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Diphtheria Disease Transmission Dynamics In Low Vaccine Coverage Setting

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Abstract

Diphtheria is a severely infectious respiratory disease which transmits through droplets and preventable by periodic vaccine programs. In this paper, A six (6) Compartmental model (S, E, I_A, I_S, Q, R) is presented to undersee the behaviour of diphtheria disease transmission within a group of people with low or zero vaccine coverage and immunity gaps. This research explores epidemiology and mathematically well-posed model. The reproduction number was analysed using the Next Generation Matrix, we underscored that a single infected individual can trigger an outbreak, and further investigation indicates that the disease will subside if the reproduction number (R_0) is less than 1, and vice versa if R_0 exceeds 1. The model captures disease mitigating strategies like maternally derived immunity, vaccination, Quarantine, and asymptomatic carriers to assess how contagious the disease is and what interventions might be most effective. To validate theoretical model predictions, we conducted numerical simulations using MATLAB 2021a software. Relevant and informative model Simulations are displayed in the full text.

Keywords: Gronwalls-Bellman inequality, Integrating factor, Routh-Hurwitz, Gaussian Elimination.
MSC2010: 00A71.

1 Introduction

Diphtheria is a deadly infectious respiratory disease caused by *Corynebacterium diphtheriae* [1], the disease is highly contagious, it transmits through coughing, sneezing, or close contact with infected fluid and can be fatal especially in children [2]. Diphtheria symptoms include breathing problems, swallowing difficulty, and potential paralysis. Despite the availability of vaccines, Diphtheria remains a global health concern. Vaccination with DTaP is the most effective way to prevent diphtheria [3], Interestingly, this vaccine combination also protects against pertussis (whooping

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cough) and tetanus [4]]. According to the study of [5], [6], [11] resurgence of vaccine-preventable infectious diseases continues to pose formidable challenges, particularly due to disruptions in basic immunization programs induced by the recent COVID-19 Pandemic. For instance, WHO and UNICEF reported that millions of children missed out of essential vaccines-preventable diseases including Diphtheria.

Recently, The Nigerian Ministry of Health declared a diphtheria outbreak in January 2023 [7], Kano and Lagos were the most affected states with several cases of zero vaccinated children. As of January 14, 2024, the World Health Organization (WHO) reports concern in the surge of suspected and fatality cases of diphtheria across African countries, with Nigeria bearing the brunt of outbreak [8]. Mathematical models offer valuable insights for analysing diphtheria transmission, evaluating control strategies, and predicting outbreaks. This review explores the diverse mathematical frameworks employed to study diphtheria transmission dynamics and its likes.

For instance, [9], [10], [11], [12] focused on diphtheria transmission in Indonesia, [9] proposed a five (5) compartmental model that captures natural immunity alongside low vaccination coverage as a major concern. Stability analysis of the model was done, results show that reducing the basic reproduction number R_0 to less than 1 via high vaccination and natural immunity is crucial to mitigating outbreaks. [10] proposed an optimal control for diphtheria outbreaks using the Pontryagin Minimum Principle and numerical methods on SEIQR (Susceptible-Exposed-Infected-Quarantined-Recovered) model of [9]. The optimal control strategy was essential in determining the most effective intervention combination for minimizing both the outbreak size and associated costs. Whereas, [11] linked post COVID-19 disruptions to increased diphtheria cases in West Java, Indonesia. Exploring the SIR (Susceptible-Infected-Recovered) model to estimate R_0 for diphtheria, concluded that while spatial analysis reveals hotspots and case cluster diffusion patterns, these findings can inform prevention and intervention strategies. Also, [12] assumed natural recovery for diphtheria with a simplified SIR model analysis. The model focuses on endemic and non-endemic conditions using the Basic Reproduction Number R_0, and Proposed an open discussion for model expansion with more complex parameters for a better and realistic representation of the disease. Similarly, [13] proposed an SEIQR model with quarantine controller to examine the impact of isolation measures, underscored that most diphtheria infections are asymptomatic or having a relatively slight clinical course. In same vein to examine the impact of quarantine on diphtheria disease, [14] developed A 5-group model SEQIR explored the effect of quarantining on exposed individuals. Results suggests prioritizing quarantine of exposed individuals is key in mitigating disease transmission.

The study of [2], [15] referred to the asymptomatic as silent infectious reservoir. For example, [2] explored imperfect vaccination coverage and its impact on transmission dynamics using a SVEAIR (Susceptible-Vaccinated - Exposed- Asymptomatic Infected - Symptomatic Infected-Recovered) diphtheria model. The study shows that the disease is eradicable with sufficient vaccination and highlighted that asymptomatic infection influence control strategies. However, [15] highlighted the two types of asymptomatic infectious carriers as Genetic carriers and infectious disease carriers. Focused on disease infectious carriers using the SI_cIR (Susceptible-Carrier-Symptomatically infectious-Removed) model. Incorporating carriers that is contagious but asymptomatic into the analysis of infectious diseases. Emphasized the model's possibility to Evaluate carriers transmission impact through simulations and the basic reproduction number.

Furthermore, [16] built a mathematical model for diphtheria based on disease epidemiology. The total population (N) at any time t was subdivided into eight (8) sub-population; Susceptible (S), Exposed (E), Symptomatic infected (I), Asymptomatic infected (A), Vaccinated individuals with complete childhood immunization (V_1), Vaccinated individuals with booster vaccine (V_2), vaccines/ Recovery Wanes (W) and Recovered humans (R). The model analyses illustrate how booster shots and a contaminated environment affects diphtheria spread. In order for optimal control of diphtheria outbreaks, [16] underlines the significant role of booster shots alongside environmental

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sanitation and treatment strategies. [17] presented a six compartmental model to examine the dynamics in COVID-19 transmission. The impact of quarantine disease controller on the symptomatic infected, asymptomatic infected and Reproduction number R_0 within a given population was studied. Normalized forward sensitivity was done on quarantine parameters to check its tangibility on the reproduction number. Results shows that reducing reproduction number will significantly reduce or eradicate disease spread. [18] built on the SEIR-type model with Treatment (T) to mitigate pertussis (whooping cough) resurgence in the post-covid-19 era. In efforts to address Tuberculosis (TB), [19] and [20] have undertaken measures aimed at reducing the burden of the disease. [19] extended the SEIR model to $S_1S_2EIR_HR$ which captured two different susceptible classes and drug resistance R_H to the first line of tuberculosis mitigation. The study significantly highlighted the influence of age groups on TB control. Also, [20] proposed a SIQRM model to assess the impact of immunity sponsored by vaccines or treatment, quarantine effectiveness, and the waning effect of immunity within a population subjected to proper disease education without restrictions.

2 Model Formulation

In this study, a Diphtheria mitigating model is formulated base on disease epidemiology, poor vaccination facility and self quarantining due to social status. in a defined population. Therefore, total population is represented by N and sub-divided into six compartments: Susceptible compartment S, Exposed compartment E, Asymptomatic infected compartment I_A , Symptomatic infected compartment I_S, Recovered compartment R and Quarantine compartment Q, respectively, therefore the Total population at time t is given by

$$N(t) = S(t) + E(t) + I_A(t) + I_S(t) + R(t) + Q(t) \tag{2.1}$$

We assume that the mixing pattern of the population is homogeneous, which infers that everybody in the population can contact disease due to diphtheria vaccines recommendations for all ages. We also assume that Individuals who do not get vaccinated flows into the Susceptible compartment (S) at rate $\kappa = bN(1-V)$, while those who are vaccinated are immune against disease and flows into the recovered compartment (R) at rate VbN respectively. A susceptible individual who have close contact with an infected individual is exposed but might not be able to transmit disease yet. The formulated model considers strong immunity of exposed individuals which is accorded to Maternally Derived Immunity MDI at rate ξ While the exposed individuals with weak MDI could get infected. In other words, Exposed individuals with maternally derived immunity are likely not to be infected by diphtheria and are assumed to flow back into the susceptible compartment. It is assumed that the potency of MDI is due to periodic vaccination during pregnancy The reservoir of diphtheria carriers is usually the Asymptomatic infected class [2]. Therefore, Susceptible population increases by influx of zero vaccinated individuals, MDI and decreases with infection force and natural death rate µ. where the infection force is represented by

$$\Phi = \frac{\beta S(\alpha I_A + I_S)}{N} \tag{2.2}$$

Here β and α represents the interaction of I_A and I_S with the susceptible compartment. Therefore, the Susceptible compartment can be mathematically written as

$$\frac{dS}{dt} = \kappa + E\xi - \frac{\beta S(\alpha I_A + I_S)}{N} - \mu S \tag{2.3}$$

The exposed compartment increases by the infection force Φ, due to symptomatic and asymptomatic interaction with susceptible individuals. The compartment decreases by diphtheria incubation parameter ϕ, maternal derived immunity ξ and natural death rate μ respectively.

$$\frac{dE}{dt} = \frac{\beta S(\alpha I_A + I_S)}{N} - (\xi + \phi + \mu)E \tag{2.4}$$

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After disease incubation period, it is assumed that some individuals who does not show clinical symptoms of disease but are able to transfer disease flows into the asymptomatic class with a proportion $(1-\rho)\phi$. The Asymptomatic infected class decrease by natural recovery rate η due to MDI against diphtheria, natural death rate μ. or decreases by yielding self for quarantined ψ.

$$\frac{dI_A}{dt} = (1-\rho)\phi E - (\eta + \psi + \mu)I_A \tag{2.5}$$

We also assume that the disease-induced death rate only occurs in symptomatic infected individuals with rate ϖ. The Symptomatic Infected compartment increases by a proportion of individuals who shows clinical symptoms of diphtheria with $\rho\phi$ after incubation in the exposed compartment. While, the symptomatic infected compartment decreases by MDI with rate γ, Quarantined and treated rate θ natural death μ and the diphtheria induced death rate ϖ.

$$\frac{dI_S}{dt} = \rho\phi E - (\gamma + \theta + \varpi + \mu)I_S \tag{2.6}$$

Quarantine/Isolation compartment increases by individuals from the symptomatic and asymptomatic compartment for treatments and immune boosting at the rates θ and ψ respectively. The Quarantined class decreases by cure rate of diphtheria Disease σ, natural mortality death μ. Also, it is assumed that some infected individuals in this compartment might decide to quarantine self due to social status or other conditions. They recover at rate Ω.

$$\frac{dQ}{dt} = \psi I_A + \theta I_S - (\sigma + \varpi + \mu)Q \tag{2.7}$$

The recovered compartment increases by individuals who have received vaccination from birth VbN, also the compartment increases by recovery of Symptomatic and asymptomatic compartment due to MDI with rates η and γ respectively. Recovery compartment increases by cure rate of individuals after quarantine process and recovery by self isolation quarantine due to certain social status conditions or others at rate σ and Ω. The recovered compartment decreases by natural death rate μ.

$$\frac{dR}{dt} = VbN + \eta I_A + \gamma I_S + (\sigma + \Omega)Q - \mu R \tag{2.8}$$

2.1 Other Model Assumptions

The following assumptions is also considered:

1. Diphtheria has constant transmission rates over time.
2. No immunity waning after vaccination, the duration for immunity is fixed.
3. The death rate in the symptomatic class is not only due to natural death but also, as a result of the disease infection.
4. The population structure or specific age bracket is not considered.
5. all parameter used are assumed to be non-negative

Based on the model description and assumptions above, the model of the diphtheria infection

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transmission dynamics is giving by the systems of non-linear differential equations. Equations (1-8).

$$\begin{aligned}
\frac{dS}{dt} &= bN(1-V) + \xi E - \frac{\beta(\alpha I_A + I_S)}{N} S - \mu S \\
\frac{dE}{dt} &= \frac{\beta(\alpha I_A + I_S)}{N} S - (\xi + \phi + \mu) E \\
\frac{dI_A}{dt} &= (1-\rho)\phi E - (\eta + \psi + \mu) I_A \\
\frac{dI_S}{dt} &= \rho\phi E - (\gamma + \theta + \varpi + \mu) I_S \\
\frac{dQ}{dt} &= \psi I_A + \theta I_S - (\sigma + \Omega + \mu) Q \\
\frac{dR}{dt} &= VbN + \eta I_A + \gamma I_S + (\sigma + \Omega) Q - \mu R
\end{aligned} \tag{2.9}$$

$S(0) > 0, E(0) > 0, I_A(0) > 0, I_S(0) > 0, R(0) > 0$, and $Q(0) > 0$

The Dynamic flow of the model is presented diagrammatically in Fig.1. Below

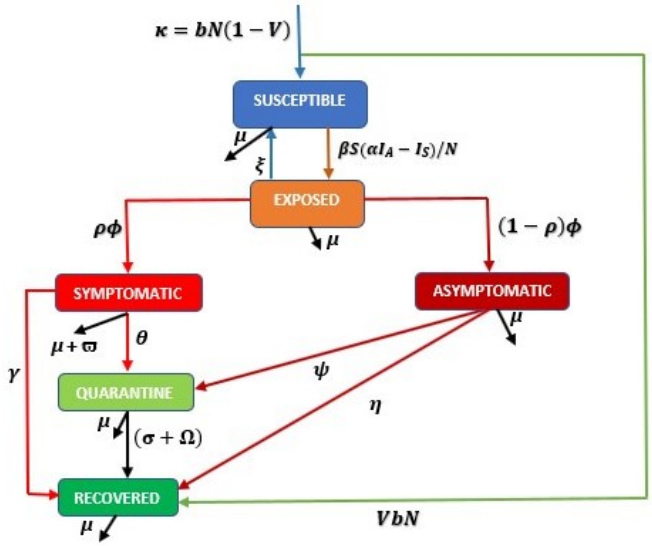

Fig.1 Schematic Diagram of Diphtheria-Quarantine Model
And the minor parameter description of model is represented in the Tab. 2. below

3 Model Analysis

3.1 *Existence and Uniqueness of Solution*

For this diphtheria transmission model, to be mathematically and epidemiologically well pose, the dynamic system must be non-negative and bounded ∀ t .We will show the boundedness and non-negativity of the system of equation (9) in the subsequent lemmas.

Lemma 1 (non-negativity). If the initial value conditions for $S(0) > 0, E(0) > 0, I_A(0) > 0, I_S(0) > 0, R(0) > 0, Q(0) > 0$ and $t(0) > 0 \forall t \epsilon [0, t_0]$ then, $S(t), E(t), I_A(t), I_S(t), R(t), Q(t)$ stays positive through out in $\Re^6_+$

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Table 1: *Parameters Description of Diphtheria-Quarantine Model*

S/N	Parameter	Description
1	β	Interaction rate of I_A with S.
2	α	Interaction rate of I_S with S.
3	ρ	Proportion of symptomatic infected population.
4	ϕ	Incubation period of diphtheria.
5	Ω	Self Quarantine
6	γ	Recovery rate of symptomatically infected population.
7	η	Recovery rate of asymptomatically infected population.
8	μ	Natural death rate.
9	κ	In flux of zero vaccinated population.
10	ϖ	Diphtheria-induced death rate.
11	σ	Cure rate of Quarantine.
12	ψ	Asymptomatic Quarantine rate.
13	θ	Symptomatic Quarantine rate.
14	V	Vaccinated.
15	b	Birth rate.

Proof: Since we are dealing with human populations, it is assumed that all the parameter used are positive. using the first equation of equation (9) that is the Susceptible compartment, we have

$$\frac{dS}{dt} = \kappa - \Phi - \mu S = -(\Phi + \mu)S$$

separating variables and applying the integrating factor method on equation (9), we obtain

$$\int_0^S \frac{dS}{S} = -\int_0^t (\Phi + \mu)dt$$
$$\ln|S(t)| > -(\Phi + \mu)S(t) + C$$
$$S(t) > Ce^{-(\Phi+\mu)S(t)}$$

Hence, $S(t) \geq 0\ \forall t > 0$ Similarly, we can show that $E(0) > 0, I_A(0) > 0, I_S(0) > 0, R(0) > 0, Q(0) > 0$.

Lemma 2 (Boundedness). The closed set B^* given by $B^* = (S, E, I_A, I_S, R, Q)\epsilon\Re_+^6 : S + E + I_A + I_S + R + Q \leq \frac{\kappa}{\mu}$ is positively invariant with respect to the dynamic system model equation (9).

Proof adding equation (9) that is the rate of change in the total population of equation (1) gives,

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dE}{dt} + \frac{dI_A}{dt} + \frac{dI_S}{dt} + \frac{dQ}{dt} + \frac{dR}{dt} \tag{3.1}$$

Adding Equation (10) and equating it to zero at time t=0 gives,

$$\frac{dN}{dt} = \kappa - \mu(S + E + I_A + I_S + Q + R) - I_S\varpi \tag{3.2}$$

Putting The Equations (11) into (10) gives

$$\frac{dN}{dt} \leq \kappa - \mu N(t) - I_S\varpi \tag{3.3}$$

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At the initial instance of outbreak, $I_S\varpi = 0$, it was assumed that there was no diphtheria disease induced death , solving and taking the like terms.

$$\frac{dN}{dt} + \mu N(t) \leq \kappa \tag{3.4}$$

by applying the integrating factor method to equation (13) and using Gronwalls-Bellman inequality, then simplifying we obtain

$$N(t) \leq \frac{\kappa}{\mu} + (N(0) - \frac{\kappa}{\mu})e^{-\mu t}, \Rightarrow N(t) \leq \frac{\kappa}{\mu}$$

as $t \to \infty$ therefore, the system equation (8) has the solution in B^*. thus the system is positively invariant.

Lemma 3 (Existence and uniqueness) From the Initial values of the Equation (9), which is $S(0) > 0,\ E(0) > 0\ I_A(0) > 0,\ I_S(0) > 0\ Q(0) > 0,\ R(0) > 0$ and $t_0 > 0$, then $t \in \Re$ the solutions $S(t)$, $E(t)$, $I_A(t)$, $I_S(t)$, $Q(t)$, $R(t)$ (t).$\exists$ in $\Re^6_+$.

Proof provided we can express model equation (8) in the form $\dot{x} = f(x)$ where,

$$\dot{x} = \begin{bmatrix} S \\ E \\ I_A \\ I_S \\ Q \\ R \end{bmatrix}, \quad f(x) = \begin{bmatrix} \kappa + \xi E - \frac{\beta S(\alpha I_A + I_S)}{N} S - \mu S \\ \frac{\beta S(\alpha I_A + I_S)}{N} S - (\xi + \phi + \mu)E \\ (1-\rho)\phi E - (\eta + \psi + \mu)I_A \\ \rho\phi E - (\gamma + \theta + \varpi + \mu)I_S \\ \psi I_A + \theta I_S - (\sigma + \Omega + \mu)Q \\ \eta I_A + \gamma I_S + (\sigma + \Omega)Q - \mu R \end{bmatrix} \tag{3.5}$$

then by the fundamental existence and uniqueness theorem, Equation (14) is proved, since f has a continuous first derivative in $\mathbb{R}^6_+$ then it is locally Lipschitz, therefore ,$\jmath$ a unique, positive and bounded solution for the system of differential Equation (9) in $\mathbb{R}^6_+$.

3.2 *Equilibrium Point Analysis*

We consider two equilibrium points which are the Disease-Free Equilibrium Point and Endemic Equilibrium points respectively. To generate the disease-free equilibrium point, all compartments in Equation (9) are equated to zero.

$$N_0 = (S_0 + E_0 + I_{A0} + I_{S0} + Q_0 + R_0) = \left(\frac{bN(1-V)}{\mu}, 0, 0, 0, 0, \frac{bNV}{\mu}\right) \tag{3.6}$$

The second steady state is the Endemic Equilibrium Point (EEP).

$$N^* = (S^* + E^* + I_A^* + I_S^* + Q^* + R^*)$$

Denoted as,

$$S^* = \frac{bN(1-V)}{\beta(\alpha I_A^* + I_S^*) + \mu},\ E^* = \frac{\beta S(\alpha I_A^* + I_S^*)}{(\xi+\phi+\mu)},\ I_A^* = \frac{(1-\rho)\phi E^*}{(\eta+\psi+\mu)},$$

$$I_S^* = \frac{\phi\rho E}{(\gamma+\theta+\varpi+\mu)},\ Q^* = \frac{\psi I_S^* + \theta I_A^*}{(\sigma+\varpi+\mu)}, R^* = \frac{VbN + \eta I_A^* + \gamma I_S^* + (\sigma+\Omega)Q^*}{\mu}$$

In the course of this study, Other endemic points will be shown in scenarios.

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3.3 Next Generation Matrix

The Basic reproduction number denoted as R_0 which is the determining parameter to whether diphtheria disease will spread or not was calculated using the Next Generation Matrix (NGM) giving by $R_0 = FV^{-1}$. Which gives us the necessary conditions for the stability of the system. Equation (16) is diphtheria transmission compartments i.e., the disease classes.

$$\begin{aligned}
\frac{dE}{dt} &= \frac{\beta(\alpha I_A + I_S)}{N} S - (\xi + \phi + \mu)E \\
\frac{dI_A}{dt} &= (1-\rho)\phi E - (\eta + \psi + \mu) I_A \\
\frac{dI_S}{dt} &= \rho\phi E - (\gamma + \theta + \varpi + \mu) I_S \\
\frac{dQ}{dt} &= \psi I_A + \theta I_S - (\sigma + \Omega + \mu) Q
\end{aligned} \tag{3.7}$$

Now, let $a = (E, I_A, I_S, Q)$ denote the initial state variable of model then, Equation (9) for the infective class becomes

$$\frac{da}{dt} = F_i(a) - V_i(a) \tag{3.8}$$

where $F_i(a)$ is the rate of new infection and $V_i(a)$ is the in and out rate of diphtheria disease infection transfer.

$$\begin{aligned}
F_i(a) &= \begin{bmatrix} \frac{\beta(\alpha I_A + I_S)}{N} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\
V_i(a) &= \begin{bmatrix} -(\xi + \phi + \mu)E \\ (1-\rho)\phi E - (\eta + \psi + \mu) I_A \\ \rho\phi E - (\gamma + \theta + \varpi + \mu) I_S \\ \psi I_A + \theta I_S - (\sigma + \Omega + \mu) Q \end{bmatrix}
\end{aligned} \tag{3.9}$$

Where The transition and transmission matrix are denoted by $F_i(a)$ and $V_i(a)$ at N_0 Respectively. Now let

$$k_1 = -(\xi + \phi + \mu), k_2 = -(\eta + \psi + \mu),\ k_3 = -(\gamma + \theta + \varpi + \mu), k_4 = (1-\rho)\phi, k_5 = -(\sigma + \Omega + \mu), k_6 = (\sigma + \Omega) \tag{3.10}$$

taking the Jacobian matrix at (DFE),

$$F_i = \begin{bmatrix} \frac{\partial F_1}{\partial E} & \frac{\partial F_1}{\partial I_A} & \frac{\partial F_1}{\partial I_S} & \frac{\partial F_1}{\partial Q} \\ \frac{\partial F_2}{\partial E} & \frac{\partial F_2}{\partial I_A} & \frac{\partial F_2}{\partial I_S} & \frac{\partial F_2}{\partial Q} \\ \frac{\partial F_3}{\partial E} & \frac{\partial F_3}{\partial I_A} & \frac{\partial F_3}{\partial I_S} & \frac{\partial F_3}{\partial Q} \\ \frac{\partial F_4}{\partial E} & \frac{\partial F_4}{\partial I_A} & \frac{\partial F_4}{\partial I_S} & \frac{\partial F_4}{\partial Q} \end{bmatrix}, D_i = \begin{bmatrix} \frac{\partial V_1}{\partial E} & \frac{\partial V_1}{\partial I_A} & \frac{\partial V_1}{\partial I_S} & \frac{\partial V_1}{\partial Q} \\ \frac{\partial V_2}{\partial E} & \frac{\partial V_2}{\partial I_A} & \frac{\partial V_2}{\partial I_S} & \frac{\partial V_2}{\partial Q} \\ \frac{\partial V_3}{\partial E} & \frac{\partial V_3}{\partial I_A} & \frac{\partial V_3}{\partial I_S} & \frac{\partial V_3}{\partial Q} \\ \frac{\partial V_4}{\partial E} & \frac{\partial V_4}{\partial I_A} & \frac{\partial V_4}{\partial I_S} & \frac{\partial V_4}{\partial Q} \end{bmatrix} \tag{3.11}$$

then Equation (20) becomes,

$$DF_i(a) = F = \begin{bmatrix} 0 & \beta\alpha S & \beta S & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, DV_i(a) = V = \begin{bmatrix} k_1 & 0 & 0 & 0 \\ -k_4 & k_2 & 0 & 0 \\ -\phi\rho & 0 & k_3 & 0 \\ 0 & -\psi & -\theta & k_5 \end{bmatrix} \tag{3.12}$$

Calculating V inverse matrix via Gaussian Elimination method, the augmented matrix becomes

$$V^{-1} = \left[\begin{array}{cccc|cccc} k_1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -k_4 & k_2 & 0 & 0 & 0 & 1 & 0 & 0 \\ -\phi\rho & 0 & k_3 & 0 & 0 & 0 & 1 & 0 \\ 0 & -\psi & -\theta & k_5 & 0 & 0 & 0 & 1 \end{array}\right]$$

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$$V^{-1} = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{k_1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{k_4}{k_1 k_2} & \frac{1}{k_2} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{\phi\rho}{k_1 k_3} & 0 & \frac{1}{k_3} & 0 \\ 0 & 0 & 0 & 1 & \frac{\psi m_4 + \theta\phi\rho}{k_1 k_3 k_5} & \frac{\psi}{k_2 k_5} & + \frac{\theta}{k_3 k_5} & \frac{1}{k_5} \end{array}\right]$$

We obtain,

$$V^{-1} = \begin{bmatrix} \frac{1}{k_1} & 0 & 0 & 0 \\ \frac{k_4}{k_1 k_2} & \frac{1}{k_2} & 0 & 0 \\ \frac{\phi\rho}{k_1 k_3} & 0 & \frac{1}{k_3} & 0 \\ \frac{\psi k_4 + \theta\phi\rho}{k_1 k_3 k_5} & \frac{\psi}{k_2 k_5} & \frac{\theta}{k_3 k_5} & \frac{1}{k_5} \end{bmatrix} \tag{3.13}$$

Finding FV^{-1} becomes,

$$FV^{-1} = \begin{bmatrix} 0 & \beta\alpha S & \beta S & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} \frac{1}{k_1} & 0 & 0 & 0 \\ \frac{k_4}{k_1 k_2} & \frac{1}{k_2} & 0 & 0 \\ \frac{\phi\rho}{k_1 k_3} & 0 & \frac{1}{k_3} & 0 \\ \frac{\psi k_4 + \theta\phi\rho}{k_1 k_3 k_5} & \frac{\psi}{k_2 k_5} & \frac{\theta}{k_3 k_5} & \frac{1}{k_5} \end{bmatrix} \tag{3.14}$$

The Reproduction number is the largest first eigenvalue,

$$FV^{-1} = \begin{bmatrix} \frac{\beta\alpha S k_4}{k_1 k_2} + \frac{\beta\phi\rho S}{k_1 k_3} & \frac{\beta\alpha S}{k_2} & \frac{\beta S}{k_3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.15}$$

The eigenvalues λ of FV^{-1} can be derived using characteristic Equation (25).

$$\left|FV^{-1} - \lambda I\right| = 0 \tag{3.16}$$

$$FV^{-1} = \begin{bmatrix} R_0 - \lambda & \frac{\beta\alpha S}{k_2} & \frac{\beta S}{k_3} & 0 \\ 0 & 0 - \lambda & 0 & 0 \\ 0 & 0 & 0 - \lambda & 0 \\ 0 & 0 & 0 & 0 - \lambda \end{bmatrix}$$

$$R_0 = \frac{\beta b(1-v)k_4}{\mu k_1 k_2} + \frac{\beta b(1-v)\phi\rho}{\mu k_1 k_3} \tag{3.17}$$

Equation (27) represents the initial reproduction number at DFE. That is $R_0 \times S_0$ we know that $S_0 = \frac{\kappa}{\mu} = \frac{b(1-v)}{\mu}$ then

$$R_0 = \frac{\beta b(1-v)\alpha(1-\rho)}{\mu(\xi+\phi+\mu)(\eta+\psi+\mu)} + \frac{\beta b(1-v)\phi\rho}{\mu(\xi+\phi+\mu)(\gamma+\theta+\varpi+\mu)} \tag{3.18}$$

Therefore, the Equation (26) is the reproduction number and can be manually calculated, the two terms of the reproduction number with zero vaccination can be represented by

$$R_0 = R_{asymptomatic} + R_{symptomatic} \tag{3.19}$$

The formula for R_0 with substituted values is:

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$$R_0 = \frac{(0.57)(0.019)(1-0)(0.3)(1-0.55)}{(0.006)(0.23+0.2+0.006)(0.1+0+0.006)} + \frac{(0.57)(0.019)(1-0)(0.2)(0.55)}{(0.006)(0.23+0.2+0.006)(0+0.1+0.05+0.05)} \tag{3.20}$$

Calculating the R_0 terms:
First Term $=R_1 = R_{asymptomatic}$

$$\begin{aligned} \text{First Term} &= \frac{(0.57)(0.019)(0.3)(0.45)}{(0.006)(0.436)(0.106)} \\ &= \frac{0.0032565}{0.00260416} \\ &\approx 1.25 \end{aligned}$$

Second Term $= R_2$=$R_{symptomatic}$

$$\begin{aligned} \text{Second Term} &= \frac{(0.57)(0.019)(0.2)(0.55)}{(0.006)(0.436)(0.256)} \\ &= \frac{0.0020211}{0.0053056} \\ &\approx 0.3811 \end{aligned}$$

After computation, we get:

$$R_0 \approx 1.6311$$

4 Stability Analysis

The stability analysis of the disease free and the endemic equilibrium points was carried out in this section.

4.1 *Local Stability Analysis of DFEP and EEP*

Theorem

Theorem 4.1. *The diseases free equilibrium point is locally asymptomatic stable if $R_0 < 1$ and unstable if $R_0 > 1$,*

Proof: Jacobian matrix of the dynamic model (9) was estimated at the disease-free equilibrium which yields

$$J_{dfe} = \begin{bmatrix} \frac{\partial F_1}{\partial S} & \frac{\partial F_1}{\partial E} & \frac{\partial F_1}{\partial I_A} & \frac{\partial F_1}{\partial I_S} & \frac{\partial F_1}{\partial Q} & \frac{\partial F_1}{\partial R} \\ \frac{\partial F_2}{\partial S} & \frac{\partial F_2}{\partial E} & \frac{\partial F_2}{\partial I_A} & \frac{\partial F_2}{\partial I_S} & \frac{\partial F_2}{\partial Q} & \frac{\partial F_2}{\partial R} \\ \frac{\partial F_3}{\partial S} & \frac{\partial F_3}{\partial E} & \frac{\partial F_3}{\partial I_A} & \frac{\partial F_3}{\partial I_S} & \frac{\partial F_3}{\partial Q} & \frac{\partial F_3}{\partial R} \\ \frac{\partial F_4}{\partial S} & \frac{\partial F_4}{\partial E} & \frac{\partial F_4}{\partial I_A} & \frac{\partial F_4}{\partial I_S} & \frac{\partial F_4}{\partial Q} & \frac{\partial F_4}{\partial R} \\ \frac{\partial F_5}{\partial S} & \frac{\partial F_5}{\partial E} & \frac{\partial F_5}{\partial I_A} & \frac{\partial F_5}{\partial I_S} & \frac{\partial F_5}{\partial Q} & \frac{\partial F_5}{\partial R} \\ \frac{\partial F_6}{\partial S} & \frac{\partial F_6}{\partial E} & \frac{\partial F_6}{\partial I_A} & \frac{\partial F_6}{\partial I_S} & \frac{\partial F_6}{\partial Q} & \frac{\partial F_6}{\partial R} \end{bmatrix} = \begin{bmatrix} -\mu & \xi & -\beta\alpha & -\beta & 0 & 0 \\ 0 & k_1 & \beta\alpha & \beta & 0 & 0 \\ 0 & k_4 & k_2 & 0 & 0 & 0 \\ 0 & \phi\rho & 0 & k_3 & 0 & 0 \\ 0 & 0 & \psi & \theta & k_5 & 0 \\ 0 & 0 & \eta & \gamma & k_6 & -\mu \end{bmatrix} \tag{4.1}$$

Then characteristic $|J_{dfe} - \lambda I| = 0$ is expanded and determined as,

$$J_{dfe} = \begin{bmatrix} -\mu-\lambda_1 & \xi & -\beta\alpha & -\beta & 0 & 0 \\ 0 & k_1-\lambda_2 & \beta\alpha & \beta & 0 & 0 \\ 0 & k_4 & k_2-\lambda_3 & 0 & 0 & 0 \\ 0 & \phi\rho & 0 & k_3-\lambda_4 & 0 & 0 \\ 0 & 0 & \psi & \theta & k_5-\lambda_5 & 0 \\ 0 & 0 & \eta & \gamma & k_6 & -\mu-\lambda_6 \end{bmatrix} \tag{4.2}$$

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A characteristic polynomial is obtained,

$$(-\mu-\lambda_1)(k_1-\lambda_2)(k_2-\lambda_3)(k_3-\lambda_4)(k_5-\lambda_5)(-\mu-\lambda_6)=0$$

Thus, from equation (31) the eigenvalues are all negative.
$\lambda_1=-\mu,\ \lambda_2=k_1,\ \lambda_3=k_2,\ \lambda_4=k_3,\ \lambda_5=k_5,\ \lambda_6=-\mu$
the three most obvious negative eigenvalues of the Jacobian matrix are $\lambda_1=\lambda_6=-\mu$ twice and $k_5=-(\sigma+\Omega+\mu)$ Hence, Routh-Hurwitz Stability criteria will be to ascertain the necessary and sufficient conditions of the negative real parts of the eigenvalues. The first three eigenvalues are the first, fifth and sixth columns. Therefore a (3×3) sub matrix emerges without the first, fifth and sixth rows and columns of system of equation (31) we get

$$J^*_{dfe}=\begin{bmatrix} k_1 & \beta\alpha & \beta \\ k_4 & k_2 & 0 \\ \phi\rho & 0 & k_3 \end{bmatrix} \tag{4.3}$$

Taking the determinants

$$|J^*_{dfe}-\lambda I|=0$$

$$|J^*_{dfe}-\lambda I|=\left|\begin{bmatrix} k_1 & \beta\alpha & \beta \\ k_4 & k_2 & 0 \\ \phi\rho & 0 & k_3 \end{bmatrix}\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right|$$

$$=\begin{vmatrix} k_1-\lambda & \beta\alpha & \beta \\ k_4 & k_2-\lambda & 0 \\ \phi\rho & 0 & k_3-\lambda \end{vmatrix}$$

$$k_1-\lambda\begin{vmatrix} k_2-\lambda & 0 \\ 0 & k_3-\lambda \end{vmatrix}-\beta\alpha\begin{vmatrix} k_4 & 0 \\ \phi\rho & k_3-\lambda \end{vmatrix}+\beta\begin{vmatrix} k_4 & k_2-\lambda \\ \phi\rho & 0 \end{vmatrix}=0$$

$$-\lambda^3-\lambda^2(k_1+k_2+k_3)+\lambda(k_1k_2+k_1k_3+k_2k_3-\beta\alpha k_3k_4+\lambda\beta\alpha k_4-\beta\phi\rho k_2+\lambda\beta\phi\rho$$

simplifying gives

$$(k_1-\lambda)(k_2-\lambda)(k_3-\lambda)-\beta\alpha(k_4(k_3-\lambda)-\beta\alpha\rho(k_2-\lambda)=0$$

$$-(\lambda^3-\lambda^2(k_1+k_2+k_3)+\lambda(k_1k_2+k_2k_3+k_1k_3-\beta\alpha k_4-\beta\phi\rho)-k_1k_2k_3+\beta\alpha k_3k_4+\beta\phi\rho k_2)=0 \tag{4.4}$$

comparing Equation (33) with Routh-Hurwitz Stability criteria Equation (34) gives:

$$f(\lambda)=\lambda^3+c_2\lambda^2+c_1\lambda+c_0 \tag{4.5}$$

$c_2=-(k_1+k_2+k_3)$
$c_1=k_1k_2+k_2k_3+k_1k_3-\beta\alpha k_4-\beta\phi\rho$
$c_0=-k_1k_2k_3+\beta\alpha k_3k_4+\beta\phi\rho k_2$

Necessary condition: R_0, is positive Since both R_1 and R_2 are positive then from the characteristics equation (34) $c_2>0$ and

$$c_1=k_1k_2(1-R_1)+k_1k_3(1-R_2)+k_2k_3>0$$

Then

$$-k_1k_2k_3+\beta\alpha k_3k_4+\beta\phi\rho k_2=k_1k_2k_3(R_0-1)>0\Leftrightarrow R_0<1$$

Sufficient condition: for $R_0<1$, $R_1<1$, $R_2<1$, by Routh-Hurwitz stability criteria, all the eigenvalues have a negative real part. hence the necessary and sufficient conditions of the disease-free equilibrium point N_0 is locally asymptotically stable $\Leftrightarrow R_0<1$

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Theorem 4.2. *The Endemic equilibrium points of (10), $N^* = (S^* + E^* + I_A^* + I_S^* + Q^* + R^*)$ is locally asymptomatically stable $\Leftrightarrow R_0 < 1$*

Proof: The Jacobian matrix J_{eep}^* of system (9) at EEP is

$$J_{eep} = \begin{bmatrix} \frac{\partial F_1}{\partial S^*} & \frac{\partial F_1}{\partial E^*} & \frac{\partial F_1}{\partial I_A^*} & \frac{\partial F_1}{\partial I_S^*} & \frac{\partial F_1}{\partial Q^*} & \frac{\partial F_1}{\partial R^*} \\ \frac{\partial F_2}{\partial S^*} & \frac{\partial F_2}{\partial E^*} & \frac{\partial F_2}{\partial I_A^*} & \frac{\partial F_2}{\partial I_S^*} & \frac{\partial F_2}{\partial Q^*} & \frac{\partial F_2}{\partial R^*} \\ \frac{\partial F_3}{\partial S^*} & \frac{\partial F_3}{\partial E^*} & \frac{\partial F_3}{\partial I_A^*} & \frac{\partial F_3}{\partial I_S^*} & \frac{\partial F_3}{\partial Q^*} & \frac{\partial F_3}{\partial R^*} \\ \frac{\partial F_4}{\partial S^*} & \frac{\partial F_4}{\partial E^*} & \frac{\partial F_4}{\partial I_A^*} & \frac{\partial F_4}{\partial I_S^*} & \frac{\partial F_4}{\partial Q^*} & \frac{\partial F_4}{\partial R^*} \\ \frac{\partial F_5}{\partial S^*} & \frac{\partial F_5}{\partial E^*} & \frac{\partial F_5}{\partial I_A^*} & \frac{\partial F_5}{\partial I_S^*} & \frac{\partial F_5}{\partial Q^*} & \frac{\partial F_5}{\partial R^*} \\ \frac{\partial F_6}{\partial S^*} & \frac{\partial F_6}{\partial E^*} & \frac{\partial F_6}{\partial I_A^*} & \frac{\partial F_6}{\partial I_S^*} & \frac{\partial F_6}{\partial Q^*} & \frac{\partial F_6}{\partial R^*} \end{bmatrix} = \begin{bmatrix} -k_8 - \mu & \xi & -\alpha k_7 & -k_7 & 0 & 0 \\ k_8 & k_1 & \alpha k_7 & k_7 & 0 & 0 \\ 0 & k_4 & k_2 & 0 & 0 & 0 \\ 0 & \phi\rho & 0 & k_3 & 0 & 0 \\ 0 & 0 & \psi & \theta & k_5 & 0 \\ 0 & 0 & \eta & \gamma & k_6 & -\mu \end{bmatrix} \tag{4.6}$$

Then characteristic $|J_{eep} - \lambda I| = 0$ is expanded and determined as,

$$J_{eep} = \begin{bmatrix} -k_8 - \mu - \lambda & \xi & -\alpha k_7 & -k_7 & 0 & 0 \\ k_8 & k_1 - \lambda & -\alpha k_7 & k_7 & 0 & 0 \\ 0 & k_4 & k_2 - \lambda & 0 & 0 & 0 \\ 0 & \phi\rho & 0 & k_3 - \lambda & 0 & 0 \\ 0 & 0 & \psi & \theta & k_5 - \lambda & 0 \\ 0 & 0 & \eta & \gamma & k_6 & -\mu - \lambda \end{bmatrix} \tag{4.7}$$

Let $k_7 = \frac{\beta S^*}{N}$, $k_8 = \frac{\alpha I_A^* + I_S^*}{N}$

Similarly, the first two negative eigenvalues are $-\mu$ and $-k_5$, the sign of the remaining part is deduced from characteristics equation (36),

$$J_{eep}^* = \begin{bmatrix} -k_8 - \mu & \xi & -\alpha k_7 & -k_7 \\ k_8 & k_1 & \alpha k_7 & k_7 \\ 0 & k_4 & k_2 & 0 \\ 0 & \phi\rho & 0 & k_3 \end{bmatrix} \tag{4.8}$$

The Determinant $|J_{eep}^* - \lambda I| = 0$ gives

$$|J_{eep}^* - \lambda I| = \left| \begin{bmatrix} -k_8 - \mu & \xi & -\alpha k_7 & -k_7 \\ k_8 & k_1 & \alpha k_7 & k_7 \\ 0 & k_4 & k_2 & 0 \\ 0 & \phi\rho & 0 & k_3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix} \right|$$

$$-k_8-\mu-\lambda \begin{bmatrix} k_1 - \lambda & \alpha k_7 & k_7 \\ k_4 & k_2 - \lambda & 0 \\ \phi\rho & 0 & k_3 - \lambda \end{bmatrix} - \xi \begin{bmatrix} k_8 & \alpha k_7 & k_7 \\ 0 & k_2 - \lambda & 0 \\ 0 & 0 & k_3 - \lambda \end{bmatrix} - \alpha k_7 \begin{bmatrix} k_8 & k_1 - \lambda & k_7 \\ 0 & k_4 & 0 \\ 0 & \phi\rho & k_3 - \lambda \end{bmatrix}$$
$$+k_7 \begin{bmatrix} k_8 & k_1 - \lambda & \alpha k_7 \\ 0 & k_4 & k_2 - \lambda \\ 0 & \phi\rho & 0 \end{bmatrix}$$

$$= (-k_8 - \mu - \lambda)(-\lambda^3 - \lambda^2(k_1 + k_2 + k_3) + \lambda(k_1k_2 + k_1k_3 + k_2k_3)) + (-k_8 - \mu - \lambda)(-\alpha k_7)(k_4k_3 - k_4\lambda) + (-k_8 - \mu - \lambda)k_7(\lambda\phi\rho - k_2\phi\rho)\lambda^2\xi k_8 + \lambda\xi(k_8k_2 + k_8k_3) - \xi k_8k_3k_2 - \alpha k_3k_4k_7k_8 + \lambda\alpha k_4k_7k_8 + \lambda k_7k_8\rho\phi - \rho\phi k_2k_7k_8 = 0 \tag{4.9}$$

$$= \lambda^3k_8 + \lambda^2k_8(k_1 + k_2 + k_3) - \lambda k_8(k_1k_2 + k_1k_3 + k_2k_3) + \lambda^3\mu + \lambda^2\mu(k_1 + k_2 + k_3) - \lambda\mu(k_1k_2 + k_1k_3 + k_2k_3) + \lambda^4 + \lambda^3(k_1 + k_2 + k_3) - \lambda^2(k_1k_2 + k_1k_3 + k_2k_3) + \alpha k_8k_7k_4k_3 - \lambda\alpha k_8k_7k_4 + \mu\alpha k_7k_4k_3 - \lambda\mu\alpha k_7k_4 + \lambda\alpha k_7k_4k_3 - \lambda^2\alpha k_7k_4 - \lambda k_8k_7\phi\rho + k_8k_7k_2\phi\rho - \lambda\mu k_7\phi\rho + \mu k_7k_2\phi\rho - \lambda^2k_7\phi\rho - \lambda k_7k_2\phi\rho + \lambda^2k_7\phi - \lambda^2\xi k_8 + \lambda\xi(k_8k_2 + k_8k_3) - \xi k_8k_3k_2 - \alpha k_3k_4k_7k_8 + \lambda\alpha k_4k_7k_8 + \lambda k_7k_8\rho\phi - \rho\phi k_2k_7k_8 = 0 \tag{4.10}$$

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$$f(\lambda) = \lambda^4 + d_3\lambda^3 + d_2\lambda^2 + d_1\lambda + d_0 \tag{4.11}$$

comparing the polynomials Equation (39) and Equation (40) we apply Routh-Hurwitz criteria.

$d_3 = \mu - (k_1 + k_2 + k_3) + k_8$

$d_2 = k_1k_2 + k_2k_3 + k_1k_3 - k_1k_8 - k_2k_8 - k_3k_8 - \mu k_1 - \mu k_2 - \mu k_3 - \phi\rho k_7 - \alpha k_4k_7 - \xi k_8$

$d_1 = \mu(k_1k_2 + k_2k_3 + k_1k_3) - k_1k_2k_3 + k_1k_2k_8 + k_1k_3k_8 + k_2k_3k_8 - \phi\rho\mu k_7 - \alpha\mu k_4k_7 + \phi\rho k_2k_7 + \alpha k_3k_4k_7 - \xi k_8k_2 - \xi k_8k_3$

$d_0 = \mu\phi\rho k_2k_7 - \mu k_1k_2k_3 - k_1k_2k_3k_8 + \alpha\mu k_3k_4k_7$

Necessary condition: The coefficient d_3 is positive and d_2, d_1, d_0 can be shown to be positive as follows:

$d_2 = \frac{k_1k_2R_2 + k_1k_3R_1}{R_0} + k_2k_3 - k_1k_8 - k_2k_8 - k_3k_8 - \mu k_2 - \mu k_3 > 0$

$d_1 = \mu k_1k_2\frac{R_2}{R_0} + \mu k_1k_3\frac{R_1}{R_0} + \mu k_2k_3 - 2k_1k_2k_3 + k_1k_2k_8 + k_1k_3k_8 + k_2k_3k_8 > 0$

$d_0 = \mu\phi\rho k_2k_7 - \mu k_1k_2k_3 - k_1k_2k_3k_8 + \alpha\mu k_3k_4k_7 = -k_1k_2k_3k_8 > 0$

Sufficient condition: Furthermore, by Routh-Hurwitz stability criteria all the eigenvalues of the characteristic equation of (40) have negative real part since it can be shown that $d_0d_3^2d_1^2 - d_1d_2d_3 > 0$, Hence, EEP

$$N^* = (S^* + E^* + I_A^* + I_S^* + Q^* + R^*)$$

is locally asymptotically stable $\Leftrightarrow R_0 > 0$

4.2 *Global Stability Analysis of DFEP and EEP*

In this subsection, we will use Lyapunov function method to show the global asymptotic stability of DFE and EEP respectively. by the following theorems

Theorem 4.3. *The DFE is globally asymptotically stable* $\Leftrightarrow R_0 \leq 1$ *then the DFE giving by*

$$N_0 = (S_0 + E_0 + I_{A0} + I_{S0} + Q_0 + R_0) = \left(\frac{\kappa}{\mu}, 0, 0, 0, 0, 0\right) \tag{4.12}$$

is stable in the positive invariant region B^* *discussed in previous subsection. now we show that equation (9) is globally asymptotically stable in the positive invariant region* B^* *for initial zero vaccination and maternally derived immunity.*

Proof: Consider a Lyapunov function candidate,

$$V(SEI_AI_SQR) = \left(S - S_0 - S_0 \ln\frac{S}{S_0}\right) + E + I_A + I_S + Q + R \tag{4.13}$$

Differentiating $V(SEI_AI_SQR)$ with respect to time in the direction of the solution of (9) gives

$$\dot{V} = \left(1 - \frac{S}{S_0}\right) + \dot{E} + \dot{I_A} + \dot{I_S} + \dot{Q} + \dot{R} \tag{4.14}$$

putting Equation (9)

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$$\dot{V} = \left(1 - \frac{S}{S_0}\right)\left(\kappa - \frac{\beta S(\alpha I_A + I_S)}{N} - \mu S\right) + \frac{\beta S(\alpha I_A + I_S)}{N}$$
$$-(\phi + \mu) + (1-\rho)\phi E - (\eta + \psi + \mu) I_A + \rho\phi E$$
$$-(\gamma + \theta + \varpi + \mu) I_S + \psi I_A + \phi I_S - (\sigma + \Omega + \mu) Q \tag{4.15}$$
$$+\eta I_A + \gamma I_S - (\sigma + \Omega) Q - \mu R$$

Expanding and collecting like terms of Equation (44) gives

$$\kappa - \frac{\beta S(\alpha I_A + I_S)}{N} - \mu S - \kappa\frac{S}{S_0} + \frac{\beta S(\alpha I_A + I_S)}{N}\frac{S}{S_0} + \mu S\frac{S}{S_0}$$
$$-\mu(E + I_A + I_S + Q + R) \tag{4.16}$$

Simplifying

$$\kappa - \kappa\frac{S}{S_0} - \kappa\frac{S_0}{S} + \frac{\beta S(\alpha I_A + I_S)}{N}\frac{S_0}{S} - \mu(E + I_A + I_S + Q + R)$$

$$\kappa\left(2 - \frac{S}{S_0} - \frac{S_0}{S}\right) + \Pi\frac{S_0}{S} - \mu(E + I_A + I_S + Q + R) \tag{4.17}$$

from Equation (1),

$\Pi = \beta\frac{(\alpha I_A + I_S)}{N}$ and since $\Pi\frac{S_0}{S}$ is non-negative we have that

$$\dot{V} \leq \kappa\left(2 - \frac{S}{S_0} - \frac{S_0}{S}\right) + \Pi\frac{S_0}{S} - \mu(E + I_A + I_S + Q + R) \tag{4.18}$$

By the inequality of arithmetic and geometric means we have,

$$\kappa\left(\frac{2SS_0 - (S_0^2 + S^2)}{SS_0}\right) - \mu(E + I_A + I_S + Q + R) \tag{4.19}$$

thus, $\dot{V} \geq 0$ When $\dot{V} = 0 \Leftrightarrow E = I_A = I_S = Q = R = 0$. therefore, it follows that the largest invariant set in $(E + I_A + I_S + Q + R) \in B^* : \dot{V} = 0$ is $N_0\left(\frac{\kappa}{\mu}, 0, 0, 0, 0, 0\right)$ Thus, by Lasalle's invariance principle the DFE, is globally asymptotically stable.

Theorem 4.4. *If $R_0 > 1$, then the equation (9) is globally asymptotically stable if $S^* = S$, $E^* = E$, $I_S^* = I_S$, $I_A^* = I_A$, $Q^* = Q$, $R^* = R$ and $\mathcal{X}_1 < \mathcal{X}_2$ Also, unstable when $R_0 \leq 1$.*

Proof: Applying the constructed Lyapunov function, suppose the basic reproductive number $R_0 > 1$, then the EEP. constructing a Lyapunov function candidate L defined by,

$$L(SEI_AI_SQR) = \left(S - S^* - S^*\ln\frac{S}{S^*}\right) + \left(E - E^* - E^*\ln\frac{E}{E^*}\right) + \left(I_A - I_A^* - I_A^*\ln\frac{I_A}{I_A^*}\right)$$
$$+\left(I_S - I_S^* - I_S^*\ln\frac{I_S}{I_S^*}\right) + \left(Q - Q^* - Q^*\ln\frac{Q}{Q^*}\right) + \left(R - R^* - R^*\ln\frac{R}{R^*}\right) \tag{4.20}$$

Differentiating L in the direction of the solution Equation (9).

$$\frac{dL}{dt} = \left(\frac{S - S^*}{S}\right)\dot{S} + \left(\frac{E - E^*}{E}\right)\dot{E} + \left(\frac{I_A^* - I_A^*}{I_A}\right)\dot{I_A} + \left(\frac{I_S^* - I_S^*}{I_S}\right)\dot{I_S} + \left(\frac{Q - Q^*}{Q}\right)\dot{Q} + \left(\frac{R - R^*}{R}\right)\dot{R}$$

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can be rewritten as

$$\frac{dL}{dt} = \left(\frac{S-S^*}{S}\right)\left(\kappa - \frac{\beta S(\alpha I_A + I_S)}{N} - \mu S\right) + \left(\frac{E-E^*}{E}\right)\left(\frac{\beta S(\alpha I_A + I_S)}{N} - (\phi+\mu)E\right) + \left(\frac{I_A^*-I_A^*}{I_A}\right)((1-\rho)\phi E$$
$$-(\eta+\psi+\mu)I_A) + \left(\frac{I_S^*-I_S^*}{I_S}\right)(\rho\phi E - (\gamma+\theta+\varpi+\mu)I_S) + \left(\frac{Q-Q^*}{Q}\right)(\psi I_A + \theta I_S - (\sigma+\Omega+\mu)Q)$$
$$+\left(\frac{R-R^*}{R}\right)(\eta I_A + \gamma I_S - (\sigma+\Omega)Q - \mu R) \tag{4.21}$$

Using the Approach of [15] and Substitution the Equation (53) into Equation (52)

$$S = S - S^*,\ E = E - E^*,\ I_S = I_S - I_S^*,\ I_A = I_A - I_S^*,$$
$$Q = Q - Q^*,\ R = R - R^* \tag{4.22}$$

$$\frac{dL}{dt} = \left(\frac{S-S^*}{S}\right)\left(\kappa - \frac{\beta S(\alpha I_A + I_S)}{N} - \mu S\right) + \left(\frac{E-E^*}{E}\right)\left(\frac{\beta S(\alpha I_A + I_S)}{N} - (\phi+\mu)E\right) + \left(\frac{I_A^*-I_A^*}{I_A}\right)((1-\rho)\phi E$$
$$-(\eta+\psi+\mu)I_A) + \left(\frac{I_S^*-I_S^*}{I_S}\right)(\rho\phi E - (\gamma+\theta+\varpi+\mu)I_S) + \left(\frac{Q-Q^*}{Q}\right)(\psi I_A + \theta I_S - (\sigma+\Omega+\mu)Q)$$
$$+\left(\frac{R-R^*}{R}\right)(\eta I_A + \gamma I_S - (\sigma+\Omega)Q - \mu R) \tag{4.23}$$

$$\frac{dL}{dt} = \left(\frac{S-S^*}{S}\right)\left(\kappa - \frac{\beta(S-S^*)(\alpha I_A + I_S)}{N} - \mu(S-S^*)\right) + \left(\frac{E-E^*}{E}\right)\left(\frac{\beta(S-S^*)(\alpha I_A + I_S)}{N} - (\phi+\mu)(E-E^*)\right)$$
$$+\left(\frac{I_A^*-I_A^*}{I_A}\right)((1-\rho)\phi E - (\eta+\psi+\mu)(I_A - I_A^*)) + \left(\frac{I_S^*-I_S^*}{I_S}\right)(\rho\phi E - (\gamma+\theta+\varpi+\mu)$$
$$(I_S - I_S^*)) + \left(\frac{Q-Q^*}{Q}\right)(\psi I_A + \theta I_S - (\sigma+\Omega+\mu)(Q-Q^*)) + \left(\frac{R-R^*}{R}\right)(\eta I_A + \gamma(I_S)$$
$$-(\sigma+\Omega)Q) - \mu(R-R^*) \tag{4.24}$$

$$\frac{dL}{dt} = \frac{(S-S^*)^2}{S}\left(\frac{\beta S(\alpha I_A + I_S)}{N} - \mu\right)\kappa + \left(\frac{\beta S(\alpha I_A + I_S)}{N} - \mu\right)\frac{S^*}{S} - \kappa\frac{S^*}{S} + \frac{\beta S(\alpha I_A + I_S)}{N} - \mu S^*$$
$$-\frac{(E-E^*)^2}{E}(\phi+\mu) + \frac{\beta S(\alpha I_A + I_S)}{N} - (\phi+\mu)E^* - \frac{E^{*2}}{E}(\phi+\mu) - \frac{(I_A^*-I_A^*)^2}{I_A}(\eta+\psi+\mu)$$
$$+(1-\rho)\phi E - \frac{I^*}{I_A}(1-\rho)\phi E - (\eta+\psi+\mu)I_A^* + \frac{I_A^{*2}}{I_A}(\eta+\psi+\mu) - \frac{(I_S-I_S^*)^2}{I_S}(\gamma+\theta+\varpi+\mu)$$
$$+\rho\phi E - \frac{I_S^*}{I_S}\rho\phi E - (\gamma+\theta+\varpi+\mu)I_S^* + \frac{I_S^{*2}}{I_S}(\gamma+\theta+\varpi+\mu) - \frac{(Q-Q^*)^2}{Q}(\sigma+\Omega + (\psi I_A + \theta I_S)$$
$$-\frac{Q^*}{Q}(\psi I_A + \theta I_S) - (\sigma+\Omega+\mu)Q^* + (\sigma+\Omega+\mu)\frac{Q^{*2}}{Q} - \frac{(R-R^*)^2}{R}\mu + \eta I_A + \gamma I_S + (\sigma+\Omega)Q$$
$$-\frac{R^*}{R}(\eta I_A + \gamma I_S + (\sigma+\Omega)Q) - \mu R^* + \frac{R^{*2}}{R}\mu \tag{4.25}$$

Re-arranging the positive terms and negative terms in the form,

$$\frac{dL}{dt} = \mathcal{X}_1 - \mathcal{X}_2 \tag{4.26}$$

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$$\mathcal{X}_1 = \kappa + \left(\frac{\beta S(\alpha I_A + I_S)}{N} - \mu\right)\frac{S^*}{S} + \frac{\beta S(\alpha I_A + I_S)}{N} + -\frac{E^{*2}}{E}(\phi + \mu) + \rho\phi E + \frac{I_A^{*2}}{I_A}(\eta + \psi + \mu) + \frac{I_S^{*2}}{I_S}(\gamma + \theta + \varpi + \mu)$$

$$+(\psi I_A + \theta I_S) + \eta I_A + \gamma I_S + (\sigma + \Omega + \mu)\frac{Q^{*2}}{Q}(\sigma + \Omega\frac{R^{*2}}{R}\mu \tag{4.27}$$

$$\frac{dL}{dt} = \frac{(S-S^*)^2}{S}\left(\frac{\beta S(\alpha I_A + I_S)}{N} - \mu\right) + \left(\kappa\frac{S^*}{S} + \frac{\beta S(\alpha I_A + I_S)}{N} - \mu S^*\right) - \frac{(E-E^*)^2}{E}(\phi + \mu) - (\phi + \mu)E^* - \frac{E^*}{E}\frac{\beta S(\alpha I_A + I_S)}{N}$$

$$+\frac{(I_A^* - I_A^*)^2}{I_A}(\eta + \psi + \mu) + (1-\rho)\phi E - \frac{I^*}{I_A}(1-\rho)\phi E - (\eta + \psi + \mu)I_A^* + \frac{(I_S - I_S^*)^2}{I_S}(\gamma + \theta + \varpi + \mu) + \rho\phi E$$

$$+\frac{I_S^*}{I_S}\rho\phi E - (\gamma + \theta + \varpi + \mu)I_S^* + \frac{Q^*}{Q}(\psi I_A + \theta I_S) + (\sigma + \Omega + \mu)Q^* + \frac{(R-R^*)^2}{R}\mu + \frac{R^*}{R}(\eta I_A + \gamma I_S$$

$$+(\sigma + \Omega)Q) + \mu R^* \tag{4.28}$$

Hence, if $\mathcal{X}_1 < \mathcal{X}_2$ then $\frac{dV}{dt} \leq 0$. It is worthy of note that at

$$\frac{dV}{dt} = 0 \Leftrightarrow S^* = S,\ E^* = E,\ I_S^* = I_S,\ I_A^* = I_A,\ Q^* = Q,\ R^* = R$$

The we conclude that the Endemic Equilibrium Point EEP is globally asymptotically stable applying the LaSalle's invariance principle.

5 Numerical Simulation

Tab.2 Value of The Model Parameters Corresponding for Diphtheria-Quarantine model.

Table 2: *Parameters Description of Diphtheria-Quarantine Model*

S/N	Parameter	Description	Values	References
1	β	Interaction rate of I_A with S	0.57	Assumed
2	α	Interaction rate of I_S with S	0.7	[2]
3	ρ	Proportion of symptomatic infected population	0.55	[2]
4	ϕ	Incubation period of diphtheria	0.3	[calculated]
5	Ω	self Quarantine rate	0.1-0.9	Assumed
6	γ	Recovery/cure rate of symptomatically infected Population	0.1	Assumed
7	η	Recovery/cure rate of asymptomatically infected Population	0.1	Assumed
8	μ	Natural death rate	0.006	[2,10]
10	θ	Quarantined rate of symptomatically infected Population	0.1/day	Assumed
9	V	Vaccination	0.1-1.2	Assumed
11	ϖ	Diphtheria induced death rate	0.05	[2,10]
12	b	Birth rate	0.019	[10]
13	ξ	maternal derived immunity	0.34	[10]
14	σ	rate of recovery	0.34	[10]
15	ψ	Quarantine rate of asymptomatic infected	0.1-0.9	Assumed

5.1 Sensitivity Analysis

In this subsection, the normalized forward sensitivity index of R_0 will be explored to determine the strength and weakness of each parameter in the models prediction. Therefore, The Normalized

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forward sensitivity index of R_0 differentiable with respect to a given parameter β defined as

$$\Upsilon_{\beta}^{R_0} = \frac{\beta}{R_0}\frac{\partial R_0}{\partial \beta} \tag{5.1}$$

where

$$R_0 = \frac{\beta b(1-v)\alpha(1-\rho)}{\mu(\xi+\phi+\mu)(\eta+\psi+\mu)} + \frac{\beta b(1-v)\phi\rho}{\mu(\xi+\phi+\mu)(\gamma+\theta+\varpi+\mu)} \tag{5.2}$$

Manual calculation of the sensitivity to parameter β

$$\frac{\beta}{R_0} = \beta(\frac{\mu(\xi+\phi+\mu)(\eta+\psi+\mu)}{\beta b(1-V)\alpha(1-\rho)} + \frac{\mu(\xi+\phi+\mu)(\gamma+\theta+\omega+\mu)}{\beta b(1-V)\phi\rho})$$

$$\frac{\partial R_0}{\partial \beta} = \frac{b(1-V)\alpha(1-\rho)}{\mu(\xi+\phi+\mu)(\eta+\psi+\mu)} + \frac{b(1-V)\phi\rho}{\mu(\xi+\phi+\mu)(\gamma+\theta+\omega+\mu)}$$

sensitivity to parameter b

$$\Upsilon_{b}^{R_0} = \frac{b}{R_0}\frac{\partial R_0}{\partial b} \tag{5.3}$$

$$\frac{b}{R_0} = b(\frac{\mu(\xi+\phi+\mu)(\eta+\psi+\mu)}{\beta b(1-V)\alpha(1-\rho)} + \frac{\mu(\xi+\phi+\mu)(\gamma+\theta+\omega+\mu)}{\beta b(1-V)\phi\rho})$$

$$\frac{\partial R_0}{\partial b} = \frac{\beta(1-V)\alpha(1-\rho)}{\mu(\xi+\phi+\mu)(\eta+\psi+\mu)} + \frac{\beta(1-V)\phi\rho}{\mu(\xi+\phi+\mu)(\gamma+\theta+\omega+\mu)}$$

sensitivity to parameter α

$$\Upsilon_{\alpha}^{R_0} = \frac{\alpha}{R_0}\frac{\partial R_0}{\partial \alpha} \tag{5.4}$$

$$\frac{\alpha}{R_0} = \alpha(\frac{\mu(\xi+\phi+\mu)(\eta+\psi+\mu)}{\beta b(1-V)\alpha(1-\rho)} + \frac{\mu(\xi+\phi+\mu)(\gamma+\theta+\omega+\mu)}{\beta b(1-V)\phi\rho})$$

$$\frac{\partial R_0}{\partial \alpha} = \frac{\beta b(1-V)(1-\rho)}{\mu(\xi+\phi+\mu)(\eta+\psi+\mu)}$$

sensitivity to parameter ρ

$$\Upsilon_{\rho}^{R_0} = \frac{\rho}{R_0}\frac{\partial R_0}{\partial \rho} \tag{5.5}$$

$$\frac{\rho}{R_0} = \rho(\frac{\mu(\xi+\phi+\mu)(\eta+\psi+\mu)}{\beta b(1-V)\alpha(1-\rho)} + \frac{\mu(\xi+\phi+\mu)(\gamma+\theta+\omega+\mu)}{\beta b(1-V)\phi\rho})$$

$$\frac{\partial R_0}{\partial \rho} = \frac{\beta\alpha b(V-1)}{\mu(\xi+\phi+\mu)(\eta+\psi+\mu)} + \frac{\beta b(1-V)\phi}{\mu(\xi+\phi+\mu)(\gamma+\theta+\omega+\mu)}$$

Sensitivity to parameter V

$$\Upsilon_{V}^{R_0} = \frac{V}{R_0}\frac{\partial R_0}{\partial V} \tag{5.6}$$

$$\frac{V}{R_0} = V(\frac{\mu(\xi+\phi+\mu)(\eta+\psi+\mu)}{\beta b(1-V)\alpha(1-\rho)} + \frac{\mu(\xi+\phi+\mu)(\gamma+\theta+\omega+\mu)}{\beta b(1-V)\phi\rho})$$

$$\frac{\partial R_0}{\partial V} = \frac{\beta\alpha b(\rho-1)}{\mu(\xi+\phi+\mu)(\eta+\psi+\mu)} + \frac{\beta b\phi\rho}{\mu(\xi+\phi+\mu)(\gamma+\theta+\omega+\mu)}$$

Sensitivity to parameter ϕ

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$$\Upsilon_{\phi}^{R_0} = \frac{\phi}{R_0}\frac{\partial R_0}{\partial \phi} \tag{5.7}$$

$$\frac{\phi}{R_0} = \phi(\frac{\mu(\xi+\phi+\mu)(\eta+\psi+\mu)}{\beta b(1-V)\alpha(1-\rho)} + \frac{\mu(\xi+\phi+\mu)(\gamma+\theta+\omega+\mu)}{\beta b(1-V)\phi\rho})$$

$$\frac{\partial R_0}{\partial \phi} = -\frac{\beta b(1-V)\alpha(1-\rho)(\mu\eta+\mu\psi+\mu^2)}{(\mu(\xi+\phi+\mu)(\eta+\psi+\mu))^2}$$

$$+\frac{\mu(\xi+\phi+\mu)(\gamma+\theta+\omega+\mu)(\beta b(1-V)\rho) - (\beta b(1-V)\phi\rho)(\mu\gamma+\mu\theta+\mu\omega+\mu^2)}{(\mu(\xi+\phi+\mu)(\gamma+\theta+\omega+\mu))^2}$$

$$\Upsilon_{\mu}^{R_0} = \frac{\mu}{R_0}\frac{\partial R_0}{\partial \mu} \tag{5.8}$$

Sensitivity to parameter μ

$$\frac{\mu}{R_0} = \mu(\frac{\mu(\xi+\phi+\mu)(\eta+\psi+\mu)}{\beta b(1-V)\alpha(1-\rho)} + \frac{\mu(\xi+\phi+\mu)(\gamma+\theta+\omega+\mu)}{\beta b(1-V)\phi\rho})$$

$$\frac{\partial R_0}{\partial \mu} = -\frac{(\beta b(1-V)\phi\rho)(\xi\gamma+\xi\theta+\xi\omega+2\mu\xi+\phi\gamma+\theta\phi+\omega\phi+2\mu\phi+2\mu\gamma+2\mu\theta+2\mu\omega+3\mu^2)}{(\mu(\xi+\phi+\mu)(\gamma+\theta+\omega+\mu))^2}$$

$$-\frac{(\beta b(1-V)\alpha(1-\rho)(\xi\eta+\xi\psi+2\mu\xi+\phi\eta+\psi\phi+2\mu\phi+2\mu\eta+2\mu\psi+3\mu^2)}{(\mu(\xi+\phi+\mu)(\eta+\psi+\mu))^2}$$

Sensitivity to parameter ξ

$$\Upsilon_{\xi}^{R_0} = \frac{\xi}{R_0}\frac{\partial R_0}{\partial \xi} \tag{5.9}$$

$$\frac{\xi}{R_0} = \xi(\frac{\mu(\xi+\phi+\mu)(\eta+\psi+\mu)}{\beta b(1-V)\alpha(1-\rho)} + \frac{\mu(\xi+\phi+\mu)(\gamma+\theta+\omega+\mu)}{\beta b(1-V)\phi\rho})$$

$$\frac{\partial R_0}{\partial \xi} = -\frac{\beta b(1-V)\alpha(1-\rho)\mu(\eta+\psi+\mu)}{(\mu(\xi+\phi+\mu)(\eta+\psi+\mu))^2} - \frac{(\beta b(1-V)\phi\rho)\mu(\gamma+\theta+\omega+\mu)}{(\mu(\xi+\phi+\mu)(\gamma+\theta+\omega+\mu))^2}$$

Sensitivity to parameter η

$$\Upsilon_{\eta}^{R_0} = \frac{\eta}{R_0}\frac{\partial R_0}{\partial \eta} \tag{5.10}$$

$$\frac{\eta}{R_0} = \eta(\frac{\mu(\xi+\phi+\mu)(\eta+\psi+\mu)}{\beta b(1-V)\alpha(1-\rho)} + \frac{\mu(\xi+\phi+\mu)(\gamma+\theta+\omega+\mu)}{\beta b(1-V)\phi\rho})$$

$$\frac{\partial R_0}{\partial \eta} = -\frac{(\beta b(1-V)\alpha(1-\rho)\mu(\xi+\phi+\mu)}{(\mu(\xi+\phi+\mu)(\eta+\psi+\mu))^2}$$

Sensitivity to parameter ψ

$$\Upsilon_{\psi}^{R_0} = \frac{\psi}{R_0}\frac{\partial R_0}{\partial \psi} \tag{5.11}$$

$$\frac{\psi}{R_0} = \psi(\frac{\mu(\xi+\phi+\mu)(\eta+\psi+\mu)}{\beta b(1-V)\alpha(1-\rho)} + \frac{\mu(\xi+\phi+\mu)(\gamma+\theta+\omega+\mu)}{\beta b(1-V)\phi\rho})$$

$$\frac{\partial R_0}{\partial \psi} = -\frac{(\beta b(1-V)\alpha(1-\rho)\mu(\xi+\phi+\mu)}{(\mu(\xi+\phi+\mu)(\eta+\psi+\mu))^2}$$

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Sensitivity to parameter γ

$$\Upsilon_{\gamma}^{R_0} = \frac{\gamma}{R_0}\frac{\partial R_0}{\partial \gamma} \tag{5.12}$$

$$\frac{\gamma}{R_0} = \gamma\left(\frac{\mu(\xi+\phi+\mu)(\eta+\psi+\mu)}{\beta b(1-V)\alpha(1-\rho)} + \frac{\mu(\xi+\phi+\mu)(\gamma+\theta+\omega+\mu)}{\beta b(1-V)\phi\rho}\right)$$

$$\frac{\partial R_0}{\partial \gamma} = -\frac{(\beta b(1-V)\rho\phi\mu(\xi+\phi+\mu)}{(\mu(\xi+\phi+\mu)(\gamma+\theta+\omega+\mu))^2}$$

Sensitivity to parameter θ

$$\Upsilon_{\theta}^{R_0} = \frac{\theta}{R_0}\frac{\partial R_0}{\partial \theta} \tag{5.13}$$

$$\frac{\theta}{R_0} = \theta\left(\frac{\mu(\xi+\phi+\mu)(\eta+\psi+\mu)}{\beta b(1-V)\alpha(1-\rho)} + \frac{\mu(\xi+\phi+\mu)(\gamma+\theta+\omega+\mu)}{\beta b(1-V)\phi\rho}\right)$$

$$\frac{\partial R_0}{\partial \theta} = -\frac{(\beta b(1-V)\rho\phi\mu(\xi+\phi+\mu)}{(\mu(\xi+\phi+\mu)(\gamma+\theta+\omega+\mu))^2}$$

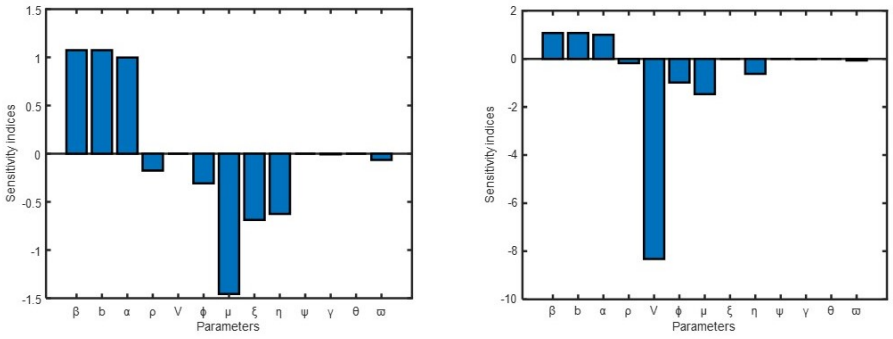
Sensitivity to parameter ω

$$\Upsilon_{\omega}^{R_0} = \frac{\omega}{R_0}\frac{\partial R_0}{\partial \omega} \tag{5.14}$$

$$\frac{\omega}{R_0} = \omega\left(\frac{\mu(\xi+\phi+\mu)(\eta+\psi+\mu)}{\beta b(1-V)\alpha(1-\rho)} + \frac{\mu(\xi+\phi+\mu)(\gamma+\theta+\omega+\mu)}{\beta b(1-V)\phi\rho}\right)$$

$$\frac{\partial R_0}{\partial \omega} = -\frac{(\beta b(1-V)\rho\phi\mu(\xi+\phi+\mu)}{(\mu(\xi+\phi+\mu)(\gamma+\theta+\omega+\mu))^2}$$

Fig. 2. illustrates the sensitivity bar chart of R_0 with increase in the maternal derived immunity of the exposed individual and vaccination at birth. That is increase in parameters ξ and V respectively.



ξ=0.9, θ= ψ = V = 0 V =0.9, θ= ξ= ψ = 0

Fig.2. Sensitivity Bar chart of R_0 with increase in Maternally derived immunity and Vaccination at birth.

Furthermore on sensitivity indices, quarantine rate of the asymptomatic and symptomatic infected individuals was increased to observe parameter tangibility on reproduction number. Fig. 3., shows the sensitivity bar chart changes as regards increase in ψ and θ respectively..

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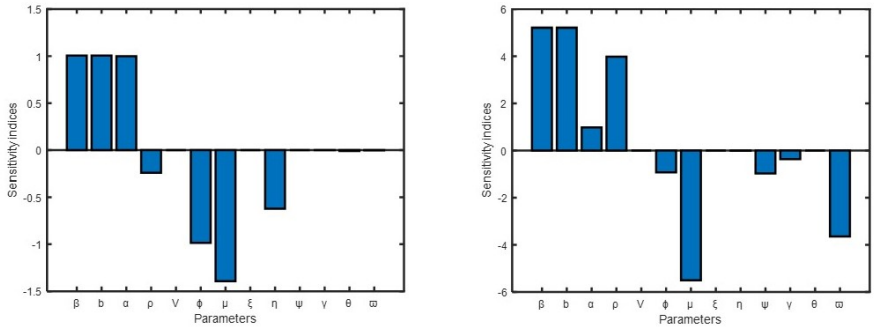
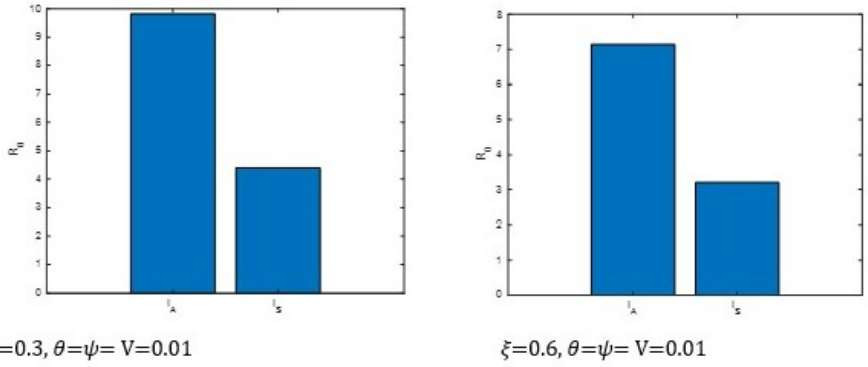


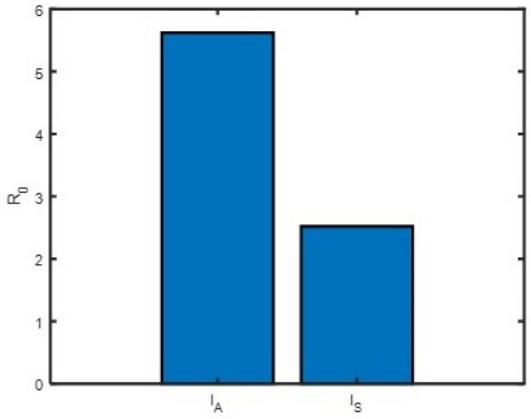
θ=0.9= ξ= V = ψ = 0 ψ =0.9= ξ= V = θ = 0

Fig.3. Sensitivity Bar chart of R_0 with increase in Quarantine of asymptomatic and symptomatic individuals..

From equation ( ) the changes in two terms of R_0 accounting for asymptomatic and symptomatic are represented in Fig. 4, Fig. 5 and Fig. 6. It is therefore vital to visualize behavioural value of increase in disease controller parameter on R_0 against the asymptomatic and symptomatic infectious individuals. For instance, Fig. 4 bar chart illustrates the corresponding results of the reproduction number against the asymptomatic and symptomatic infected individuals with increase in maternal derived immunity parameter at $\xi = 0.3, 0.6, 0.9$ respectively.



ξ=0.3, θ=ψ= V=0.01 ξ=0.6, θ=ψ= V=0.01



ξ=0.9, θ=ψ= V=0.01

Fig.4. Impact of increase in maternally derived immunity parameter in R_0 on Asymptomatic and Symptomatic Infectious Individuals

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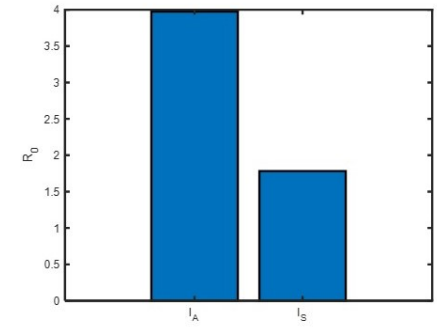
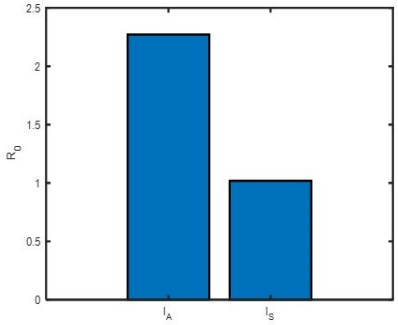
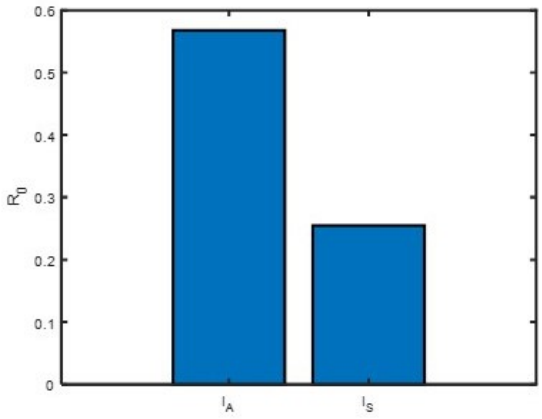
In same vein, Fig. 5 displays a combo increase bar chart of maternally derived immunity and vaccination.



V=0.3, ξ=0.9, $\theta = \psi = 0.01$



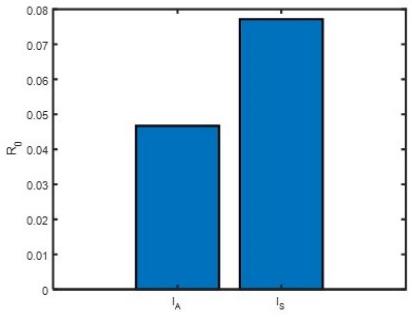
V=0.6, ξ=0.9, $\theta = \psi = 0.01$



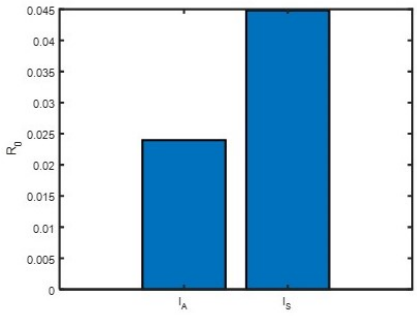
V=0.9, ξ=0.9, $\theta = \psi = 0.01$

Fig.5. Combo impact of increase in Maternally derived immunity and vaccination at birth.

Furthermore, Fig. 6 illustrates simultaneous increase in quarantine parameters, that is V, ψ and ξ respectively. It was observed that the combination of this three disease controller on reproduction number representing Asymptomatic term tended towards zero, which infers that disease in the Asymptomatic term almost died out.



V=0.9, ξ=0.9, $\theta = \psi = 0.3$



V=0.9, ξ=0.9, $\theta = \psi = 0.6$

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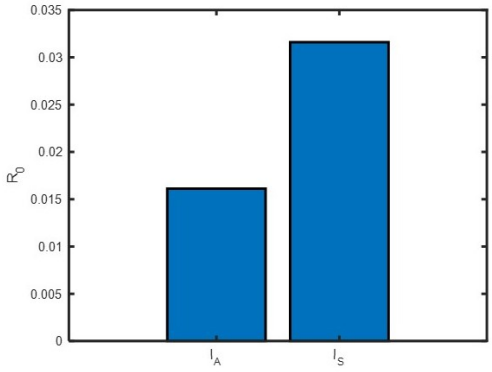

V=0.9, ξ=0.9, θ= ψ = 0.9

Fig.6. Combo impact of increase in Maternally derived immunity, vaccination at birth and Quarantine rate parameter

In the quest to examine the most appropriate diphtheria disease controller, the parameters of vaccination at birth was increased for both Symptomatic and Asymptomatic infectious individuals. Fig. 7. illustrates the impact of increase in by $V = 0, 0.1, 0.3, 0.6, 0.9$ (different levels of Vaccination at birth).

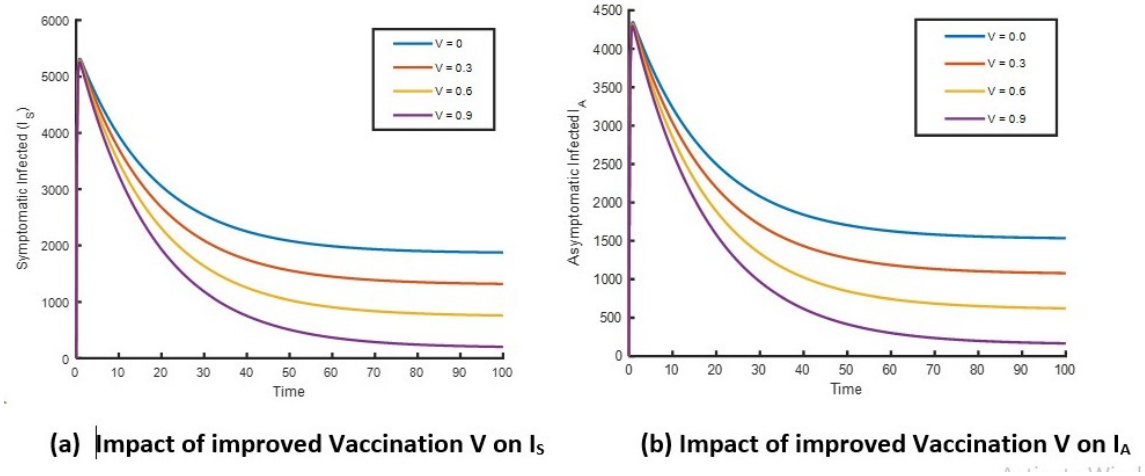

Fig.7. Improved Vaccination impact on asymptomatic and symptomatic infectious individual.

Also, Fig. 8., shows increase in quarantine control measure on I_S and I_A for parameter $\psi = 0, 0.1, 0.3, 0.6, 0.9$.

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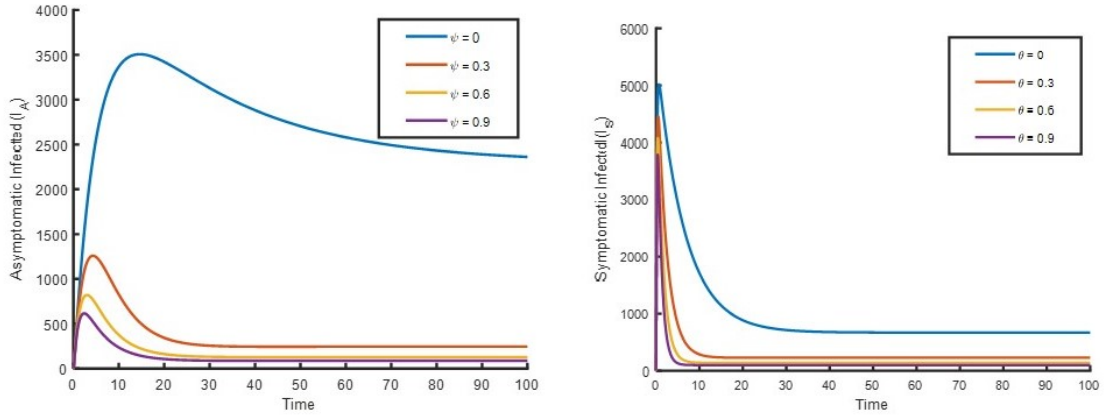
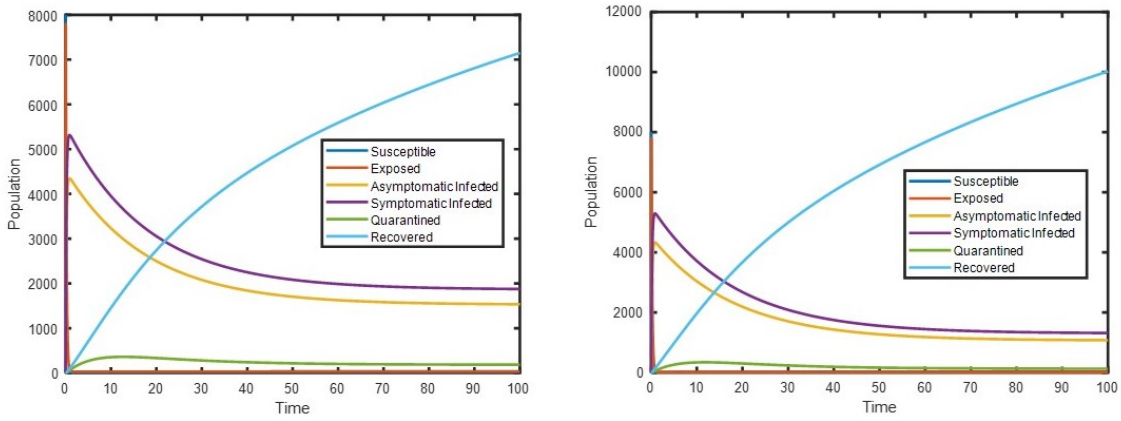


(c) Quarantine impact (ψ) on I_A **(d) Quarantine impact (θ) on I_S**

Fig.8. Improved Quarantine impact on asymptomatic and symptomatic infectious individual.

Furthermore, a combination of high and low diphtheria disease control rate was explored on SEI_AI_SQR model. At the initial instance of disease spread, the total population was $N(0) = 10001$, where the Susceptible compartment $S(0) = 8000$, Exposed compartment $E(0) = 2000$, asymptomatic infected compartment $I_A(0) = 0$, symptomatic infected compartment $I_S(0) = 1$, Quarantined compartment $Q(0) = 0$ and Recovered compartment $R(0) = 0$.

Fig.9 and Fig.10, displays the SEI_AI_SQR dynamic model with low and high Vaccination at birth.



(e) V = 0 (f) V = 0.3

Fig.9. Impact of Low vaccination at birth on SEI_AI_SQR model

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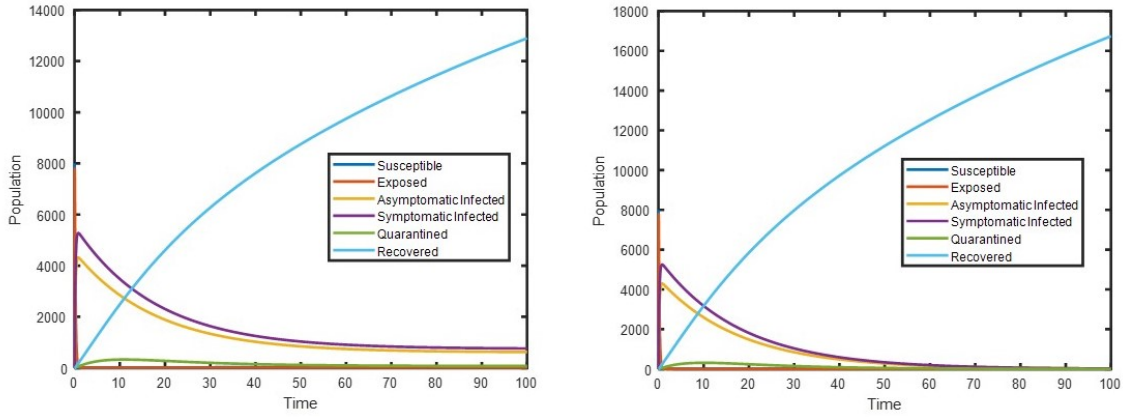
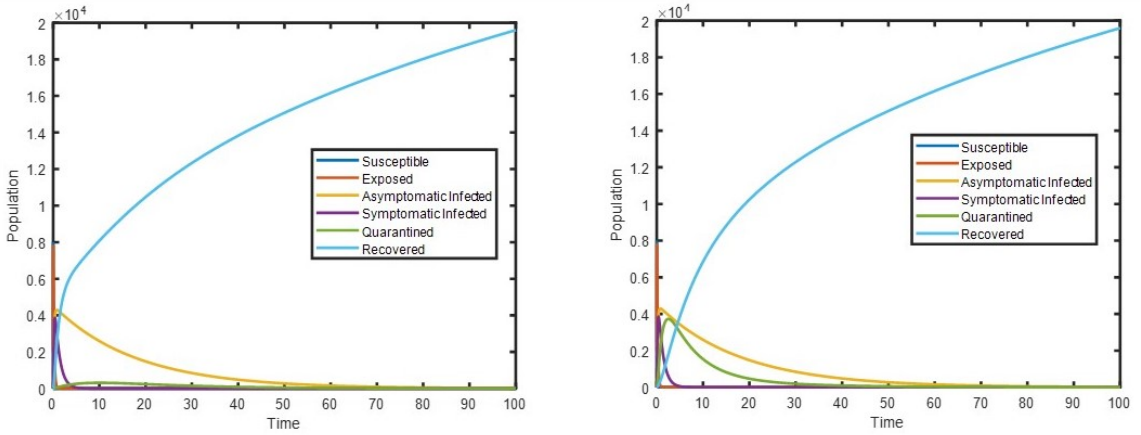


(g) V = 0.6 (h) V = 1

Fig.10. Impact of High Vaccination at birth on SEI_AI_SQR Model

Fig. 11 (i) displays the impact of High Vaccination at birth on the formulated diphtheria transmission model with improved quarantine rate of asymptomatic infectious compartment and Fig. (j) shows the improved quarantine on symptomatic compartment.



(i) V = 1, ψ = 0.9, $\theta = 0$ (j) V = 1, ψ = 0, $\theta = 0.9$

Fig.11 Impact of High Vaccination at birth with increase in quarantine on Symptomatic and Asymptomatic infectious individuals SEI_AI_SQR

Since it has been highlighted that the reservoir of infection lies in the asymptomatic infectious individuals, this infers that mitigating disease spread amongst the I_A compartment will significantly mitigate disease spread, there fore Fig. 12, Fig. 13, Fig. 14 and Fig. 15 shows the 3 D sub plot (SEI_A) effect of low/High asymptomatic quarantine rate in combination with low/High vaccination at birth. Fig. 12 (k) shows no disease control and low quarantine impact while (l) displays low quarantine impact on the sub model.

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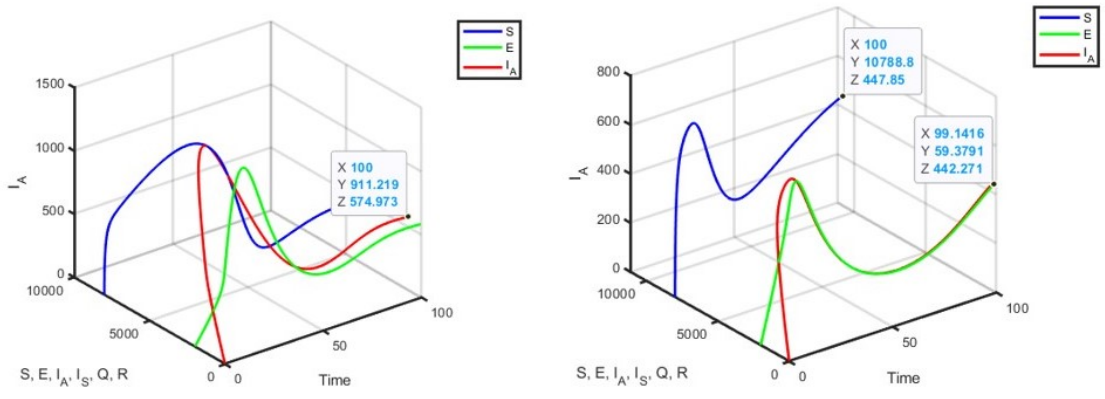
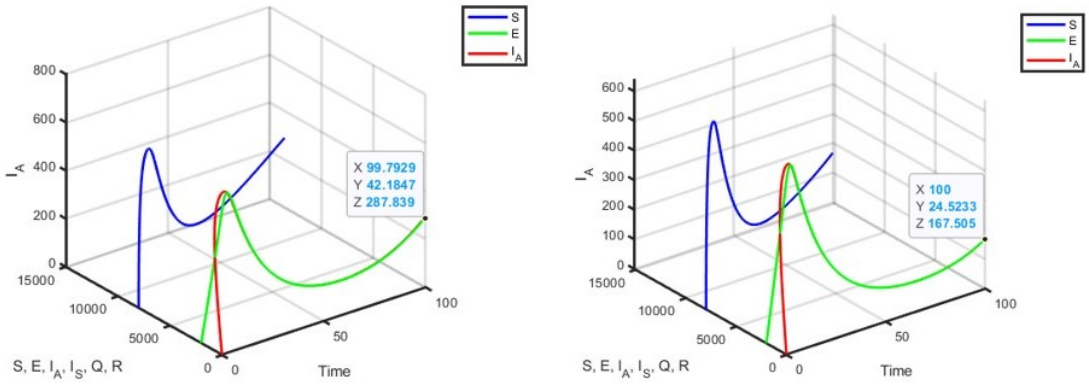


(k) V = 0, ψ = 0. (l) V = 0, ψ = 0.3

Fig.12. Impact of no disease control and low quarantine on SEI_A Sub-Model

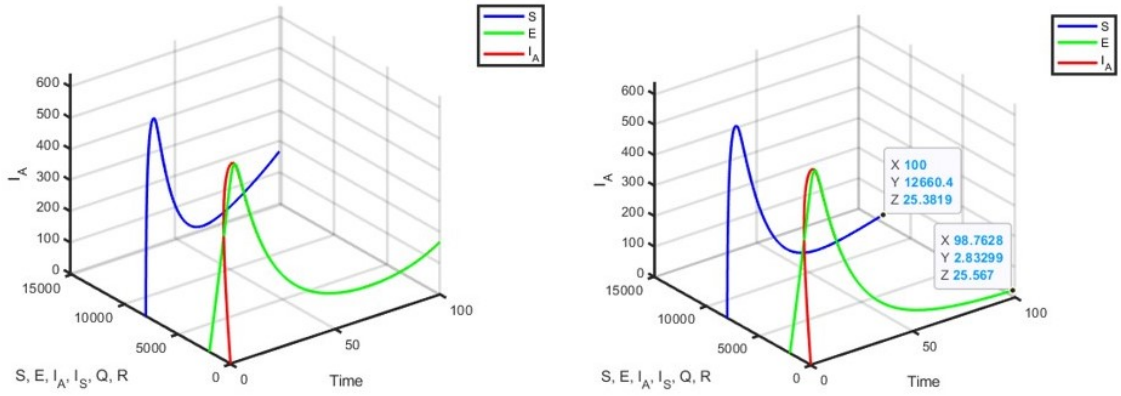
Further experiment was carried out on the sub-plot with high quarantine rate on the SEI_A sub-model. Graphical results are displayed in Fig. 13 below.



(m) V = 0, ψ = 0.6 (n) V = 0, ψ = 0.9

Fig.13.. Impact of high quarantine on SEI_A Sub-Model

The impact of high quarantine rate on I_A with low vaccination at birth with no maternal derived immunity was imputed for the sub plot. Fig. 14 displays results of changes in the graph.



(o) V = 0, ψ = 1.2 (p) V = 0.3, ψ = 1.2

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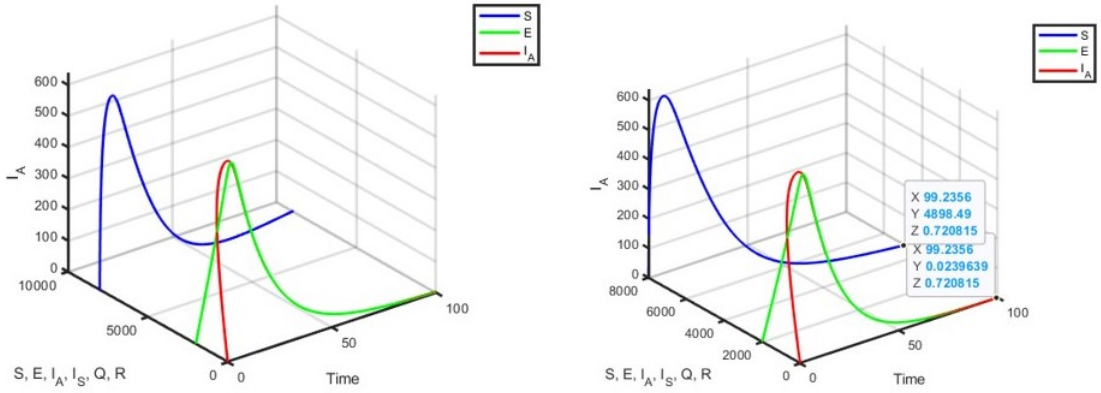
Fig.14. Impact of high quarantine on I_A and low Vaccination at birth for SEI_A Sub Model



(q) V = 0.6, ψ = 1.2 (r) V = 0.9, ψ = 1.2

Fig.15. Impact of high quarantine on I_A and improved Vaccination at birth for SEI_A Sub Model

6 Discussion of Results and Conclusion

The theoretical solutions indicates that the developed model is epidemiologically and mathematically well posed. Graphical Experiment was performed on the dynamics of disease transmission using Equation (9) with MATLAB 2021a software.

The reproduction number was determined using the Next Generation Matrix method with initial minor parameters value obtained from diphtheria disease outbreak in Thailand and Indonesia [2,12], others were assumed or fitted. R_0 representing two terms was derived to be approximately 1.6311 for the given scenario. The first term $R_{asymptomatic}$ shows reservoir of infection with value approx. 1.25, bar chart plot of I_A and I_S illustrate that combination of vaccination and quarantine will significantly die out the disease. The sensitivity analysis on R_0 parameters was manually calculated to gain a deeper understanding of the strengths and weaknesses of diphtheria predictive model. similarly, The experimental results illustrated that increasing vaccination coverage at birth (V) and implementing effective quarantine measures for both asymptomatic (ψ) and symptomatic (θ) individuals show promising results in limiting disease transmission.

Furthermore, A thorough analysis of different disease control combinations was done on SEI_AI_SQR model to evaluate their efficacy in curbing diphtheria spread. At the initial instance of disease spread, the total population was $N(0) = 10001$, where the Susceptible compartment $S(0) = 8000$, Exposed compartment $E(0) = 2000$, asymptomatic infected compartment $I_A(0) = 0$, symptomatic infected compartment $I_S(0) = 1$, Quarantined compartment $Q(0) = 0$ and Recovered compartment $R(0) = 0$. In same vein, due to the impact of asymptotic (silent disease spreader) to disease spread, a 3D sub-plots was harnessed to visualize the effects of low and high asymptomatic quarantine rates in combination with low and high vaccination at birth on the SEI_A sub-model, highlighting the importance of controlling disease spread among asymptomatic individuals

Our findings suggests that increase in vaccination rates at birth emerges as the most potent disease mitigating strategy, leading to a higher number of recovered individuals. Also, Quarantine measures demonstrated effectiveness, with maternal derived immunity showing promising results as well (That is vaccination during pregnancy is key). However, the most impactful approach involves a combination of all three strategies.

in a nut shell, These models offer valuable insights for public health professionals. Continued research will refine existing models, incorporate additional complexities, and contribute to more

effective diphtheria control strategies.

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8 Conflict of Interest

The authors declare no conflict of interest.

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