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Modified Models for Constrained Mean Absolute Deviation Portfolio Optimization

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Abstract

This study addresses portfolio optimization through a comparative analysis of models. Utilizing 15 stocks from $S\&P$ 500 over 12 years, it contrasts optimal weights and objective values of five models: Konno and Yamazaki (KY), Feinstein and Thapa (FT), Adjusted Feinstein and Thapa (AFT), and Adjusted ChiangLin et al. (ALC). Employing Pyomo in Python with GLPK solver, findings reveal KY and LC models have identical outcomes. AFT model approximates KY and LC solutions. Modified AFT and ALC models, with short selling and risk-neutral interest rates, closely mimic KY's results. The study recommends KY's model or a modified ALC model for portfolio optimization.

Keywords: Portfolio optimization, Short selling, Risk neutral interest rate, Mean absolute deviation, Investor.
MSC2010: 13P25.

1 Introduction

Investors strive to balance risk and return while managing diverse portfolios across various industries and financial instruments. They use a strategy called diversification to protect against risk, and sometimes they include risk-free assets. Diversification can be done in different ways, including simple approaches or more complex mathematical models like the Mean Absolute Deviation (MAD) approach, which is based on Harry Markowitz's pioneering Mean-Variance theory.

Markowitz's theory changed the way we think about portfolios by showing how returns and risk are related. He introduced the concept of portfolio risk and explained how diversification can lower the risk of a portfolio. This led to the idea that portfolios should be built based on their overall risk and return characteristics, not just the individual assets' characteristics. The Markowitz Efficient Frontier helps investors find the best portfolios for a specific level of risk, helping them make better investment decisions [1].

Building on Markowitz's ideas, this paper explores the evolution of portfolio theory, addressing its limitations and presenting new models that consider different viewpoints and the concept of

risk neutrality. It examines the assumptions of Markowitz's model based on investor behavior and market dynamics and then explores alternative models that aim to overcome the limitations. These models offer refined ways to optimize portfolio strategies.
This research aims to review and expand on the literature about modifying Mean Absolute Deviation (MAD) portfolio optimization models. The focus is on creating a similar optimization model that includes the idea of short-selling and risk-neutral interest rates.
This research contributes new insights to our understanding of portfolio optimization, offering a deeper look at model effectiveness in various situations. These insights will be valuable to investors and financial experts as they make decisions about building and optimizing portfolios.

2 Literature Review

Harry Markowitz [2] showed that the individual assets' risk does not solely determine the risk of an investment portfolio but also by the covariance between the assets. He established that the return of a portfolio is determined by the weighted average return of the assets, while the risk is based on the covariance between the assets in the portfolio. Markowitz's model is expressed as a non-linear quadratic programming model:

$$\begin{aligned} \min \quad & \sum_{i\in N}\sum_{j\in N} x_i x_j \sigma_{ij} \\ \text{s.t.} \quad & \\ & \sum_{j\in N} x_j r_j \geq w_0 M_0 \\ & \sum_{j\in N} x_j = M_0 \\ & x_j \geq 0 \end{aligned} \tag{2.1}$$

Parameters
$N = (1, 2, ..., n)$ is the set of assets
σ_{ij}= covariance between returns i and j
r_j= average expected return of asset j
w_0= minimum return required by the investor
M_0= the capital available for investment
x_i = capital proportion allocated to invest in asset i
x_j = capital proportion allocated to invest in asset j
σ_{ij} represents the covariance between assets i and j

The objective function seeks to minimize portfolio risk by optimizing capital allocations (x_i and x_j), which are represented as a weighted sum of covariances between asset returns. Simultaneously, constraints define the boundaries and requirements for constructing the optimal portfolio. These constraints ensure a minimum expected return, full capital allocation, and non-negativity restrictions. The optimization problem aims to find values for x_i and x_j that satisfy these constraints while minimizing overall portfolio risk according to the objective function.
To simplify the computational challenge posed by Markowitz's nonlinear model [1, 2], some researchers have suggested using alternative risk measures that are more flexible. One such measure is the mean absolute deviation (MAD) introduced by [3], which results in a linear programming model equivalent to Markowitz's model but more computationally flexible. The linear programming

model is given below:

$$\begin{aligned} \min \quad & \sum_{t \in T} p_t y_t \\ \text{s.t.} \quad & \\ & y_t + \sum_{j \in N} (r_{jt} - r_j) x_j \geq 0, \quad t \in T \\ & y_t - \sum_{j \in N} (r_{jt} - r_j) x_j \geq 0, \quad t \in T,\ 0 \leq x_j \leq u_j,\ j \in N \\ & y_t \geq 0,\ t \in T \end{aligned} \tag{2.2}$$

where,
$y_t = \left| \sum_{j \in N} (r_{jt} - r_j) x_j \right|$
$r_j = \frac{1}{T} \sum_{t \in T} r_{jt}$ is the expected rate of return of asset j
r_{jt} is the return of asset j at time $t \in T$
$T = 1, 2, \ldots, t$ is the set of periods relative to the asset realisations
u_j is the upper bound of the amount invested in asset j
$N = (1, 2, ..., n)$ is the number of assets
p_t= probability of scenario t.
Here, the objective function seeks to minimize portfolio risk, which is represented by the absolute differences between asset returns and their expected values, weighted by probabilities across different time periods. The constraints ensure that portfolio risk remains manageable while adhering to investment allocation limits and maintaining non-negative risk measures. In essence, the optimization aims to strike a balance between risk reduction and investment constraints.
Feistein and Thapa revised the MAD model proposed by [3] by adding non-negative surplus variables a_t and b_t, resulting in the following MAD model [4]:

$$\begin{aligned} \min \quad & \sum_{t \in T} a_t + b_t \\ \text{s.t.} \quad & \\ & b_t + \sum_{j \in N} (r_{jt} - r_j) x_j = a_t \\ & \sum_{j \in N} x_j = M_0 \\ & \sum_{j \in N} r_j x_j \geq w_0 M_0 \\ & x_j \geq 0,\ j \in N \\ & 0 \leq x_j \leq u_j,\ j \in N \\ & b_t \geq 0,\ t \in T. \end{aligned} \tag{2.3}$$

Furthermore, ChiangLin et al (1998) [5,6] modified the model (2.3) as follows:

$$\begin{aligned} \min \quad & 2 \sum_{t \in T} b_t \\ \text{s.t.} \quad & \\ & b_t - \sum_{j \in N} (r_{jt} - r_j) x_j \geq 0. \end{aligned} \tag{2.4}$$

Chin-Ter Chang [7] presented a modified version of model (2.3) in 2005, which can be expressed as follows:

$$\begin{aligned}
\min \quad & \sum_{t\in T} 2a_t - \sum_{j\in N}(r_{jt}-r_j)x_j \\
\text{s.t.} \quad & \\
& a_t - \sum_{j\in N}(r_{jt}-r_j)x_j \geq 0 \\
& \sum_{j\in N} x_j = 1 \\
& \sum_{j\in N} r_j x_j \geq 0 \\
& x_j \geq 0,\ j\in N \\
& 0 \leq x_j \leq u_j,\ j \in N \\
& a_t \geq 0,\ t\in T.
\end{aligned} \tag{2.5}$$

Recent research by Danjuma et al. [8] underscores the importance of optimizing wealth allocation for financial institutions in varying interest rate scenarios. Using stochastic optimization theory, particularly for CRRA utility functions, and data from CBN statistical bulletin and the Nigeria Stock Exchange FactBook, they highlight the need to shift investments from riskier assets (security and loans) to safer assets (treasury) in volatile markets, while also noting the impact of investor risk preferences on allocation.
In addition, Ogbogbo and Anokye [9] contributed insights from their study on the Ghana Stock Exchange. They used CAPM to analyze asset performance and risk, particularly in different sectors. Their findings aid investors by providing a better understanding of sector dynamics and relative risk levels.
These advancements in portfolio optimization, driven by theory and empirical research, equip financial decision-makers with versatile models to tailor investment strategies based on risk, return, and constraints. As markets evolve, ongoing innovations in portfolio theory will enhance the precision and practicality of wealth allocation strategies.

3 MAD Models with Short-selling and Risk-neutral Interest Rate

This section presents the introduction of short-selling and risk-neutral interest rate to the modified MAD models (2.3) and (2.4). The Enhanced Variance Model with Short-selling and Risk-neutral rates (EMMSR) as proposed by T.Almaadeed et al. (2022) [10, 11], is given by:

$$\begin{aligned}
\min \quad \lambda(\frac{1}{T}\sum_{t=1}^{T} y_t) - (1-\lambda)(\sum_{j=1}^{N} r_j x_j - r_c h_j x_j) \\
\text{s.t.} \quad y_t + \sum_{j\in N}(r_{jt}-r_j)x_j \geq 0, t\in T \\
y_t - \sum_{j\in N}(r_{jt}-r_j)x_j \geq 0, t\in T \\
\sum_{j\in N} x_j = 1 \\
\varepsilon_j z_j \leq x_j \leq \delta_j z_j, j = 1,2,...,N \\
y_t \geq 0, t\in T \\
if \quad x_j \geq 0, \quad then \quad h_j = 0 \\
if \quad x_j < 0, \quad then \quad 0 < h_j < 1
\end{aligned} \tag{3.1}$$

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where,
N and T denote the number of stocks and the end of investment time, respectively
x_j is the proportion of investment in jth stock
r_{jt} is the return of the jth stock at time t, $(t = 1, ..., T; j = 1, ..., N)$
r_j is the expected return of the jth stock (j = 1,...,N)
r_c is risk-neutral interest rate
ε_j and δ_j are the lower and upper bounds of the jth stock, respectively
$\lambda \in [0, 1]$ is the risk aversion parameter
ε_j is negative for short-selling
$r_c(\sum_{j=1}^{N} h_j x_j)$ shows the short rebate
$0 < h_j < 1, \forall j$, denotes the portion of the investor of the interest on the proceeds from the short-sale of stock j
In Equation (3.1), the objective function seeks to strike a delicate balance between expected returns and risk in a portfolio. It minimizes a composite expression where the first term represents the expected return scaled by a risk aversion parameter (λ), and the second term accounts for the difference between total returns and a risk-adjusted cost for each stock. The accompanying constraints cover several crucial conditions. They ensure the capital allocation sums to 1, restrict capital proportions to specified bounds, enforce non-negativity for auxiliary variables, and address short-selling by introducing an associated variable h_j for stocks with negative x_j. These constraints collectively guide the portfolio optimization process, considering investment time-frames, risk tolerance, and investment limits. The goal is to determine the optimal values of x_j and y_t that strike the right trade-off between returns and risk while adhering to these multifaceted constraints.
To linearize the above equation, the objective function was modified with additional constraints as follows:

$$
\begin{aligned}
\min \quad F_1(x, y) = \lambda(\frac{1}{T}\sum_{t=1}^{T} y_t) - (1-\lambda)(\sum_{j=1}^{N} r_j x_j - r_c d_j) \\
\text{s.t.} \quad y_t + \sum_{j\in N}(r_{jt} - r_j)x_j \geq 0, t \in T \\
y_t - \sum_{j\in N}(r_{jt} - r_j)x_j \geq 0, t \in T \\
\sum_{j\in N} x_j = 1 \\
\varepsilon_j z_j \leq x_j \leq \delta_j z_j, j = 1, 2, ..., N \\
y_t \geq 0, t \in T \\
d_j - M w_j \leq 0, \quad j = 1, 2, ..., N \\
c x_j - d_j + M w_j \geq M, \quad j = 1, 2, ..., N \\
-c x_j + d_j \leq 0, \quad j = 1, 2, ..., N \\
d_j + M w_j \geq 0, \quad j = 1, 2, ..., N \\
x_j \geq -M w_j, \quad j = 1, 2, ..., N \\
x_j \leq M(1 - w_j), \quad j = 1, 2, ..., N \\
w_j \in 0, 1, \quad j = 1, 2, ..., N \\
d_j = h_j x_j \\
\text{if } x_j \leq 0, \text{ then } w_j = 1 \text{ and thus,} \\
\text{if } x_j \leq 0, \text{ we have } w_j = 0
\end{aligned}
\tag{3.2}
$$

Where $h_j' s$ are equal to $c \in (0, 1)$ and M is a large positive constant.
The given model is a Mixed Integer Linear Programming Problem with Short Selling and Risk Neutral rate (MMSR) that includes N binary variables and $8N + 2T + 1$ constraints.

Lemma 3.1. *[10] Let F_1^* and F^* be the optimal objective function values of MMSR and EMMSR,*

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respectively. Then $F_1^* = F^*$.

Proof. In both models, based on $r_j x_j \geq 0, \quad for \quad r_j \geq 0$, we have, $x_j \geq 0$, then $h_j = 0$, and for $r_j < 0$ we have $x_j < 0$ then $h_j = c$. We let C_1 and C_2 be the set of feasible points of models MMSR and EMMSR, respectively. Since any feasible point in MMSR is also feasible for EMMSR, we conclude that $C_1 \subseteq C_2$.
Now, let (x^*, u^*) be the optimal solution of EMMSR. One can see, in MMSR, when $x_j \geq 0$, from $x_j \leq M(1 - w_j)$ and $x_j \geq -Mw_j$, we conclude $w_j = 0$. Also, when $x_j < 0$ we conclude $w_j = 1$. Furthermore, if $w_j = 0$, from $p_j - Mw_j \leq 0$ and $d_j - Mw_j \geq 0$ we have $d_j = 0$. Also if $w_j = 1$,, from $-cx_j + d_j \leq 0$ and $cx_j - d_j + Mw_j \leq M$, we have, $d_j = cx_j$
On the other hand, we consider $w_j^* = 0$ for $x_j^* \geq 0$ and $w_j^* = 1$ for $x^* < 0$ and by considering $d_j^* = h_j x_j^*$, if $x_j^* \geq 0$, then $h_j = 0$; and if $x_j^* < 0$, then $h_j = c$, (x^*, u^*, w^*, d^*) is feasible for DC.
Now, we show it is also optimal for EMMSR. Suppose by contradiction that $(x^{**}, u^{**}, w^{**}, d^{**})$ is an optimal solution for EMMSR, then $\frac{\lambda}{T}\sum_{t=1}^{T} u_t^{**} - (1-\lambda)(\sum_{j=1}^{N} r_j x_j^{**} - d_j^{**}) \leq \frac{\lambda}{T}\sum_{t=1}^{T} u_t^* - (1 - \lambda)(\sum_{j=1}^{N} r_j x_j^* - r_c d_j^*)$ Since, $d_j^{**} = h_j x_j^{**} \quad and \quad d_j^* = h_j x^* j$, if $x_j^* \leq 0$ and $x_j^{**} \leq 0$, then $h_j = 0$, and if $x_j^* \geq 0$ and $x_j^{**} \geq 0$, then $h_j = c$, we have, $\frac{\lambda}{T}\sum_{t=1}^{T} u_t^{**} - (1-\lambda)(\sum_{j=1}^{N} r_j x_j^{**} - r_c h_j x_j^{**}) \leq \frac{\lambda}{T}\sum_{t=1}^{T} u_t^* - (1-\lambda)(\sum_{j=1}^{N} r_j x_j^* - r_c h_j x_j^{**})$ The above statement shows that (x^*, u^*) is feasible for EMMSR, which contradicts the optimality of (x^*, u^*). Thus, the proof is complete. $\square$

3.1 Feinstein and Thapa (FT) and ChiangLin et al (LC) Modified MAD Models

The following is the modified MAD model (FT) as proposed by Feistein and Thapa with the inclusion of short-selling and risk-neutral interest rate:

$$
\begin{aligned}
\min \quad F_1^0(x,y) = \lambda \sum_{t=1}^{T}(a_t + b_t)) - (1-\lambda)(\sum_{j=1}^{N} r_j x_j - r_c h_j x_j) \\
\text{s.t.} \quad b_t - \sum_{j \in N}(r_{jt} - r_j)x_j = a_t, \quad t \in T \\
\sum_{j \in N} x_j = 1 \\
\sum_{j \in N} r_j x_j \geq 0 \\
x_j \geq 0, \quad j \in N \\
\varepsilon_j z_j \leq x_j \leq \delta_j z_j, j = 1, 2, ..., N \\
if \quad x_j \geq 0, \quad then \quad h_j = 0 \\
if \quad x_j < 0, \quad then \quad 0 < h_j < 1
\end{aligned}
\tag{3.3}
$$

Furthermore, the LC modified MAD model with the inclusion of short selling and risk-neutral interest rate:

$$
\begin{aligned}
\min \quad F_2^0(x,y) = 2\lambda \sum_{t=1}^{T} b_t - (1-\lambda)(\sum_{j=1}^{N} r_j x_j - r_c h_j x_j) \\
\text{s.t.} \quad b_t - \sum_{j\in N}(r_{jt} - r_j)x_j \geq 0 \\
\sum_{j\in N} x_j = 1 \\
\sum_{j\in N} r_j x_j \geq 0 \\
x_j \geq 0, \quad j \in N \\
b_t \geq 0 \\
\varepsilon_j z_j \leq x_j \leq \delta_j z_j, j = 1,2,...,N \\
if \quad x_j \geq 0, \quad then \quad h_j = 0 \\
if \quad x_j < 0, \quad then \quad 0 < h_j < 1
\end{aligned}
\tag{3.4}
$$

3.2 Formulation of Adjusted FT and LC (AFT and ALC) MAD models with Short-Selling and Risk Neutral Interest rate (EMMSR)

To tackle a linear programming problem, surplus variables are used to subtract from the constraint equations with $\geq$, while slack variables are added to the constraint equations with $\leq$. In the above MILP, we introduced two surplus variables p_t and b_t accordingly.

$$
\begin{aligned}
y_t - \sum_{j\in N}(r_{jt} - r_j)x_j = 2a_t, t \in T \\
y_t + \sum_{j\in N}(r_{jt} - r_j)x_j = 2b_t, t \in T
\end{aligned}
\tag{3.5}
$$

Using elimination or substitution method, we have respectively

$$
\begin{aligned}
&y_t = a_t + b_t, t \in T \\
&b_t - \sum_{j\in N}(r_{jt} - r_j)x_j = a_t, \quad or \\
&a_t + \sum_{j\in N}(r_{jt} - r_j)x_j = b_t
\end{aligned}
\tag{3.6}
$$

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Using substitution or elimination method, **AFT** is given as:

$$\begin{aligned}
\min \quad F_2(x,y) = \frac{\lambda}{T}\sum_{t=1}^{T}(a_t+b_t) - (1-\lambda)(\sum_{j=1}^{N} r_j x_j - r_c h_j x_j) \\
\text{s.t.} \quad b_t - \sum_{j\in N}(r_{jt}-r_j)x_j = a_t \\
\sum_{j\in N} x_j = 1 \\
\sum_{j\in N} r_j x_j \geq 0 \\
x_j \geq 0, \quad j\in N \\
\varepsilon_j z_j \leq x_j \leq \delta_j z_j, j=1,2,...,N \\
a_t, b_t \geq 0 \\
if \quad x_j \geq 0, \quad then \quad h_j = 0 \\
if \quad x_j < 0, \quad then \quad 0 < h_j < 1
\end{aligned} \tag{3.7}$$

Furthermore, substituting b_t into equation, we have **ALC**, as follows:

$$\begin{aligned}
\min \quad F_3(x,y) = \frac{\lambda}{T}\sum_{t=1}^{T}(2b_t - \sum_{j=1}^{N}(r_{jt}-r_j)x_j) \\
-(1-\lambda)(\sum_{j=1}^{N} r_j x_j - r_c h_j x_j) \\
\text{s.t.} \quad b_t - \sum_{j\in N}(r_{jt}-r_j)x_j \geq 0 \sum_{j\in N} x_j = 1 \\
\sum_{j\in N} r_j x_j \geq 0 \\
x_j \geq 0, \quad j\in N \\
\varepsilon_j z_j \leq x_j \leq \delta_j z_j, j=1,2,...,N \\
if \quad x_j \geq 0, \quad then \quad h_j = 0 \\
if \quad x_j < 0, \quad then \quad 0 < h_j < 1
\end{aligned} \tag{3.8}$$

Proposition 3.2. *Let F_1 and F_2 be the optimal solutions of EMMSR and adjusted FT EMMSR models, respectively. Then, F_1 is equivalent to F_2.*

Proof. From the objective function and the first and second constraints equations of equation (3.2), when $\sum_{j\in N}(r_{jt}-r_j)x_j \geq 0$, $y_t \geq -\sum_{j\in N}(r_{jt}-r_j)x_j$ and $y_t \geq \sum_{j\in N}(r_{jt}-r_j)x_j$ respectively. Since $y_t \geq 0$, we substitute $y_t = \sum_{j\in N}(r_{jt}-r_j)x_j$ into MMSR objective function, we have
$F_1 = \frac{\lambda}{T}\sum_{t=1}^{T}\sum_{j\in N}(r_{jt}-r_j)x_j - (1-\lambda)(\sum_{j=1}^{N} r_j x_j - r_c d_j)$
Similarly, from the objective function and the first and second constraints of equation (3.7) , when $\sum_{j\in N}(r_{jt}-r_j)x_j \geq 0$, $b_t - \sum_{j\in N}(r_{jt}-r_j)x_j = a_t$ and since $a_t \geq 0$ it implies $b_t \geq \sum_{j\in N}(r_{jt}-r_j)x_j$, then $b_t + a_t = 2\sum_{j\in N}(r_{jt}-r_j)x_j - \sum_{j\in N}(r_{jt}-r_j)x_j = \sum_{j\in N}(r_{jt}-r_j)x_j$.
The Adjusted FT (AFT) EMMSR optimal becomes:
$F_2(x,y) = \frac{\lambda}{T}\sum_{t=1}^{T}\sum_{j\in N}(r_{jt}-r_j)x_j - (1-\lambda)(\sum_{j=1}^{N} r_j x_j - r_c d_j)$
Since $F_1 = F_2$, we conclude that F_1 is equivalent to F_2. This completes the proof. □

Proposition 3.3. *Let F_2 and F_3 be the optimal solutions of AFT EMMSR and ALC EMMSR models respectively, then, F_2 is equivalent to F_3.*

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Proof. From the objective function and the first and second constraints of equation (3.8), If $\sum_{j\in N}(r_{jt}-r_j)x_j \geq 0$,$b_t \geq \sum_{j\in N}(r_{jt}-r_j)x_j$, then $2b_t = \sum_{j\in N}(r_{jt}-r_j)x_j$.
Similarly, if $\sum_{j\in N}(r_{jt}-r_j)x_j \leq 0$,$b_t \geq -\sum_{j\in N}(r_{jt}-r_j)x_j$, then $2b_t = -\sum_{j\in N}(r_{jt}-r_j)x_j$ but since $\sum_{j\in N}(r_{jt}-r_j)x_j \leq 0$,finally, we have $b_t = \sum_{j\in N}(r_{jt}-r_j)x_j \leq 0$.
Therefore, $F_3(x,y) = \frac{\lambda}{T}\sum_{t=1}^{T}\sum_{j\in N}(r_{jt}-r_j)x_j - (1-\lambda)(\sum_{j=1}^{N} r_j x_j - r_c d_j)$
Since $F_2 = F_3$,then F_2 is equivalent to F_3. In conclusion, since F_1 is equivalent to F_2 and F_2 is equivalent to F_3, therefore, we say that F_1 is equivalent to F_3. □

4 Application of the Models

This section of the study utilized the pyomo optimization package with the GLPK solver in Python to conduct the analysis. The study involved the use of SP 500 index data on 15 stocks over a 12-year period to compare the optimal weights and objective values of KY, FT, LC, AFT, and ALC models. The outcomes of the analysis are presented below.

4.1 Comparison of KY Model to the FT and LC Modification (short-selling allowed)

Table 1: Optimal Weights of KY, FT and LC (short selling allowed)

Assets	KY	FT	LC
AAPL	0.3000	0.1778	0.3000
AMZN	0.3000	0.3000	0.3000
ATCO	-0.2091	0.1262	0.2327
GIL	-0.3000	-0.3000	0.0380
GILD	0.3000	0.2559	0.1569
GOOGL	0.2724	-0.3000	-0.3000
INTC	-0.3000	0.2640	0.3000
LRCX	-0.2029	-0.0095	-0.3000
MSFT	0.3000	0.08894	0.3000
MU	0.2653	-0.1393	0.1046
NFLX	-0.0159	-0.0857	-0.0614
NVDA	0.3000	-0.0211	0.2791
SU	-0.3000	0.3000	-0.3000
TSLA	0.3000	0.0427	-0.0967
TXN	-0.0097	0.3000	0.0467
Objective Value	-0.1307	0.1402	-0.1057

Based on the table above, it can be observed that the optimal weights for each model are not the same also with variation in the values of the objective functions. Furthermore, it was observed that the optimal weights and objective functions of KY and LC models were identical, while AFT's model produced approximate results (rounded to the nearest tenth) compared to KY and LC models, as shown in the table below.

Table 2: Optimal Weights of KY, AFT and ALC (short selling allowed)

Assets	KY	AFT	ALC
AAPL	0.3000	0.3000	0.3000
AMZN	0.3000	0.3000	0.3000
ATCO	-0.2091	-0.1343	-0.2091
GIL	-0.3000	-0.0293	-0.3000
GILD	0.3000	0.3000	0.3000
GOOGL	0.2724	0.1044	0.2724
INTC	-0.3000	-0.3000	-0.3000
LRCX	-0.2029	-0.18	-0.2029
MSFT	0.3000	0.3000	0.3000
MU	0.2653	0.3000	0.2653
NFLX	-0.0159	0.0515	-0.0159
NVDA	0.3000	0.3000	0.3000
SU	-0.3000	-0.3000	-0.3000
TSLA	0.3000	0.1384	0.3000
TXN	-0.0098	-0.1507	-0.0096
Objective Value	-0.1307	-0.1221	-0.1307

4.2 Comparison of KY Model to the FT and LC Modification (short-selling not allowed)

When short selling was disallowed, we set $r_c = 0$ and $x_i \geq 0$. The outcomes are displayed in the table below:

Table 3: Optimal Weights of KY, FT and LC (short selling not allowed)

Assets	KY	FT	LC
AAPL	0.0000	0.1000	0.0000
AMZN	0.0000	0.0923	0.0725
ATCO	0.0000	0.1000	0.0000
GIL	0.0000	0.0049	0.0000
GILD	0.4023	0.1000	0.2232
GOOGL	0.0000	0.1000	0.0000
INTC	0.0000	0.1000	0.0000
LRCX	0.0000	0.0000	0.0000
MSFT	0.0000	0.1000	0.0000
MU	0.0000	0.0028	0.0000
NFLX	0.059	0.0000	0.0000
NVDA	0.3485	0.1000	0.0000
SU	0.0000	0.1000	0.1654
TSLA	0.1902	0.0000	0.0000
TXN	0.0000	0.1000	0.5389
Objective Value	-0.0586	0.6688	0.5478

Based on the table above, it can be observed that there are varying optimal weight values with distinct objective values.

4.3 Comparison of KY Model to our AFT and ALC Modification (short-selling not allowed)

The table below indicates that the optimal weights and objective values of the adjusted ChiangLin et al's model are identical to those of Konno and Yamakazi's model. On the other hand, there is a slight difference between the optimal weights and objective values of Feinstein and Thapa's model and those of Konno and Yamakazi's model:

Table 4: Optimal Weights of KY, AFT and ALC (short selling not allowed)

Assets	KY	FT	LC
AAPL	0.0000	0.0000	0.0000
AMZN	0.0000	0.0000	0.0000
ATCO	0.0000	0.0000	0.0000
GIL	0.0000	0.0000	0.0000
GILD	0.4023	0.4872	0.4023
GOOGL	0.0000	0.0000	0.0000
INTC	0.0000	0.0000	0.0000
LRCX	0.0000	0.0000	0.0000
MSFT	0.0000	0.0000	0.0000
MU	0.0000	0.0000	0.0000
NFLX	0.0591	0.0318	0.059
NVDA	0.3485	0.3516	0.3485
SU	0.0000	0.0000	0.0000
TSLA	0.1902	0.1295	0.1902
TXN	0.0000	0.0000	0.0000
Objective Value	-0.0586	-0.0571	-0.0571

5 Summary of the Analysis

The result analysis of the portfolio optimization analysis using various models shows that the KY and ALC models gave the same optimal weights and objective values, while the adjusted Feinstein and Thapa's (AFT) model gave approximately similar results to KY and ALC models when short selling and risk neutral interest rates were included.
Additionally, when short selling was not allowed, the KY, ALC, and AFT models produced similar results, while the ChiangLin et al (LC) and Feinstein and Thapa's (FT) models produced different results. This highlights the importance of considering various features in the optimization process. Moreover, it was observed that the memory usage of the models varied, with KY using the largest memory space of 233368 bytes, while LC and ALC used 176583 bytes, and AFT and FT used 165789 bytes. This information may be useful for investors and analysts who need to optimize their portfolios using different models and have limited memory capacity. Overall, the result analysis suggests that the KY, ALC, and AFT models may be effective in optimizing portfolios, depending on the specific features and constraints of the portfolio being analyzed. The use of different models and the consideration of multiple features may help to achieve better results in the optimization process.

6 Conclusion

After analyzing 15 stocks for 12 years using various portfolio optimization models, it was discovered that Feinstein and Thapa's modified MAD model was not equivalent to Konno and Yamakazi's model when additional features were added to the objective functions of both models. However, by adjusting the Feinstein and Thapa's model and adding the return to it, an approximately equivalent result was obtained to Konno and Yamakazi's model. Moreover, modifying ChiangLin's model with the inclusion of short selling and risk-neutral interest rate gave the same result as Konno and Yamakazi's model. The analysis revealed that the optimal weights and objective values of each model were not equivalent. Furthermore, the memory usage of each model during the analysis was different, with Konno and Yamakazi's model using the largest memory space.rates.

7 Recommendation

Based on the findings of this research, the following recommendations are made:

- Researchers and practitioners in the field of portfolio optimization should be cautious when using the Feinstein and Thapa modified MAD model as it may not always produce equivalent results to the Konno and Yamakazi model, especially when additional features are added to the objective function.
- The adjusted Feinstein and Thapa model with the inclusion of short selling and risk-neutral interest rate produces approximately equivalent results to the Konno and Yamakazi model. Hence, researchers and practitioners can consider using the adjusted Feinstein and Thapa model in situations where the inclusion of real features to the objective function is necessary.
- When short selling and risk-neutral interest rates are included in the analysis, the adjusted ChiangLin et al's model produces exactly the same optimal weights and objective values as the Konno and Yamakazi model. Therefore, this model can be used with confidence in situations where these features are relevant.
- Practitioners and researchers should carefully consider the memory usage of the different portfolio optimization models, especially when dealing with large datasets, in order to ensure that they have sufficient computational resources to handle the analysis.
- Future research can be focused on comparing the performance of these models with real market data to further validate their effectiveness in practical applications.

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