtained as a Green function.

The Green function $\varphi(\mathbf{x}, \mathbf{a})$ *which solves the perturbed fractional partial differential equation*

$$[(-\Delta)^{1+\alpha} + V(\mathbf{x})]\Psi = \delta(\mathbf{x}-\mathbf{a}), \alpha \in (0,1)$$

is

$$\varphi(\mathbf{x}, \mathbf{a}) = \int_0^\infty \Psi(s,a) J_\nu(as) J_\nu(ar)\mathrm{d}s,$$

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where, $\nu = \frac{n}{2} - 1$ *and the kernel*

$$\Psi(s,a) = \begin{cases} \sum_{m,n=0}^{+\infty}{}' \dfrac{c_a s^{\frac{n}{2}}}{q_{m,n}(s^2-a^2)^{\gamma_{m,n}} + d_n s^{\kappa_n(\alpha)}} & , \quad s > a \\ \sum_{m,n=0}^{+\infty}{}' c_a s^{\frac{n}{2}-\kappa_n(\alpha)} & , \quad s = a \\ \sum_{m,n=0}^{+\infty}{}' \dfrac{c_a s^{\frac{n}{2}}}{d_n s^{\kappa_n(\alpha)} - q_{m,n}(s^2-a^2)^{\gamma_{m,n}}} & , \quad s < a \end{cases}$$

with

$$\begin{aligned} c_a &= \frac{1}{2}a^{\frac{n}{2}}, \\ \gamma_{m,n} &= -(4m + \frac{13n}{2} + 2), \\ \kappa_n(\alpha) &= 2\alpha + \frac{n}{2} + 1 \\ q_{m,n} &= c_{m,n}\frac{2^{4m+\frac{13n}{2}+2}a^{4m+7n+2}}{\Gamma(-(4m+\frac{13n}{2}+1))}, \textit{ and} \\ d_n &= (-1)^{1+\alpha}\frac{n(n+1)}{2}(2\pi)^{-\frac{n}{2}} \end{aligned}$$

Proof. The Fourier transform of equation (2.4) using equation (2.7) and Lemma 2.5 becomes

$$\Big[(2\pi)^{\frac{n}{2}} s^{1-\frac{n}{2}} \sum_{m,n=0}^{\infty}{}' c_{m,n} \frac{2^{4m+\frac{13n}{2}+2}a^{4m+7n+2}(s^2-a^2)^{-(4m+\frac{13n}{2}+2)}}{s^{\frac{n}{2}-1}\Gamma(-(4m+\frac{13n}{2}+1))} + (-1)^{1+\alpha}\frac{n(n+1)}{2}s^{2(1+\alpha)}\Big]\widehat{\varphi}^{\omega}(s) = \frac{1}{2}(2\pi)^{\frac{n}{2}} a^{\frac{n}{2}} s^{1-\frac{n}{2}} \mathcal{J}_{\frac{n}{2}-1}(as). \quad (3.1)$$

Thus,

$$\widehat{\varphi}^{\omega}(s) = \sum_{m,n=0}^{+\infty}{}' \frac{c_a \mathcal{J}_{\frac{n}{2}-1}(as)}{q_{m,n}(s^2-a^2)^{\gamma_{m,n}} + d_n s^{\kappa_n(\alpha)}}$$

where, $s > a$

$$\begin{aligned} c_a &= \frac{1}{2}a^{\frac{n}{2}}, \\ \gamma_{m,n} &= -(4m + \frac{13n}{2} + 2), \\ \kappa_n(\alpha) &= 2\alpha + \frac{n}{2} + 1 \\ q_{m,n} &= c_{m,n}\frac{2^{4m+\frac{13n}{2}+2}a^{4m+7n+2}}{\Gamma(-(4m+\frac{13n}{2}+1))}, \text{ and} \\ d_n &= (-1)^{1+\alpha}\frac{n(n+1)}{2}(2\pi)^{-\frac{n}{2}} \end{aligned}$$

are real numbers. Thus, by Fourier inversion formula

$$f(r) = \mathscr{F}_{\omega}^{-1}[F(s)] = \int_0^{\infty} F(s) s^{\frac{n}{2}} \mathcal{J}_{\frac{n}{2}-1}(ar)\mathrm{d}s,$$

we have the Fourier-Bessel function

$$\begin{aligned}
\varphi(\mathbf{x},\mathbf{a}) &= \sum_{m,n=0}^{+\infty}{}' \mathscr{F}_{\omega}^{-1}\left[\frac{c_a \mathcal{J}_{\frac{n}{2}-1}(as)}{q_{m,n}(s^2-a^2)^{\gamma_{m,n}} + d_n s^{\kappa_n(\alpha)}}\right] \\
&= \sum_{m,n=0}^{+\infty}{}' \int_0^{\infty} \frac{c_a s^{\frac{n}{2}} \mathcal{J}_{\frac{n}{2}-1}(as)\mathcal{J}_{\frac{n}{2}-1}(ar)}{q_{m,n}(s^2-a^2)^{\gamma_{m,n}} + d_n s^{\kappa_n(\alpha)}} \mathrm{d}s \\
&= \int_0^{\infty} \Psi(s,a)\mathcal{J}_{\nu}(as)\mathcal{J}_{\nu}(ar)\mathrm{d}s.
\end{aligned}$$

Here, the kernel

$$\Psi(s,a) = \begin{cases} \displaystyle\sum_{m,n=0}^{+\infty}{}' \frac{c_a s^{\frac{n}{2}}}{q_{m,n}(s^2-a^2)^{\gamma_{m,n}} + d_n s^{\kappa_n(\alpha)}} & , \quad s > a \\ \displaystyle\sum_{m,n=0}^{+\infty}{}' \frac{c_a}{d_n} s^{\frac{n}{2}-\kappa_n(\alpha)} & , \quad s = a \\ \displaystyle\sum_{m,n=0}^{+\infty}{}' \frac{c_a s^{\frac{n}{2}}}{d_n s^{\kappa_n(\alpha)} - q_{m,n}(s^2-a^2)^{\gamma_{m,n}}} & , \quad s < a \end{cases}$$

Hence, the result follows. □

4 Conclusion

The Green functions associated with the Laplacian with fractional order generates an integral equation which is solvable under the condition that the potential function is a multiple of a real valued weighted Bessel function. The solution of the FPDE given in terms of integral

$$\varphi(\mathbf{x},\mathbf{a}) = \int_0^{\infty} \Psi(s,a)\mathcal{J}_{\nu}(as)\mathcal{J}_{\nu}(ar)\mathrm{d}s$$

which have been obtained through the combination of Fourier and Hankel transform techniques adds to the glossary of Fourier integrals which are not listed in the work of Erdelyi ([7]) and can be studied further by upcoming researchers with ample consultation with the works of E.C. Titchmarsh [17]. The overall approach in solving the FPDE here is similar to Green's function integral equation method (GFIM) [16].

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