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Common Fixed Point Theorems of Some Certain Generalized Contractive Conditions in Convex Metric Space Settings

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Abstract

This paper investigates convergence results for fixed point iterations to the unique common fixed point of sequences of Akram-Jungck and M_J type contractive operators, using Jungck-Schaefer-like iterative technique. We also demonstrate the applicability of these results to solving optimization problems.

Keywords: Akram-Jungck type Contraction, M_J contraction, Jungck-Schaefer type Iteration, Common Fixed Point and Convex Metric Space.
MSC2010: 54H25.

1 Introduction

Let X be a nonempty set and a mapping $H : X \to X$. A point $x \in X$ is a fixed point of H if $H(x) = x$ and $Fix_H = \{x \in X : H(x) = x\}$ denotes the set of all fixed points of H in X.
Given a complete metric space (X, d) and a self mapping H on X satisfying:

$$d(Hx, Hy) \leq \aleph d(x, y), \ \forall \ x, y \ \in X \tag{1.1}$$

$\aleph$ is a non-negative real number less than 1. Banach [1], established that H has a unique fixed point in X.
Motivated by Banach work, Rakotch [2], generalized Banach's fixed point theorem by introducing a monotone decreasing function $\beta : (0, \infty) \to [0, 1)$ such that, for each $x, y \in X, x \neq y$,

$$d(Hx, Hy) \leq \beta(d(x, y)) \tag{1.2}$$

Maia [3], extended Banach's fixed point theorem on complete metric space by using the notion of equivalent metrics. In generalizing Banach's theorem, Kannan [4] proved that the operator H need not continuous to have a fixed point. He established his results using the following contractive

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definition: there exists $\aleph \in [0, 1/2)$ such that

$$d(Hx, Hy) \leq \aleph[d(x, Hx) + d(y, Hy)], \ \forall \ x, \ y \in X. \tag{1.3}$$

Akram et al. [5], give a more general class of contractive definitions than (1.1) to (1.3) above and many more others in literature using the definitions below:
Definition 1.1 [5]: A self-map $H : X \to X$ of a metric space (X, d) is said to be A-contraction if it satisfies the condition:

$$d(Hx, Hy) \leq \phi(d(x, y), d(x, Hx), d(y, Hy)) \tag{1.4}$$

for all $x, y \in X$ and some $\phi \in (A)$, where (A) is the set of all functions $\phi : \mathbb{R}_+^3 \to \mathbb{R}_+$ satisfying:
i) ϕ is continuous on the set $\mathbb{R}_+^3$ (with respect to Euclidean metric on $\mathbb{R}^3$);
ii) if any of the conditions $a \leq \phi(a, b, b)$, or $a \leq \phi(b, b, a)$, or $a \leq \phi(b, a, b)$ holds for some $a, b \in \mathbb{R}_+$, then there exist $\aleph \in [0, 1)$ such that $a \leq \aleph b$.

Literature abounds with several generalizations and extensions of classical Banach's fixed point theorem, interested reader can see [6–9] and the references there in.

More over, in 1976, Gerald Jungck [10] established the notion of common fixed point of mappings. He proved that if (S, G) are pair of self-mappings defined on a complete metric space (X, d), with $G(X) \subset S(X)$ and S is continuous. Then, S *and* G have a unique common fixed point, if there exists $\aleph \in (0, 1)$ such that,

$$d(Gx, Gy) \leq \aleph d(Sx, Sy) \forall x, y \in X, \tag{1.5}$$

Definition 1.2: Let X be a non-empty set. Two mappings $S, H\colon M \to M$ are said to commute iff $SH = HS$.
Example 1.3: Consider $S, H\colon M \to M$ such that, $Hx = x, Sx = 1 - x, \forall x \in M$.
$H(S(x)) = H(1 - x) = 1 - x$
$S(H(x)) = Sx = 1 - x$. Thus, S and H commute.
Definition 1.4 [10]: Let M be a complete metric space, and suppose $G, S : M \to M$. For $x_0 \in M$, sequence $\{Sx_n\}_{n=0}^{\infty} \subset M\}$ defined by

$$Sx_{n+1} = Gx_n, \ n \geq 0, \tag{1.6}$$

is called Jungck iterative process.

Using idea of Jungck, many authors have improved on the existing iterative techniques.
Definition 1.5 [8]: Let B be a Banach space, and the pair of operator $U, G : B \to B$. For any $x_0 \in B$, the sequence $\{Ux_n\}_{n=0}^{\infty}$, defined by

$$Ux_{n+1} = (1 - \alpha)Ux_n + \alpha Gx_n, \quad n \geq 0, \quad \alpha \in (0, 1). \tag{1.7}$$

is called Jungck-Schaefer iteration.
For more on Jungck-type iterative algorithms, interested reader can see [9, 11–16] and references therein.

Recently, Olatinwo and Omidire [17] established unique common fixed point of generalized M_J contraction defined below.
Definition 1.6 [17]: Let (X, d) be a metric space and $H, S : X \to X$ such that

$$\begin{aligned} d(Hx, Hy) \quad \leq \quad & \phi(d(Sx, Sy), d(Sx, Hx), d(Sy, Hy) \\ & [d(Sx, Hx)]^r [d(Sy, Hx)]^p d(Sx, Hy), d(Sy, Hx)[d(Sx, Hx)]^m) \\ & \forall x, y \in X; r, p, m \in \mathbb{R}_+ \end{aligned} \tag{1.8}$$

and that (8) is satisfied by the set of all functions $\phi : \mathbb{R}_+^5 \to \mathbb{R}_+$ such that:
(i) ϕ is continuous on the set $\mathbb{R}_+^5$ (with respect to Euclidean metric on $\mathbb{R}^5$);
(ii) if any of the conditions $a \leq \phi(a, b, b, b, b)$, or $a \leq \phi(b, b, a, b, b)$, or $a \leq \phi(b, b, a, c, c)$ holds for

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some $a, b, c \in \mathbb{R}_+$, then there exists a constant $\aleph \in [0, 1)$ such that $a \leq \aleph b$.

Definition 1.7: Let (X, d) be a metric space and $H, S : X \to X$ such that

$$d(Hx, Hy) \leq \phi(d(Sx, Sy), d(Sx, Hx), d(Sy, Hy)) \forall\ x,\ y \in X \tag{1.9}$$

and that (9) is satisfied by the set of all functions $\phi : \mathbb{R}_+^3 \to \mathbb{R}_+$ such that:
(i) ϕ is continuous on the set $\mathbb{R}_+^3$ (with respect to Euclidean metric on $\mathbb{R}^3$);
(ii) if any of the conditions $a \leq \phi(a, b, b,)$, or $a \leq \phi(b, b, a,)$, or $a \leq \phi(b, a, b)$ holds for some $a, b, c \in \mathbb{R}_+$, then there exists a constant $\aleph \in [0, 1)$ such that $a \leq \aleph b$.

Meanwhile, Takahashi [18] introduced the notion of convex metric space by defining convex structure on a metric space. He showed (with examples) that all normed spaces and their convex subsets are embeded in convex metric space.
Definition 1.8 [14, 18]: Let (X, d) is a metric space. A map $W : X \times X \times [0, 1] \longrightarrow X$ is said to be a convex structure on X if for each $(x, y, \lambda) \in X \times X \times [0, 1]$ and $t \in X$,

$$d(t, W(x, y, \lambda)) \leq \lambda d(t, x) + (1 - \lambda)d(t, y)$$

A metric space X together with the convex structure W is called a convex Metric Space.
Lemma 1.9 [18]: For $x, y \in X$ and $\lambda \in [0, 1]$,

$$d(x, y) = d(x, W(x, y; \lambda)) + d(W(x, y; \lambda), y).$$

2 Preliminary

In this and next section, let $H_i : C \to C$ be sequence of operators and $(S, \{H_i\}_{i=1}^k)$ pair of commuting operators. And let $(S, H_i) = (S, H_1), (S, H_2), (S, H_3), \cdots, (S, H_k)$.

Definition 2.1: A pair of self-map $(S, \{H_i\}_{i=1}^k)$ on a convex metric space (C, d, W) is said to be generalized M_J Type contraction if

$$\begin{aligned} d(H_i x, H_i y) \quad \leq \quad & \phi(d(Sx, Sy), d(Sx, H_i x), d(Sy, H_i y), [d(Sx, H_i x)]^r [d(Sy, H_i x)]^p \\ & d(Sx, H_i y), d(Sy, H_i x)[d(Sx, H_i x)]^m),\ \forall\ x,\ y \in C; r, p, m \in \mathbb{R}_+ \end{aligned}$$

for all $x, y \in C$ and some ϕ , where (ϕ) is the set of all functions $\phi : \mathbb{R}_+^5 \to \mathbb{R}_+$ satisfying:
i) ϕ is continuous on the set $\mathbb{R}_+^5$ (with respect to Euclidean metric on $\mathbb{R}^5$);
ii) if any of the conditions $a \leq \phi(a, b, b, b, b)$, or $a \leq \phi(b, b, a, c, c)$, or $a \leq \phi(b, b, a, b, b)$ holds for some $a, b, c \in \mathbb{R}_+$, there exist $\aleph \in [0, 1)$ such that $a \leq \aleph b$.

Definition 2.2: A pair of self-map $(S, \{H_i\}_{i=1}^k)$ on a convex metric space (C, d, W) is said to be Akram-Jungck Type contraction if it satisfies the conditions:

$$d(H_i x, H_i y) \leq \phi(d(Sx, Sy), d(Sx, H_i x), d(Sy, H_i y)) \forall\ x,\ y \in C \tag{2.1}$$

for all $x, y \in C$ and some ϕ , where ϕ is the set of all functions $\phi : \mathbb{R}_+^3 \to \mathbb{R}_+$ satisfying:
i) ϕ is continuous on the set $\mathbb{R}_+^3$ (with respect to Euclidean metric on $\mathbb{R}^3$);
ii) if any of the conditions $a \leq \phi(a, b, b)$, or $a \leq \phi(b, b, a)$, or $a \leq \phi(b, a, b)$ holds for some $a, b \in \mathbb{R}_+$, then there exists $\aleph \in [0, 1)$ such that $a \leq \aleph b$.

Remark 2.3:

(i) If $k = 1$, Definitions 2.1 and 2.2 reduce to Definitions 1.6 and 1.7 respectively.

(ii) If $k = 1$ and $S = I$(Identity operator) Definition 2.2 reduces to $A-$ contraction of Akram et al.

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(iii) If $k=1$ and $\phi(d(Sx,Sy),d(Sx,Hx),d(Sy,Hy)) = a(Sx,Sy),\ a\in(0,1)$, Definition 2.2 reduces to Jungck contraction.

The following lemma and definitions shall be required in the sequel:
Lemma 2.4:[19] Let (X,d,W) be a convex metric space. For each $x,y\in X$ and $\lambda,\lambda_1,\lambda_2\in[0,1]$, we have the following:
(i) $W(x,x;\lambda)=x; W(x,y;0)=y$ *and* $W(x,y;1)=x$; and
(ii) $|\lambda_1\lambda_2|d(x,y)\le d(W(x,y;\lambda_1),W(x,y;\lambda_2))$
Definition 2.5: [20] A function $S:\mathbb{R}^n\mapsto\mathbb{R}$ is said to be convex if for any $x,y\in\mathbb{R}^n$ and $\aleph\in[0,1]$ we have that

$$S(x+\aleph(y-x))\le\aleph S(x)+(1-\aleph)Sy.$$

3 Main Result

In this section we present some fixed point theorems in convex metric space settings for mappings defined in section 2 above. These results include the analogues, generalization and extension of some certain results in [5], [10] and [17]

Theorem 3.1: Let (C,d,W) be a complete convex metric space, and let $S:C\to C$ be a continuous operator commuting with each $\{H_i\}_{i=1}^k:C\to C$ such that $H_i(C)\subseteq S(C)$ for each i. If $\{H_i\}_{i=1}^k:C\to C$ is a sequence of operator satisfying definition (2.1). Then;
(i) all H_i and S have a unique common fixed point $x^*\in C$;
(ii) for any $x_0\in C$ the sequence $\{Sx_n\}_{n=0}^\infty$ defined by

$$Sx_{n+1}=W(Sx_n,H_ix_n;\lambda)\qquad(\lambda\in[0,1)) \tag{3.1}$$

converges to x^* the unique common fixed point of H_i and S.

Proof: let $x_1\in C$ and for each i, $\ Sx_n=H_ix_{n-1},\forall\ x\in C$.
Therefore, since each $H_i(i=1,2,\cdots,k)$ satisfy definition (2.1), then we have by iteration (3.1);

$$\begin{aligned}
d(Sx_n,Sx_{n+1}) &= d(Sx_n,W(Sx_n,H_ix_n;\lambda))\\
&\le \lambda d(Sx_n,Sx_n)+(1-\lambda)d(Sx_n,H_ix_n)\\
&= (1-\lambda)d(Sx_n,H_ix_n)\\
&= (1-\lambda)d(H_ix_{n-1},H_ix_n)
\end{aligned}$$

Then,

$$\begin{aligned}
d(Sx_n,Sx_{n+1}) &\le (1-\lambda)d(H_1x_0,H_1x_1),(1-\lambda)d(H_2x_1,H_2x_2),\cdots,(1-\lambda)d(H_kx_{n-1},H_kx_n)\\
&\le (1-\lambda)[\phi(d(Sx_{n-1},Sx_n),d(Sx_{n-1},H_ix_{n-1}),d(Sx_n,H_ix_n),\\
&\quad [d(Sx_{n-1},H_ix_{n-1})]^r[d(Sx_n,H_ix_{n-1})]^pd(Sx_{n-1},H_ix_n),\\
&\quad d(Sx_n,H_ix_{n-1})[d(Sx_{n-1},H_ix_{n-1})]^m)]\\
&= (1-\lambda)[\phi(d(Sx_{n-1},Sx_n),d(Sx_{n-1},Sx_n),d(Sx_n,Sx_{n+1})\\
&\quad [d(Sx_{n-1},Sx_n)]^r[d(Sx_n,Sx_n)]^pd(Sx_{n-1},Sx_{n+1}),\\
&\quad d(Sx_n,,Sx_n)[d(Sx_{n-1},Sx_n)]^m))]\\
&= (1-\lambda)[\phi(d(Sx_{n-1},Sx_n),d(Sx_{n-1},Sx_n),d(Sx_n,Sx_{n+1}),0,0)]\\
&\le (1-\lambda)\aleph(d(Sx_{n-1},Sx_n)\le\aleph(d(Sx_{n-1},Sx_n).
\end{aligned}$$

i.e;

$$d(Sx_1,Sx_2)=(1-\lambda)d(H_1x_0,H_1x_1)\le\aleph(d(Sx_0,Sx_1);$$

and, in like manner,

$$d(Sx_2, Sx_3) \leq \aleph(d(Sx_1, Sx_2) = \aleph^2(d(Sx_0, Sx_1),$$

$$\vdots$$

continue till $i = k$, we have

$$d(Sx_n, Sx_{n+1}) \leq \aleph^k(d(Sx_0, Sx_1).$$

Therefore,

$$d(Sx_{n+1}, Sx_n) \leq \aleph^k d(Sx_1, Sx_0),$$

so, for any $m > n$, inductively, we have

$$\begin{aligned} d(Sx_m, Sx_n) &\leq \sum_{q=n}^{m-1} d(Sx_{q+1}, Sx_q) \\ &\leq \sum_{q=n}^{m-1} \aleph^q d(Sx_1, Sx_0) \\ &= \aleph^k d(Sx_1, Sx_0) \sum_{q=0}^{m-n-1} \aleph^q \\ &\leq \aleph^k d(Sx_1, Sx_0) \sum_{q=0}^{\infty} \aleph^q \\ &= \aleph^k d(Sx_1, Sx_0) \frac{1}{1-\aleph}. \end{aligned} \quad (3.2)$$

We have from (3.2) that $\{Sx_n\}_{n=0}^{\infty}$ is a Cauchy sequence in C (a complete convex metric space) then, there exists $x^* \in C$ such that for each i

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} H_i x_{n-1} = x^*.$$

With continuity of S and its commutaivity with each H_i, we have the following:

$$Sx^* = S(\lim_{n\to\infty} Sx_n) = \lim_{n\to\infty} S^2 x_n \quad (3.3)$$

$$Sx^* = S(\lim_{n\to\infty} H_i x_n) = \lim_{n\to\infty} (SH_i x_n) = \lim_{n\to\infty} (H_i S x_n) \quad (3.4)$$

Thus, using equation (3.3), (3.4) and definition (2.1) again with $x = Sx_n,\ y = x^*$, we have, for each H_i

$$\begin{aligned} d(H_i(Sx_n), H_i x^*) \leq\ & \phi(d(S(Sx_n), Sx^*), d(S(Sx_n), H_i(Sx_n)), d(Sx^*, H_i x^*), \\ & [d(S(Sx_n), H_i(Sx_n))]^r [d(Sx^*, H_i(Sx_n))]^p d(S(Sx_n), H_i x^*), \\ & d(Sx^*, H_i(Sx_n))[d(S(Sx_n), H_i(Sx_n))]^m). \end{aligned}$$

Using the continuity of, S and metric as well as taking limit in the above together with the application of (3.3) and (3.4) yield,

$$\begin{aligned} d(S^2x_n, H_i x^*) \leq\ & \phi(d(S^2x_n, Sx^*), d(S^2x_n, H_i(Sx_n)), d(Sx^*, H_i x^*), \\ & [d(S^2x_n, H_i(Sx_n)]^r [d(Sx^*, H_i(Sx_n))]^p d(S^2x_n, Tx^*), \\ & d(Sx^*, H_i(Sx_n))[d(S^2x_n, H_i(Sx_n)]^m) \end{aligned}$$

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as $n \to \infty$ we have,

$$\begin{aligned} d(Sx^*, H_ix^*) &\leq \phi(d(Sx^*, Sx^*), d(Sx^*, Sx^*), d(Sx^*, H_ix^*), [d(Sx^*, Sx^*]^r[d(Sx^*, Sx^*)]^p d(Sx^*, H_ix^*), \\ &\quad d(Sx^*, Sx^*)[d(S^2x_n, Sx^*]^m) = \phi(0, 0, d(Sx^*, H_ix^*), 0, 0) \\ &\leq \aleph^k.0 = 0. \end{aligned}$$

Hence, $Sx^* = H_ix^*$.
And this implies that $Sx^* = x^* = H_ix^*(for\ each\ i,\ i = 1, 2, 3, \cdots, k)$.
Now, for the uniqueness of the fixed point. Suppose not, then there exists
$H_ix^* = Sx^* = x^*$, and $H_iy^* = Sy^* = y^*$, such that $x^* \neq y^*$.
Therefore,

$$\begin{aligned} 0 &< d(x^*, y^*) \\ &= d(H_ix^*, H_iy^*) \\ &\leq \phi(d(Sx^*, Sy^*), d(Sx^*, H_ix^*), d(Sy^*, H_iy^*), \\ &\quad [d(Sx^*, H_ix^*)]^r[d(Sy^*, H_ix^*)]^p d(Sx^*, H_iy^*), d(Sy^*, H_ix^*)[d(Sx^*, H_ix^*)]^m) \\ &= \phi(d(x^*, y^*), d(x^*, x^*), d(y^*, y^*), \\ &\quad [d(x^*, x^*)]^r[d(y^*, x^*)]^p d(x^*, y^*), d(y^*, x^*)[d(x^*, x^*)]^m) \\ &= \phi(d(x^*, y^*), 0, 0, 0, 0) \leq \aleph^k.0 = 0. \end{aligned}$$

That is, $d(x^*, y^*) \leq 0 \Rightarrow d(x^*, y^*) = 0$, hence $x^* = y^*$.

Theorem 3.2: Let (C, d, W) be a complete convex metric space, and let $S : C \to C$ be a continuous operator commuting with each $H_i : C \to C (i = 1, 2, \cdots, k,)$ such that $H_i(C) \subseteq S(C)$ for each i. If $\{H_i\}_{i=1}^k : C \to C$ is a sequence of operator satisfying definition (2.2). Then:
(i) all H_i and S have a unique common fixed point $x^* \in C$;
(ii) for any $x_0 \in C$ the sequence $\{Sx_n\}_{n=0}^{\infty}$ defined by (3.1) converges to x^*.

Proof: The proof line follows the same line of argument as proof of Theorem 3.1.

Remark: Our results here shown that the sequences of generalized M_J and Akram-Jungck type contractions have a unique common fixed point. This is a generalization to some results contain in [5], [10], [19] and many other related ones in literature.

4 Application

We consider the basic mathematical programming problem

$$min\ S(x) \qquad subject\ to \quad x \in C \qquad (MP)$$

where $S : \mathbb{R}^n \to \mathbb{R}$ and $C \subset \mathbb{R}^n$. The function S is the objective function and the set C is the constrain set.
Definition 4.1 [20]: "A point $x^* \in C$ is said to be a local minimum of (MP) if there exists $\delta > 0$ such that $S(x) \geq S(x^*)$ for all $x \in B_\delta(x^*) \cap C$."

Definition 4.2 [20]: "A point $x^* \in C$ is said to be a minimum or global minimum of (MP) if $S(x) \geq S(x^*)$ for all $x \in C$."

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Consider a minimization problem represented as follows:

$$min\ S(x) \quad subject\ to \quad H_i(x)\ satisying\ definition\ 2.1\ (and\ 2.2), \quad x \in C \subseteq \mathbb{R}^n, \tag{4.1}$$

and

$$\phi(t) := \lambda \int_a^b K(u_1, u_2, t)F(u_1, u_2, \phi(u))du$$

$F : [a, b] \times [a, b] \times C[a, b] \to \mathbb{R}$ is the given nonlinear function from $\mathbb{R}^n \to \mathbb{R}$, $\phi \in C[a, b]$ is the unknown function, $K : \mathbb{R}^n \to \mathbb{R}$ is the kernel, $\lambda \in \mathbb{R}$.
Let d be a convex metric space induced by the norm $||\cdot||$, i.e

$$d(\phi_1, \phi_2) = ||\phi_1(t) - \phi_2(t)|| = max_{1\leq i\leq n}|\phi_i(t) - \phi_{i+1}(t)|.$$

The usual metric in $C[a, b]$. Then the space $C[a, b]$ is a complete convex metric space with respect to metric defined above.
We define $H_i : C[a, b] \to C[a, b]$ as

$$H_i(\phi(t)) = \lambda \int_a^b K(u_1, u_2, t)F(u_1, u_2, \phi(u))du \tag{4.2}$$

Assuming S and H_i have a common fixed point. Thus, the solution set to the above minimization problem is the fixed point of H_i, which is also the fixed point of S i.e

$$Fix_{S\cap H_i} = \{x^* \in C : Sx^* = H_i x^* = x^*\}$$

the common fixed point of all H_i and S, where S is the objective function.

Theorem 4.4: Let the pair (H_i, S) be self map on $C \subset \mathbb{R}^n$ satisfying assumptions of Theorem (3.1). Let $C = [a, b]$ a finite interval and $\phi \in C[a, b]$. Assume for each $\{\phi_i\}_{i=1}^n \in C[a, b]$ and $u \in [a, b]$ the function $F(u_1, u_2, \phi(u))$ satisfies

$$|F(u_1, u_2, \phi_1(u)) - F(u_1, u_2, \phi_2(u))| \leq |f(u)||\phi_1 - \phi_2| \quad \forall\ u \in [0, 1]$$

such that

$$\int_a^b |K(u_1, u_2, t)f(u)|^2 du \leq L^2$$

is bounded by $(0, 1)$ where

$$0 < |\lambda|L\sqrt{b - a} < 1.$$

Then, the operator H_i as define by (4.2) has a fixed point. And our minimization problem (4.1) has a solution in $C[a, b]$. Therefore, for any $\phi_0 \in C[a, b]$, the sequence $\{H_i^n x_0\}$ converges to the solution set of (4.1).
Since the fixed point set of $(S \cap H_i)$ is a singleton by statement of the Theorems 3.1 (i), therefore, x^* is the minimum of (4.1).

Remark:
We have proved that sequences of generalized M_J and Akram-Jungck type contractions have a unique common fixed point. It has also been demonstrated that these results have applications in optimization theory. Therefore, this paper has shown the importance of fixed point theory in solving optimization problems. However, further studies can be carried out on the application to numerical examples to finding approximate solution of such problem.

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