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On the core of some classes of generalised Bol-Moufang loops

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Abstract

This study centers on properties of the core of some classes of generalized Bol-Moufang loops wherein a construction of a generalized Bol loop is presented. Several properties of the core of generalized Bol loops are established. Some of these properties inform that the core of generalized Bol loops belongs to the variety of Left Symmetric Left Distributive (LSLD) groupoid and therefore, a rack. A necessary and sufficient condition for the core of half-Bol loops to be medial is given with proof. The core of the inverse property (IP) generalized Moufang loop with universal α-elasticity is examined.

Keywords: Generalised Bol loops; Generalised Moufang loops; Core.
MSC2010: 20N02, 20N05.

1 Introduction

A groupoid $(G, \cdot)$ consists of a set G together with a binary operation '$\cdot$' on G. For $x \in G$, define the left (right) translation of y by $L_x(y) = x \cdot y$ $(R_x(y) = y \cdot x)$, for all $x, y \in G$. A quasigroup is a groupoid $(G, \cdot)$ with a binary operation $\cdot$ such that for each $a, b \in G$ the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions $x, y \in G$. A loop is a quasigroup with a two-sided identity element. A left loop in which all right translations are bijective is also called a loop. For basic facts on loops and quasigroups, we refer the reader to [1–6]. A loop satisfying the right Bol identity

$$((xy)z)y = x((yz)y)$$

or equivalently

$$R_y R_z R_y = R_{(yz)y}$$

for all $x, y, z \in G$ is called a right Bol loop. A loop satisfying the mirror identity $(x(yx))z = x(y(xz))$ for all $x, y, z \in G$ is called a left Bol loop, and a loop which is both left and right Bol is a Moufang loop. For this paper, the term "Bol loop" will refer to right Bol loop; all statements about right Bol loops dualize trivially to left Bol loops. For basic facts on Bol loops, we refer the reader to [3, 7].

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Ajmal introduced and studied generalised Bol loops in [8] wherein they were called right B-loops. Since right B-loops were first used to connote bruck loops of odd order [9], we maintain generalised Bol loops for Ajmal generalisation to be consistent with recent literatures which have adopted Glauberman use of B-loops for Bruck loops of odd order [10].
Since the introduction of generalised Bol loops by Ajmal [8], several studies have been undertaken to reveal the structures and properties of generalized Bol loops. Some interesting results on generalised Bol-Moufang loops can be found in [11–15].
In an attempt to construct invariant of classes of isotopic Moufang loops, R. H. Bruck [1] introduced the concept of core as an invariants of isotopic Moufang loops. Suppose G is a Moufang loop, Bruck defined the core of $(G, \cdot)$ as the groupoid $(G, +)$ consisting of the elements of G equipped with the binary operation '+' defined by $x + y = xy^{-1}x$ for all $x, y \in G$. Robinson studied the core of Bol loops in [7] and asserted that a necessary but not sufficient condition for the core to be a quasigroup is that $x \mapsto x \cdot x$ is a permutation of the Bol loops. Certain properties of generalised Bol loops were presented in a recent study of generalised Bol loops [16].
In [17], the core of a right Bol loop (RBL) was shown to be elastic and right idempotent and the core of a middle Bol loop was shown to possess the left inverse property (automorphic inverse property, right idempotence respectively) if and only if its corresponding RBL has the super anti-automorphic inverse property (automorphic inverse property, exponent 2 respectively). Several other connections between RBL and MBL were established. Jaíyéolá et al. [18] linked the Bryant-Schneider group of a middle Bol loop with some of the isostrophy-group invariance of its corresponding right (left) Bol loop while Oyebo et al. [19] deduced subgroups of the crypto-automorphism group of a middle Bol loop.
The core of generalised Bol loops was first studied by Adéníran et. al. in [16] wherein isotopy characterization of generalised Bol loops was examined, and isotopic generalised Bol loops were shown to have isomorphic cores. It was also shown that the set of semi-automorphisms of generalised Bol loops $(G, \cdot)$ were the automorphisms of the core $(G, +)$. Recently, Adéníran et. al. in [20] characterised generalised Bol loops using the notion of isotopy. Among other results, It was shown that a generalised Bol loop can be constructed using a group and a subgroup of it. A right conjugacy closed σ-generalised Bol loop is shown to be a σ-generalised right central loop.
The organization of the paper is as follows. Section 1 gives a brief introduction of the paper and section 2 contains necessary definitions of terms used throughout the paper. Section 3 and section 4 contain the main result; in section 3 the core of generalised Bol loops is found to satisfy $(x + y)^{-1} = x^{-1} + y^{-1}$, left key law, left distributive law. The core of generalised Bol loops is thus found to belong to the variety of Left Symmetric Left Distributive (LSLD) groupoid, and therefore, a rack. A necessary and sufficient condition for the core of half-Bol loops to be medial is provided. We examine the core of IP generalised Moufang loop with universal α-elasticity. Finally in section 4, a construction of generalised Bol loop from alternative division ring which is not associative is presented and it is proved that the resulting generalised Bol loop is not a generalised Moufang loop.

2 Preliminaries

In this section, we give some basic notions and terminologies that are needed throughout this study. Some of these notions can be found in [3]. Some earlier results in the study of the core of generalized Bol loops are also provided in this section.

Definition 2.1 ([3]). *A loop $(G, \cdot)$ is called a left inverse property loop if it satisfies the left inverse property (LIP) given by:*

$$x^{\lambda}(xy) = y.$$

Definition 2.2 ([3]). *A loop $(G, \cdot)$ is called a right inverse property loop if it satisfies the right inverse property (RIP) given by:*

$$(yx)x^{\rho} = y.$$

Definition 2.3 ([3]). *A loop $(G,\cdot)$ is called an automorphic inverse property (AIP) loop if it satisfies the automorphic inverse property given by:*

$$(xy)^{-1} = x^{-1}y^{-1}.$$

Definition 2.4 ([3]). *A loop $(G,\cdot)$ is called an anti-automorphic inverse property (AAIP) loop if it satisfies the automorphic inverse property given by:*

$$(xy)^{-1} = y^{-1}x^{-1}.$$

Definition 2.5 ([3]). *Let $(G,\cdot)$ be a loop, the left nucleus, N_λ, the middle nucleus, N_μ and the right nucleus, N_ρ are defined as follows:*

$$N_\lambda = \{x \in G | x \cdot yz = xy \cdot z \ \forall \ y,z \in G\}$$
$$N_\mu = \{y \in G | x \cdot yz = xy \cdot z \ \forall \ x,z \in G\}$$
$$N_\rho = \{z \in G | x \cdot yz = xy \cdot z \ \forall \ x,y \in G\}$$

Definition 2.6 ([21]). *A groupoid $(G,*)$ is called*

1. *left distributive if $x*(y*z) = (x*y)*(x*z)$,*
2. *medial if $(x*y)*(u*v) = (x*u)*(y*v)$,*
3. *idempotent if $x*x = x$,*
4. *left key (left symmetric or left involutory) if $x*(x*y) = y$*

Definition 2.7 ([22], [23]). *A groupoid $(G,*)$ is called a left involutory quandle if it satisfies the following*

1. *Left involutory law*
2. *Left distributive law*
3. *Idempotent law*

Definition 2.8 ([22]). *A left (right) quasigroup $(G,*)$ is called a rack if it satisfies left (right) distributive law.*

Definition 2.9. *[13] A loop is a generalised Bol loop if it satisfies the relation*

$$(xy\cdot z)y^\alpha = x(yz\cdot y^\alpha)$$

where y^α is the image of y under some mapping α of the loop onto itself.

The dual of the relation in Definition 2.9 is given by

$$(y^\alpha \cdot zy)x = y^\alpha(z\cdot yx)$$

and it is called half-Bol loops.

Definition 2.10 ([25]). *A loop is a generalised Moufang loop if it satisfies the relation*

$$(xy\cdot z)y^\alpha = x(y\cdot zy^\alpha)$$

where y^α is the image of y under some mapping α of the loop onto itself.

Definition 2.11 ([16]). *Let $(G,\cdot)$ be a generalised Bol loop, for all $x,y\in G$ define $x+y$ by*

$$x+y = xy^{-1}\cdot x^\alpha.$$

The groupoid $(G,+)$ is called the core of $(G,\cdot)$.

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The groupoid $(G, +)$ is defined since any generalised Bol loop has two-sided inverse. It must be noted that the core of half-Bol loops is given by $x \cdot y^{-1}x^{\alpha}$. It is known from the work of Vanžurová [24] that the core of a (left) Bol loop is medial if and only if the following identity is satisfied;

$$y(x(z(u(z(xy))))) = z(x(y(u(y(xz))))).$$

Definition 2.12 ([3]). *Let $(G, \cdot)$ be a quasigroup and let $f, g \in G$. For all $x, y \in G$, let*

$$x \circ y = xR_{(g)^{-1}} \cdot yL_{(f)^{-1}}$$

The $(G, \circ)$ is a loop called the principal isotope of the quasigroup $(G, \cdot)$.

Definition 2.13 ([25]). *Let $(G, \cdot)$ be a generalised Bol loop. $(G, \cdot)$ is said to be α-elastic if the identity $(y \cdot z) \cdot y^{\alpha} = y \cdot (z \cdot y^{\alpha})$, holds in G.*

Definition 2.14 ([25]). *Let $(G, \cdot)$ be a generalised Bol loop. $(G, \cdot)$ is called a right α-alternative loop if it satisfies $(xy) \cdot y^{\alpha} = x \cdot (yy^{\alpha})$ and it is called left α-alternative loop if it satisfies $(y^{\alpha}y) \cdot x = y^{\alpha} \cdot (yx)$.*

Definition 2.15 ([25]). *Let $(G, \cdot)$ be a generalised Bol loop. $(G, \cdot)$ is called an α-alternative loop if it is both right and left α-alternative.*

3 On the core of some generalised loops

Here, we examine properties of the core of generalised Bol loops, half-Bol loops and generalised Moufang loops.

Lemma 3.1. *Let $(G, \cdot)$ be a commutative generalised Bol loop and let α preserves inverse. The core $(G, +)$, associated with $(G, \cdot)$, satisfies the identity $(x + y)^{-1} = (x^{-1} + y^{-1})$.*

Proof. We want to show that

$$(x + y)^{-1} = (x^{-1} + y^{-1})$$

Of course,

$$\begin{aligned}
(x + y)(x^{-1} + y^{-1}) &= (xy^{-1} \cdot x^{\alpha})(x^{-1}y \cdot (x^{-1})^{\alpha}) \\
&= (xy^{-1} \cdot x^{\alpha})(x^{-1}y \cdot (x^{\alpha})^{-1}) \\
&= (xy^{-1} \cdot x^{\alpha})((x^{\alpha})^{-1} \cdot x^{-1}y) \qquad \text{(commutativity and GBP)} \\
&= (xy^{-1} \cdot x^{-1})y \\
&= (x^{-1} \cdot xy^{-1})y \\
&= e
\end{aligned}$$

□

Theorem 3.2. *Let $(G, \cdot)$ be a generalized Bol loops with automorphic inverse property such that '$\cdot$' is right distributive over '$+$' and α preserves inverse, then $(G, +)$ satisfies the following;*

1. *the left key property, $x + (x + y) = y$,*
2. *the left distributive property, $x + (y + z) = (x + y) + (x + z)$,*

Proof. 1.

$$\begin{aligned}
x + (x + y) &= x + (xy^{-1} \cdot x^{\alpha}) \\
&= x(xy^{-1} \cdot x^{\alpha})^{-1} \cdot x^{\alpha} \\
&= x(x^{-1}y \cdot (x^{\alpha})^{-1}) \cdot x^{\alpha} \\
&= y \cdot (x^{\alpha})^{-1} \cdot x^{\alpha} \\
&= y
\end{aligned}$$

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2.

$$\begin{aligned} x(y+z)^{-1}\cdot x^{\alpha} &= x(yz^{-1}\cdot y^{\alpha})^{-1}\cdot x^{\alpha} \\ &= x(y^{-1}z\cdot (y^{\alpha})^{-1})\cdot x^{\alpha} \\ &= x(y^{-1}+z^{-1})\cdot x^{\alpha} \\ &= (xy^{-1}\cdot x^{\alpha})+(xz^{-1}\cdot x^{\alpha}) \\ &= (x+y)+(x+z) \end{aligned}$$

□

Corollary 3.3. *Let $(G,\cdot)$ be a generalized Bol loops with automorphic inverse property such that '$\cdot$' is right distributive over '$+$' and α preserves inverse. The core, $(G,+)$, associated with $(G,\cdot)$, belongs to the variety of LSLD groupoid.*

Proof. The proof follows from Theorem 3.2. □

Corollary 3.4. *Let $(G,\cdot)$ be a generalized Bol loop. The core, $(G,+)$ associated with $(G,\cdot)$, is a rack.*

Proof. The proof follows from Definition 2.8 and Theorem 3.2. □

Now, we give a necessary and sufficient condition for the core of half-Bol loops to be medial. A result similar to that contained in [24] is obtained for the core of half-Bol loops under a new feature for the self map, α and summarised in the theorem below. We denote the half-Bol loops by $\mathcal{HBL}$.

Theorem 3.5. *Let $(G,\cdot)$ be a half-Bol loops, $\mathcal{HBL}$ and α is a homomorphic self map. Since the core $(G,+)$ of $(G,\cdot)$ has the AIP, $(G,+)$ is medial if and only if the following identity holds in $\mathcal{HBL}$.*

$$y(x^{\alpha}(z(u(z^{\alpha}(x^{\alpha}y^{\alpha}))))) = z(x^{\alpha}(y(u(y^{\alpha}(x^{\alpha}z^{\alpha}))))).\quad x,y,z,u\in\mathcal{HBL}. \tag{3.1}$$

Proof. Mediality, $(x+y)+(z+u)=(x+z)+(y+u)$ for $\mathcal{GBL}$ takes the form

$$\begin{aligned} (x(y^{-1}x^{\alpha}))\cdot((z+u)^{-1}\cdot(x+y)^{\alpha}) &= (x(z^{-1}x^{\alpha}))\cdot((y+u)^{-1}\cdot(x+z)^{\alpha}) \\ x(y^{-1}(x^{\alpha}(z+u)^{-1}\cdot(x+y)^{\alpha})) &= x(z^{-1}(x^{\alpha}(y+u)^{-1}\cdot(x+z)^{\alpha})) \\ (y^{-1}(x^{\alpha}(z+u)^{-1}\cdot(x+y)^{\alpha})) &= (z^{-1}(x^{\alpha}(y+u)^{-1}\cdot(x+z)^{\alpha})) \\ y(x^{\alpha}(z+u^{-1})\cdot(x(yx^{\alpha}))^{\alpha}) &= z(x^{\alpha}(y+u^{-1})\cdot(x(zx^{\alpha}))^{\alpha}) \\ y(x^{\alpha}((z+u^{-1})\cdot(x^{\alpha}(y^{\alpha}(x^{\alpha})^{\alpha})))) &= z(x^{\alpha}((y+u^{-1})\cdot(x^{\alpha}(z^{\alpha}(x^{\alpha})^{\alpha})))) \\ y((x^{\alpha}((z(uz^{\alpha}))\cdot x^{\alpha}))\cdot y^{\alpha}) &= z((x^{\alpha}((y(uy^{\alpha}))\cdot x^{\alpha}))\cdot z^{\alpha}) \end{aligned}$$

Using half-Bol identity twice, we obtain (3.1). □

In what follows we examine the core of IP generalized Moufang loops with universal α-elasticity. The universal α-elasticity property was introduced and studied for generalized Moufang loops by the authors in [25].

Theorem 3.6. *The core $G(+)$ of a commutative IP generalized Moufang loop with universal α-elasticity property is a quasigroup if and only if the mapping $x\mapsto xx^{\alpha}$ is a permutation on G.*

Proof. Let $G(+)$ be a quasigroup. Then, for each $a\in G$, there exist a unique element $x\in G$ such that $x+e=a$, where e is the unity of the loop $(G,\cdot)$. So, the equation $xe^{-1}x^{\alpha}=xx^{\alpha}=a$ has unique solution in $(G,\cdot)$ and consequently $x\mapsto xx^{\alpha}$ is a permutation on G.
Conversely, suppose that the $\phi: x\longrightarrow xx^{\alpha}$ is a permutation on G. The equation $a+x=b$ where $a,b\in G$, then, $ax^{-1}a^{\alpha}=b$ has unique solution $x=a^{\alpha}b^{-1}a$. Also, consider the equation $x+a=b$ i.e. $xa^{-1}x^{\alpha}=b$, where $a,b\in G$. Since $(G,\cdot)$ is commutative, the last equation is equivalent to $a^{-1}xx^{\alpha}=b$ which implies $xx^{\alpha}=ab$. Thus, $\phi(x)=ab$ which has a unique solution since ϕ is a permutation on G. □

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It must be noted that the law of α-elasticity does not hold in the core of an IP generalised Moufang loop with universal α-elasticity law. This is unlike what is obtainable in classical IP loop with universal elasticity property.

Theorem 3.7. *The core $G(+)$ of an IP generalized Moufang loop $G(\cdot)$ with universal α-elasticity law, where α is a homomorphism, is a left distributive groupoid if and only if the following identity*

$$x(y \cdot xzx^\alpha \cdot y^\alpha)x^\alpha = xyx^\alpha \cdot z \cdot (xyx^\alpha)^\alpha \tag{3.2}$$

holds in $G(\cdot)$.

Proof. Let $G(+)$ be a left-distributive groupoid, that is,

$$x + (y + z) = (x + y) + (x + z)$$

It thus follows that

$$\begin{aligned} x(y + z)^{-1} \cdot x^\alpha &= (x + y)(x + z)^{-1} \cdot (x + y)^\alpha \\ x(yz^{-1} \cdot y^\alpha)^{-1} \cdot x^\alpha &= (xy^{-1} \cdot x^\alpha)(xz^{-1} \cdot x^\alpha)^{-1} \cdot (xy^{-1} \cdot x^\alpha)^\alpha \\ x \cdot y^{-1}z(y^\alpha)^{-1} \cdot x^\alpha &= xy^{-1}x^\alpha x^{-1}z(x^\alpha)^{-1}(xy^{-1}x^\alpha)^\alpha \end{aligned}$$

Replace y^{-1} by y and z by xzx^α in the above equation to obtain

$$x(y \cdot xzx^\alpha \cdot y^\alpha)x^\alpha = xyx^\alpha \cdot z \cdot (xyx^\alpha)^\alpha$$

□

Theorem 3.8. *The core $G(+)$ of an IP generalized Moufang loop $G(\cdot)$ with universal α-elasticity, where α is a homomorphism, is right distributive if and only if the identity*

$$xyx^\alpha \cdot z \cdot (xyx^\alpha)^\alpha = xzx^\alpha \cdot yz^{-1}y^\alpha \cdot (xzx^\alpha)^\alpha. \tag{3.3}$$

holds in $G(\cdot)$.

Proof. From the law of right distributivity,

$$(x + y) + z = (x + z) + (y + z),$$

it follows that,

$$\begin{aligned} (xy^{-1}x^\alpha) + z &= (xz^{-1}x^\alpha) + (yz^{-1}y^\alpha) \\ (xy^{-1}x^\alpha)z^{-1}(xy^{-1}x^\alpha)^\alpha &= (xz^{-1}x^\alpha)(yz^{-1}y^\alpha)^{-1} \cdot (xz^{-1}x^\alpha)^\alpha \\ xy^{-1}x^\alpha \cdot z^{-1} \cdot (xy^{-1}x^\alpha)^\alpha &= xz^{-1}x^\alpha \cdot y^{-1}z(y^\alpha)^{-1} \cdot (xz^{-1}x^\alpha)^\alpha. \end{aligned}$$

On replacing y^{-1} by y, z^{-1} by z and z by yzy^α in the last equation respectively, we obtain

$$xyx^\alpha \cdot z \cdot (xyx^\alpha)^\alpha = xzx^\alpha \cdot yz^{-1}y^\alpha \cdot (xzx^\alpha)^\alpha$$

□

4 A construction of generalised Bol loop

In the following, we give a construction of generalised Bol loop from 5-tuple alternative division ring. The construction follows Robinson's technique of constucting Bol loop in [26]. The generalised Moufang loop mentioned here, Definition 2.10, is different from various versions of generalised Mounfang loops already found in literatures.

Theorem 4.1. *Let $G = R \times R \times R \times R \times R$, where R is a non-associative alternative division ring. Let $x = (a_1, b_1, c_1, d_1, e_1)$, $y = (a_2, b_2, c_2, d_2, e_2)$ and $z = (a_3, b_3, c_3, d_3, e_3)$ are elements of G. Let $\alpha : G \longrightarrow G$ be a projection map defined as*

$$\alpha(a, b, c, d, e) = (0, b, 0, d, 0).$$

α is a projection onto the second and fourth component. Define

$$x \cdot y = (a_1 + a_2, b_1 + b_2, c_1 + c_2 + a_1b_2, d_1 + d_2 + b_1b_2, e_1 + e_2 + a_1d_2 + c_1b_2) \tag{4.1}$$

and

$$x - y = (a_1 - a_2, b_1 - b_2, c_1 - c_2, d_1 - d_2, e_1 - e_2). \tag{4.2}$$

Then, $(G, \cdot)$ is a generalszed Bol loop.

Proof. $(G, \cdot)$ constructed above is non-associative since, $xy \cdot z \neq x \cdot yz$ as shown below;

$$\begin{aligned} xy \cdot z =&(a_1 + a_2 + a_3, b_1 + b_2 + b_3, c_1 + c_2 + c_3 + a_1b_2 + (a_1 + a_3)b_3, d_1 + d_2 + d_3 + b_1b_2 \\ &+ (b_1 + b_2)b_3, e_1 + e_2 + e_3 + a_1d_2 + c_1b_2 + (a_1 + a_2)d_3 + (c_1 + c_2 + a_1b_2)b_3) \end{aligned}$$

and

$$\begin{aligned} x \cdot yz =&(a_1 + a_2 + a_3, b_1 + b_2 + b_3, c_1 + c_2 + c_3 + a_2b_3 + a_1(b_2 + b_3), d_1 + d_2 + d_3 + b_2b_3 \\ &+ b_1(b_2 + b_3), e_1 + e_2 + e_3 + a_2d_3 + c_2b_3 + a_1(d_2 + d_3 + b_2b_3) + c_1(b_2 + b_3)) \end{aligned}$$

It is obvious that $(G, \cdot)$ is a loop with identity $(0, 0, 0, 0, 0)$. Let $x, y, z \in G$ and using the equations (4.1) and (4.2), we obtain

$$\begin{aligned} (xy \cdot z)y^\alpha =&\Big(a_1 + a_2 + a_3, b_1 + b_2 + b_3, c_1 + c_2 + c_3 + a_1b_2 + (a_1 + a_2)b_3 + (a_1 + a_2 + a_3)b_2, \\ &d_1 + d_2 + d_3 + b_1b_2 + (b_1 + b_2)b_3 + (b_1 + b_2 + b_3)b_2, e_1 + e_2 + e_3 + a_1d_2 + c_1b_2 + \\ &(a_1 + a_2)d_3 + (c_1 + c_2 + a_1b_2)b_3 + (a_1 + a_2 + a_3)d_2 + \\ &(c_1 + c_2 + c_3 + a_1b_2 + (a_1 + a_2)b_3)b_2\Big) \end{aligned}$$

and

$$\begin{aligned} x(yz \cdot y^\alpha) =&\Big(a_1 + a_2 + a_3, b_1 + b_2 + b_3, c_1 + c_2 + c_3 + a_2b_3 + (a_2 + a_3)b_2 + a_1(2b_2 + b_3), \\ &d_1 + d_2 + d_3 + b_2b_3 + (b_2 + b_3)b_2 + b_1(2b_2 + b_3), e_1 + e_2 + e_3 + a_2d_3 + c_2b_3 + \\ &(a_2 + a_3d_2) + (c_2 + c_3 + a_2b_3)b_2 + a_1(2d_2 + d_3 + b_2b_3 + (b_2 + b_3)b_2) + c_1(2b_2 + b_3)\Big) \end{aligned}$$

Thus, $(xy \cdot z)y^\alpha - x(yz \cdot y^\alpha) = (0, 0, 0, 0, a_1b_2 \cdot b_3 + a_1b_3 \cdot b_2 - a_1 \cdot b_2b_3 - a_1 \cdot b_3b_2)$. The right alternative property of R gives

$$a_1(b_2 + b_3)^2 = [a_1(b_2 + b_3)]\,(b_2 + b_3).$$

Using linearization technique, we obtain

$$a_1b_2 \cdot b_3 + a_1b_3 \cdot b_2 - a_1 \cdot b_2b_3 - a_1 \cdot b_3b_2 = 0.$$

Thus, $(xy \cdot z)y^\alpha = x(yz \cdot y^\alpha)$ and $(G, \cdot)$ is a generalized Bol loop. □

Lemma 4.2. *Let $(G, \cdot)$ be a generalised Bol loop. G is α-flexible if and only if G is generalised Moufang.*

Proof. Since G is generalised Bol loop, it satisfies $(xy \cdot z)y^\alpha = x(yz \cdot y^\alpha)$. Using the α-flexible property in the generalised Bol identity gives $(xy \cdot z)y^\alpha = x(y \cdot zy^\alpha)$ which is the generalised Moufang identity.
Conversely, let G be generalised Moufang loop. On setting $x = 1$ in Definition 2.10, the result follows. □

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Remark 4.3. *The generalised Moufang identity in lemma 4.2 emanated from non proper right Bol loops [27] (that is, right Bol loops that are Moufang). It is obvious that when α is an identity map in the two lemmas, we return to Bol loop and Moufang loop context accordingly.*

Theorem 4.4. *The generalised Bol loop $(G,\cdot)$ constructed above is not generalised Moufang.*

Proof. We shall endeavour to show the proof in two cases:

CASE 1; If the generalized Bol loop $(G,\cdot)$ constructed emanated from proper right Bol loops (right Bol loops that are not Moufang), then $(G,\cdot)$ is not generalized Moufang as the generalized Bol identity $(xy\cdot z)y^\alpha = x(yz\cdot y^\alpha)$ yields a tautology on setting $x=1$.

CASE 2; If the generalized Bol loop $(G,\cdot)$ constructed emanated from non proper right Bol loops, we give the following computation to show that $(G,\cdot)$ is not flexible (in fact α-flexible will be appropriate in this context where α is the projection map in our construction) which implies that it is not generalized Moufang.

$$\begin{aligned} yz =& \Big(a_2+a_3, b_2+b_3, c_2+c_3+a_2b_3, d_2+d_3+b_2b_3, e_2+e_3+a_2d_3+c_2b_3\Big)\\ (yz)y^\alpha =& \Big(a_2+a_3, 2b_2+b_3, c_2+c_3+a_2b_3, 2d_2+d_3+b_2b_3+b_2(b_2+b_3),\\ & e_2+e_3+a_2d_3+c_2b_3+(a_2+a_3)d_2+(c_2+c_3+a_2b_3)b_2\Big) \end{aligned}$$

Also,

$$\begin{aligned} zy^\alpha =& \Big(a_3, b_2+b_3, c_3+a_3b_2, d_3+d_2+b_3b_2, e_3+a_3d_2+c_3b_2\Big)\\ y(zy^\alpha) =& \Big(a_2+a_3, 2b_2+b_3, c_2+c_3+a_2b_3+a_2(b_2+b_3), 2d_2+d_3+b_2b_3+\\ & b_2(b_2+b_3), e_2+e_3+a_3d_2+c_3b_2+a_2(d_2+d_3+b_2b_3)+c_2(b_2+b_3)\Big) \end{aligned}$$

Thus, $(yz)y^\alpha \neq y(zy^\alpha)$ and $(G,\cdot)$ is therefore not generalized Moufang based on Lemma 4.2

□

We compute the core of the generalised Bol loops constructed as follows:
Recall that the core of generalised Bol loops is given in Definition 2.11 as $x+y = xy^{-1}\cdot x^\alpha$. This can be re-written using the right translation map as $xR_{y^{-1}}\cdot x^\alpha$.
Now let $x,y\in G$ defined above, we can let $x=(a_1,b_1,c_1,d_1,e_1)$, $y=(a_2,b_2,c_2,d_2,e_2)$. Thus,

$$xR_{y^{-1}} = (a_1-a_2, b_1-b_2, c_1-(c_2+a_3b_2), d_1-(d_2+b_3b_2), e_1-(e_2+a_2d_3+c_3b_2))$$

and therefore,

$$xR_{y^{-1}}\cdot x^\alpha = (a_1-a_2, 2b_1-b_2, c_1-(c_2+a_3b_2), 2d_1-(d_2+b_3b_2), e_1-(e_2+a_3d_2+c_3b_2))$$

5 Conclusion

This study is concluded with some research directions based on the contents of this study and references cited therein.

1. The results of this study suggest that the core of generalised Bol loops could be studied as a rack and if the core is assumed to be right cancellative, we have an involutory quandle as a consequence of left distributive law. Thus, the core could be studied as a symmetric space in the sense of Ottmar Loos [28]. We therefore invite experts in symmetric spaces to consider the core of generalised Bol loops perhaps further feature(s) of the self map could be revealed. It will also be interesting to know what the core of generalised Bol loops could be used for under the new feature for the self map.

2. A study of the half-homomorphism [29] of generalised Bol loop with a particular attention to the self map, α might also be interesting in its own right.

3. The core of Moufang loop is known to be left distributive groupoid (see [30]) and in [30], it has been proved that commutative IP loop with universal elasticity is a commutative Moufang loop if and only if its core is a left distributive quasigroup. Based on this fact we propose the following If $G(\cdot)$ is a commutative IP generalised Bol loop with universal elasticity for which the mapping $x \mapsto xx^{\alpha}$ is a bijection, then $G(\cdot)$ is a commutative M_{α}-loop if and only if its core $G(+)$ is a left distributive quasigroup.

Competing Interests

The authors declare that they have no competing interests.

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