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The Cauchy Problem for Nonlinear Higher Order Partial Differential Equations Using Projected Differential Transform Method

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Abstract

This study applies the Projected Differential Transform Method (**PDTM**) to solve nonlinear higher-order partial differential equations (**PDEs**). The Projected Differential Transform (**PDT**) series solutions converge to exact solutions with relative ease. Numerical problems of fourth- and sixth-order nonlinear hyperbolic equations and nonlinear wave-like equations with variable coefficients are solved to show that **PDTM** can efficiently provide exact solutions for nonlinear **PDEs** of higher order with initial conditions. The results demonstrate that the **PDTM** is exceptionally accurate, efficient, and reliable and that it can be applied to many other types of nonlinear higher-order **PDEs**. Compared to the Modified Decomposition Method, the Homotopy Analysis Approach, and the Homotopy Perturbation Method, this method significantly reduces numerical computations and outperforms in accuracy.

Keywords: Projected differential transform method, Cauchy problem, Nonlinear Higher Order Partial Differential equation, Hyperbolic Equation, Wave-like Equation.
MSC2010: 35G25, 35Q70, 35Q79.

1 Introduction

The Cauchy problem for nonlinear higher order partial differential equations (**PDEs**) with source term is considered in this paper given as follows:

$$\left(\frac{\partial^2}{\partial t^2}-\alpha\frac{\partial^2}{\partial x^2}\right)^k w(x,t)=Nw(x,t)+h(x,t), \qquad k\geq 1, \tag{1.1}$$

given the initial conditions

$$\frac{\partial^i w(x,0)}{\partial t^i}=f_i(x), \qquad i=0,\dots,2k-1, \tag{1.2}$$

where $w = w(x,t)$ is the unknown function, $Nw(x,t)$ represents the nonlinear term with nonlinear differential operator N, $h(x,t)$ is the in-homogeneous or source term and $\alpha = \alpha(x,t)$ may be a constant or function of x or t. When $k > 1$, eqn.(1.1) becomes a higher order nonlinear hyperbolic equation [1], while for $k = 1$, is reduced into a wave-like equation [2, 3]. Some nonlinear problems, such as earthquake stresses, elastic waves in soil, and coupling currents in a planar multi-strand two-layer superconducting cable, are described by models similar to eqn. (1.1) - (1.2) [4], [5], [6].

Special cases of eqn. (1.1) and (1.2) have been considered in [1–3] using the Modified Decomposition Method (MDM), Homotopy Perturbation Method (HPM) and Homotopy Analysis Approach (HAA) respectively. One of the main difficulties in the use of the Adomian Decomposition Method (ADM) and its modification, MDM, is the complex calculations of the Adomian polynomials [7], [8]. While HPM needs some restrictive assumptions and functional equations to be solved in each iteration which is tricky in case of nonlinear problems [9, 10]. The HAA as applied by [3] requires initial guesses and the choice of an auxiliary linear operator is computationally stressful [10]. The projected differential transform method (PDTM) known for its simplicity and versatility in solving nonlinear initial valued problems was first presented by Jang [11]. This method does not require the computation of Adomian polynomials, any restrictive assumptions and auxiliary linear operators, or an initial guess like MDM, HPM, and HAA respectively.

The PDTM typically involves transformations and a series of iterative computations that can be carried out with the aid of symbolic computational software such as Maple or Mathematica. It requires selecting appropriate basis functions to transform each differential and function involved in the given problem and the associated initial conditions and solving a set of algebraic equations iteratively [11], [12]. The solutions obtained are usually infinite power series for suitable initial conditions, which easily converge naturally to the exact solution of the differential equations [13]. PDTM can provide accurate solutions when applied correctly, particularly for problems with well-behaved solutions [13]. It is well-suited for handling nonlinear PDEs, which are prevalent in various scientific and engineering applications. PDTM has been applied to a wide range of PDEs in various fields, including fluid dynamics, heat transfer, solid mechanics, and quantum mechanics [13], [14]. It has been used to study problems such as the nonlinear Schrödinger equation, the Korteweg-de Vries equation, and the Burgers' equation [14], [15], [16].

In recent years, researchers have continued to exploit the efficiency and robustness of PDTM in handling non-linearity in PDEs for improving other similar methods, such as variations and hybrid methods that combine PDTM with other analytic techniques, such as Laplace and Elzaki transforms, among others that cannot handle nonlinearity, for solving a broader range of PDEs [17], [18], [19], [20], [21]. Therefore, using the projected differential transform method (PDTM) proposed in [11], this paper is interested in obtaining analytical solutions to the model eqn. (1.1) - (1.2). The remaining sections are organized as follows. In the following section, we describe the method for $(n+1)$-dimensional differential transform. In "section" 3, the PDTM for nonlinear higher-order differential equations is defined along with its properties. In "section" 4, PDTM is applied to solve eqn. (1.1) - (1.2). Section 5 applies this method to some special cases of eqn.(1.1) - (1.2). In "section" 6, we conclude this paper with a concise discussion.

2 Description of $(n+1)$-Dimensional Differential Transform Method

The fundamentals of $(n+1)$-dimensional differential transform method (DTM) are presented by Jang [11] as follows:
Let the function $w(\mathbf{x},t)$ be analytic at $(\tilde{x},\tilde{t})$, then $w(\mathbf{x},t)$ can be depicted by the Taylor series,

$$w(\mathbf{x},t) = \sum_{q_1=0}^{\infty} \dots \sum_{q_n=0}^{\infty} \sum_{p=0}^{\infty} \frac{1}{q_1!\dots q_n!p!} \left[\frac{\partial^{q_1+..+q_n+p} w(\tilde{x},\tilde{t})}{\partial x_1^{q_1} \dots \partial x_n^{q_n} \partial t^p} \right] \left(\prod_{i=1}^{n} (x_i - \tilde{x}_i)^{q_i} \right) (t-\tilde{t})^p \tag{2.1}$$

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Definition 2.1. *The* $(n+1)$ *dimensional differential transform* $W(\boldsymbol{q},p)$ *of* $w(\boldsymbol{x},t)$ *at* $(\tilde{x},\tilde{t})$ *is defined by*

$$W(\mathbf{q},p)=\frac{1}{q_1!...q_n!p!}\left[\frac{\partial^{q_1+..+q_n+p}w(\tilde{x},\tilde{t})}{\partial x_1^{q_1}...\partial x_n^{q_n}\partial t^p}\right] \tag{2.2}$$

Definition 2.2. *The inverse differential transform of* $W(\boldsymbol{q},p)$ *is defined by*

$$w(\mathbf{x},t)=\sum_{q_1=0}^{\infty}...\sum_{q_n=0}^{\infty}\sum_{p=0}^{\infty}W(\mathbf{q},p)\left(\prod_{i=1}^{n}(x_i-\tilde{x}_i)^{q_i}\right)(t-\tilde{t})^p \tag{2.3}$$

It follows that, the coefficient of the Taylor series expansion of the function $w(\mathbf{x},t)$ is what is being referred to as the $(n+1)$ dimensional differential transform $W(\mathbf{q},p)$.

3 Basic Idea of the PDTM

The following fundamental definitions of PDTM are presented in [11]. Consider the Taylors series of $w(\mathbf{x},t)$ with respect to (w.r.t) some variables $X_k \in (X_1,...,X_n,t)$. Without a loss of generality, let $X_k=t$. Since $w(\mathbf{x},t)$ is analytic at $(\tilde{x},\tilde{t})$, then we can write eqn. (2.1) as follows:

$$w(\mathbf{x},t)=\sum_{p=0}^{\infty}\frac{1}{p!}\left[\frac{\partial^p}{\partial t^p}w(x_1,...,x_n,\tilde{t})\right](t-\tilde{t})^p. \tag{3.1}$$

Let P_{DT} denote the projected differential transform operator and P_{DT}^{-1} the inverse projected differential operator. The basic definitions and operations of the PDTM are introduced below.

Definition 3.1. *The projected differential transform* ***(PDT)*** $W(\boldsymbol{x},p)$ *of* $w(\boldsymbol{x},t)$ *with respect to the variable* t *at* $\tilde{t}$ *is defined by*

$$W(\mathbf{x},p)=\frac{1}{p!}\left[\frac{\partial^p}{\partial t^p}w(\mathbf{x},\tilde{t})\right]. \tag{3.2}$$

Definition 3.2. *The inverse projected differential transform of* $W(\ x,p)$ *with respect to the variable* t *is defined by*

$$w(\mathbf{x},t)=\sum_{p=0}^{\infty}W(\ \mathbf{x},p)(t-\tilde{t}_i)^p. \tag{3.3}$$

From eqn. (3.2) and (3.3), it follows that

$$w(\mathbf{x},t)=\sum_{p=0}^{\infty}\frac{1}{p!}\left[\frac{\partial^p}{\partial t^p}w(x_1,...,x_n,\tilde{t})\right](t-\tilde{t})^p. \tag{3.4}$$

which is the Taylor series of the multivariable function $w(\mathbf{x},t)$ at $\tilde{t}=0$. Based on the preceding definitions, the following table outlines the PDTM's fundamental operations.

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Table 1: Basic Operations of the PDTM

Original Function	Transformed Function
$u(x,t)=\alpha w(x,t)\pm\beta v(x,t)$	$U(x,p)=\alpha W(x,p)\pm\beta V(x,p)$
$u(x,t)=t^m x^n$	$U(x,p)=\delta(p-m)\times x^n$
$u(x,t)=\mathrm{e}^{\mathrm{t}}x^{\mathrm{n}}$	$U(x,p)=\dfrac{1}{p!}\times x^n$
$u(x,t)=\dfrac{\partial}{\partial t}w$	$U(x,p)=(p+1)W(x,p+1)$
$u(x,t)=\dfrac{\partial^m}{\partial t^m}w$	$U(x,p)=(p+1)(p+2)\dots(p+m)W(x,p+m)$
$u(x,t)=\dfrac{\partial}{\partial x}w$	$U(x,p)=\dfrac{\partial}{\partial x}W(x,p)$
$u(x,t)=\dfrac{\partial^n}{\partial x^n}w$	$U(x,p)=\dfrac{\partial^n}{\partial x^n}W(x,p)$
$u(x,t)=\alpha\dfrac{\partial^{m+n}}{\partial t^m\partial x^n}w$	$U(x,p)=\alpha(p+1)(p+2)\dots(p+m)\dfrac{\partial^n}{\partial x^n}W(x,p)$
$u(x,t)=w^2(x,t)$	$U(x,p)=\sum_{s=0}^{p}W(x,s)W(x,p-s)$
$u(x,t)=w^3(x,t)$	$U(x,p)=\sum_{r=0}^{p}\sum_{s=0}^{p-r}W(x,p-r-s)W(x,r)W(x,s)$
$u(x,t)=\left(\dfrac{\partial}{\partial t}w\right)^2$	$U(x,p)=\sum_{s=0}^{p}(p-s+1)(s+1)W(x,p-s+1)W(x,s+1)$
$u(x,t)=\left(\dfrac{\partial^2}{\partial t^2}w\right)^2$	$U(x,h)=\sum_{s=0}^{p}(p-s+1)(p-s+2)(s+1)(s+2)W(x,p-s+2)W(x,s+2)$
$u(x,t)=\left(\dfrac{\partial^2}{\partial x^2}w\right)^2$	$U(x,p)=\sum_{s=0}^{p}\left(\dfrac{\partial^2}{\partial x^2}W(x,s)\right)\left(\dfrac{\partial^2}{\partial x^2}W(x,p-s)\right)$
$u(x,t)=w\dfrac{\partial}{\partial t}w$	$U(x,p)=\sum_{s=0}^{p}(p+1)W(x,s)W(x,p+1-s)$
$u(x,t)=w\dfrac{\partial^2}{\partial t^2}w$	$U(x,p)=\sum_{s=0}^{p}(p-s+1)(p-s+2)W(x,p-s+2)W(x,s)$
$u(x,t)=w\dfrac{\partial}{\partial x}w$	$U(x,p)=\sum_{s=0}^{p}W(x,p-s)\dfrac{\partial}{\partial x}W(x,s)$
$u(x,t)=w\dfrac{\partial^2}{\partial x^2}w$	$U(x,p)=\sum_{s=0}^{p}W(x,p-s)\dfrac{\partial^2}{\partial x^2}W(x,s)$

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4 Solution of Equations (1.1)-(1.2) Using PDTM

We apply the PDTM to eqn. (1.1) in standard form. By expanding the LHS of eqn. (1.1) to k^{th} term using binomial expansion, we have

$$\left(\frac{\partial^2}{\partial t^2}-\alpha\frac{\partial^2}{\partial x^2}\right)^k w(x,t)=\sum_{r=0}^{k}(-\alpha)^{k-r}\binom{k}{r}\frac{\partial^{2r}w}{\partial t^{2r}}.\frac{\partial^{2k-2r}w}{\partial x^{2k-2r}}, \tag{4.1}$$

where the Binomial coefficient is given by $\binom{k}{r}=\dfrac{k!}{r!(k-r)!}\qquad 0\le r\le k.$
Simplifying the R.H.S of eqn. (4.1), we have

$$\left(\frac{\partial^2}{\partial t^2}-\alpha\frac{\partial^2}{\partial x^2}\right)^k w(x,t)=\frac{\partial^{2k}w}{\partial t^{2k}}+\sum_{r=0}^{k-1}(-\alpha)^{k-r}\binom{k}{r}\frac{\partial^{2k}w}{\partial t^{2r}\partial x^{2k-2r}}.$$

Thus eqn.(1.1) becomes

$$\frac{\partial^{2k}w}{\partial t^{2k}}+\sum_{r=0}^{k-1}(-\alpha)^{k-r}\binom{k}{r}\frac{\partial^{2k}w}{\partial t^{2r}\partial x^{2k-2r}}=Nw+h(x,t). \tag{4.2}$$

Rearranging eqn. (4.2) the following is obtained

$$\frac{\partial^{2k}w}{\partial t^{2k}}=Nw+h(x,t)-\sum_{r=0}^{k-1}(-\alpha)^{k-r}\binom{k}{r}\frac{\partial^{2k}w}{\partial t^{2r}\partial x^{2k-2r}}. \tag{4.3}$$

Now taking the PDT of each term of eqn. (4.3) using Table 1 above, we have

$$P_{DT}\left\{\frac{\partial^{2k}w}{\partial t^{2k}}\right\}=(p+1)(p+2)\dots(p+2k)W(x,p+2k), \tag{4.4}$$

$$P_{DT}\left\{Nw(x,t)\right\}=NW(x,p), \tag{4.5}$$

$$P_{DT}\left\{h(x,t)\right\}=H(x,p), \tag{4.6}$$

$$\begin{aligned}P_{DT}\left\{\sum_{r=0}^{k-1}(-\alpha)^{k-r}\binom{k}{r}\frac{\partial^{k}w}{\partial t^{2r}\partial x^{2k-2r}}\right\}&=(-\alpha)^k\frac{\partial^{2k}W(x,p)}{\partial x^{2k}}+(-\alpha k)^{k-1}(p+1)\times\\ &(p+2)\frac{\partial^{2k-2}W(x,p)}{\partial x^{2k-2}}+(-\frac{\alpha k(k-1)}{2})^{k-2}(p+1)(p+2)(p+3)(p+4)\frac{\partial^{2k-4}W(x,p)}{\partial x^{2k-4}}+\\ &\dots+(-\alpha k)(p+1)(p+2)\dots(p+2k-2)\frac{\partial^2 W(x,p)}{\partial x^2}.\end{aligned} \tag{4.7}$$

Plugging the transformed functions eqn. (4.4) - (4.7) into eqn. (4.3), we have

$$\begin{aligned}(p+1)(p+2)\dots(p+2k)W(x,p+2k)&=NW(x,p)+H(x,p)+(-\alpha)^k\frac{\partial^{2k}}{\partial x^{2k}}W(x,p)+\\ &(-\alpha k)^{k-1}(p+1)(p+2)\frac{\partial^{2k-2}}{\partial x^{2k-2}}W(x,p)+(-\frac{\alpha k(k-1)}{2})^{k-2}(p+1)(p+2)(p+3)(p+4)\times\\ &\frac{\partial^{2k-4}}{\partial x^{2k-4}}W(x,p)+\dots+(-\alpha k)(p+1)(p+2)\dots(p+2k-2)\frac{\partial^2}{\partial x^2}W(x,p),\end{aligned} \tag{4.8}$$

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and rearranging the transformed eqn. (4.8) reduces to a recurrence relation as follows:

$$W(x,p+2k) = \frac{1}{(p+1)(p+2)\dots(p+2k)}\Big(NW(x,p) + H(x,p) + (-\alpha)^k \frac{\partial^{2k}}{\partial x^{2k}} W(x,p) + (-\alpha k)^{k-1}(p+1)(p+2)\frac{\partial^{2k-2}}{\partial x^{2k-2}} W(x,p) + (-\frac{\alpha k(k-1)}{2})^{k-2}(p+1)(p+2)(p+3)(p+4) \times \frac{\partial^{2k-4}}{\partial x^{2k-4}} W(x,p) + \cdots + (-\alpha k)(p+1)(p+2)\cdots(p+2k-2)\frac{\partial^2}{\partial x^2} W(x,p)\Big). \tag{4.9}$$

We note here that the values of $p = 0, 1, 2, \dots, 2k-1$.
Now transforming the initial conditions, eqn. (1.2), using eqn. (3.2) we have

$$W(x,0) = F_0(x), \tag{4.10}$$

$$W(x,1) = F_1(x), \tag{4.11}$$

$$W(x,2) = \frac{1}{2!}F_2(x), \tag{4.12}$$

$$W(x,3) = \frac{1}{3!}F_3(x), \tag{4.13}$$

$$\vdots \tag{4.14}$$

$$W(x,2k-1) = \frac{1}{(2k-1)!}F_{2k-1}(x). \tag{4.15}$$

Substituting eqn. (4.10)-(4.15) into the recurrence relation eqn. (4.9) we have

$$W(x,2k) = \frac{1}{2k!}F_{2k}(x), \tag{4.16}$$

$$W(x,2k+1) = \frac{1}{(2k+1)!}F_{2k+1}(x), \tag{4.17}$$

$$\vdots \tag{4.18}$$

Using the inverse projected differential transform given in eqn. (3.3) and replacing the values of $W(x,p)$, we have

$$w(x,t) = \sum_{p=0}^{\infty} W(x,p)(t-\tilde{t})^p = F_0(x) + F_1(x)t + \frac{1}{2!}F_2(x)t^2 + \frac{1}{3!}F_3(x)t^3 + \cdots + \frac{1}{2k!}F_{2k}(x)t^{2k} + \frac{1}{(2k+1)!}F_{2k+1}(x)t^{(2k+1)} \tag{4.19}$$

Hence, the general solution of eqn. (1.1) and (1.2) is given as

$$w(x,t) = \sum_{p=0}^{\infty} \frac{1}{p!}F_p(x)t^p$$

.

5 Numerical Problems

In this section, three numerical examples of nonlinear higher-order hyperbolic equations and two nonlinear wave-like equations with variable coefficients are solved using the projected differential transform method.

5.1 Nonlinear Higher Order Hyperbolic Equations

Problem 1. If $k = 2$, $\alpha = 1$, $Nw = w - \left(\frac{\partial w}{\partial t}\right)^2$ and we set $h(x,t) = 0$, then eqn. (1.1) - (1.2) becomes

$$\frac{\partial^4 w}{\partial t^4} - 2\frac{\partial^4 w}{\partial t^2 \partial x^2} + \frac{\partial^4 w}{\partial x^4} = w - \left(\frac{\partial w}{\partial t}\right)^2, \tag{5.1}$$

Subject to the initial conditions

$$w(x,0) = \frac{\partial w(x,0)}{\partial t} = \frac{\partial^2 w(x,0)}{\partial t^2} = \frac{\partial^3 w(x,0)}{\partial t^3} = \mathrm{e}^x. \tag{5.2}$$

This is a Cauchy problem for the hyperbolic equation of fourth order [1].

On using the PDT method, eqn. (5.1) becomes

$$\begin{aligned}(p+1)(p+2)(p+3)(p+4)W(x,p+4) - 2(p+1)(p+2)\frac{\partial^2}{\partial x^2}W(x,p+2) + \frac{\partial^4}{\partial x^4}W(x,p)\\ = \sum_{s=0}^{p} W(x,p-s)W(x,s) - \sum_{s=0}^{p}(p-s+1)(s+1)W(x,p-s+1)W(x,s+1)\end{aligned} \tag{5.3}$$

Simplifying eqn. (5.3) reduces it to a recurrence relation as follows:

$$\begin{aligned}W(x,p+2) = \frac{1}{(p+1)(p+2)(p+3)(p+4)}\Big(2(p+1)(p+2)\frac{\partial^2}{\partial x^2}W(x,p+2) - \frac{\partial^4}{\partial x^4}W(x,p) +\\ \sum_{s=0}^{p} W(x,p-s)W(x,s) - \sum_{s=0}^{p}(p-s+1)(s+1)W(x,p-s+1)W(x,s+1)\Big)\end{aligned} \tag{5.4}$$

Transforming the initial conditions eqn. (5.2), using eqn. (3.2) we have

$$W(x,0) = \mathrm{e}^x \tag{5.5}$$

$$W(x,1) = \mathrm{e}^x \tag{5.6}$$

$$W(x,2) = \frac{1}{2}\mathrm{e}^x \tag{5.7}$$

$$W(x,3) = \frac{1}{6}\mathrm{e}^x \tag{5.8}$$

Substituting eqn. (5.5)-(5.8) into the recurrence relation eqn. (5.4), we have
$W(x,4) = \frac{1}{24}\mathrm{e}^x$, $W(x,5) = \frac{1}{120}\mathrm{e}^x$, $W(x,6) = \frac{1}{720}\mathrm{e}^x$, $W(x,7) = \frac{1}{5040}\mathrm{e}^x$,
$W(x,8) = \frac{1}{40320}\mathrm{e}^x$, $W(x,9) = \frac{1}{362880}\mathrm{e}^x, \ldots$
On using the inverse PDT eqn. (3.3), we have

$$\begin{aligned}w(x,t) &= \sum_{p=0}^{\infty} W(x,p)(t-\tilde{t})^p\\ &= W(x,0) + W(x,1)t + W(x,2)t^2 + W(x,3)t^3 + W(x,4)t^4 + W(x,5)t^5 + \cdots\end{aligned} \tag{5.9}$$

Substituting the values of $W(x,p)$ into the inverse projected differential transform, and factoring e^x, we obtain

$$w(x,t) = \mathrm{e}^x\left(1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \frac{1}{5!}t^5 + \frac{1}{6!}t^6 + \frac{1}{7!}t^7 + \frac{1}{8!}t^8 + \frac{1}{9!}t^9 + \ldots\right) \tag{5.10}$$

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Hence the required closed-form solution is obtained as

$$w(x,t) = \mathrm{e}^{x+t}$$

as also obtained using MDM by [1] and is presented graphically in Figure 1.

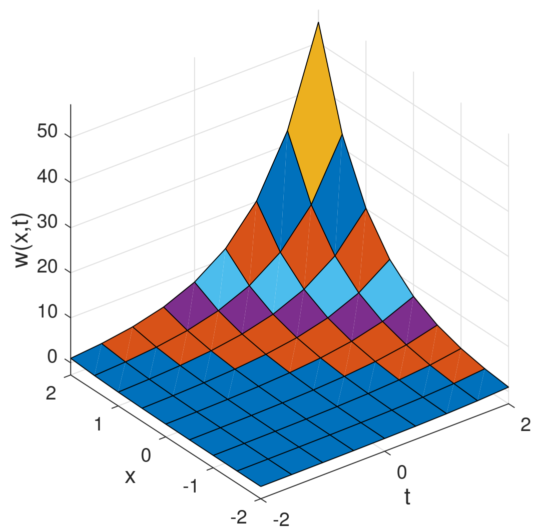

Figure 1: The Graph of Exact Solution to Problem 1 via PDTM.

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Problem 2.

If $k = 2$, $\alpha = 1$, and $Nw = \left(\frac{\partial^2}{\partial t^2} w\right)^2 - \left(\frac{\partial^2}{\partial x^2} w\right)^2 - 144w$ and $h(x,t) = 0$, then eqn. (1.1) and (1.2) becomes

$$\frac{\partial^4 w}{\partial t^4} - 2\frac{\partial^4 w}{\partial t^2 \partial x^2} + \frac{\partial^4 w}{\partial x^4} = \left(\frac{\partial^2 w}{\partial t^2}\right)^2 - \left(\frac{\partial^2 w}{\partial x^2}\right)^2 - 144w \tag{5.11}$$

with the initial conditions

$$w(x,0) = -x^4, \quad \frac{\partial w(x,0)}{\partial t} = \frac{\partial^2 w(x,0)}{\partial t^2} = \frac{\partial^3 w(x,0)}{\partial t^3} = 0 \tag{5.12}$$

which is the Cauchy problem for hyperbolic equations of fourth order [1].
Taking the **PDT** of eqn. (5.11) with respect to t, we have

$$\begin{aligned}
&(p+1)(p+2)(p+3)(p+4)W(x,p+4) - 2(p+1)(p+2)\frac{\partial^2}{\partial x^2}W(x,p+2) \\
&\quad + \frac{\partial^4}{\partial x^4}W(x,p) = \sum_{s=0}^{p}(p-s+1)(p-s+2)(s+1)(s+2)W(x,p-s+2)W(x,p-s+2)W(x,s+2) \\
&\quad - \sum_{s=0}^{p}\frac{\partial^2}{\partial x^2}W(x,s)\frac{\partial^2}{\partial x^2}W(x,p-s) - 144W(x,p)
\end{aligned} \tag{5.13}$$

Simplifying eqn. (5.13), reduces to a set of recurrence relations as follows:

$$\begin{aligned}
W(x,p+4) &= \frac{1}{(p+1)(p+2)(p+3)(p+4)}\Big(2(p+1)(p+2)\frac{\partial^2}{\partial x^2}W(x,p+2) - \frac{\partial^4}{\partial x^4}W(x,p) \\
&\quad + \sum_{s=0}^{p}(p-s+1)(p-s+2)(s+1)(s+2)W(x,p-s+2)W(x,p-s+2)W(x,s+2) \\
&\quad - \sum_{s=0}^{p}\frac{\partial^2}{\partial x^2}W(x,s)\frac{\partial^2}{\partial x^2}W(x,p-s) - 144W(x,p)\Big)
\end{aligned} \tag{5.14}$$

Transforming the initial conditions (5.12), using eqn. (3.2) we have

$$W(x,0) = -x^4 \tag{5.15}$$
$$W(x,1) = 0 \tag{5.16}$$
$$W(x,2) = 0 \tag{5.17}$$
$$W(x,3) = 0 \tag{5.18}$$

Substituting eqn. (5.15)-(5.18) into the recurrence relation eqn.(5.14), we have
$W(x,4) = 1, \quad W(x,5) = W(x,6) = W(x,7) = W(x,8) = W(x,9) = \cdots = 0.$
On using the inverse **PDT** eqn. (3.3), it follows that

$$\begin{aligned}
w(x,t) &= \sum_{p=0}^{\infty} W(\mathbf{x},p)(t-\tilde{t})^p \\
&= W(x,0) + W(x,1)t + W(x,2)t^2 + W(x,3)t^3 + W(x,4)t^4 + W(x,5)t^5 + \dots
\end{aligned} \tag{5.19}$$

Substituting the values of $W(x,p)$ into eqn.(5.19), we obtained the required exact solution as

$$w(x,t) = -x^4 + t^4.$$

This exact solution is also obtained using MDM by [1] and is depicted graphically in Figure 2.

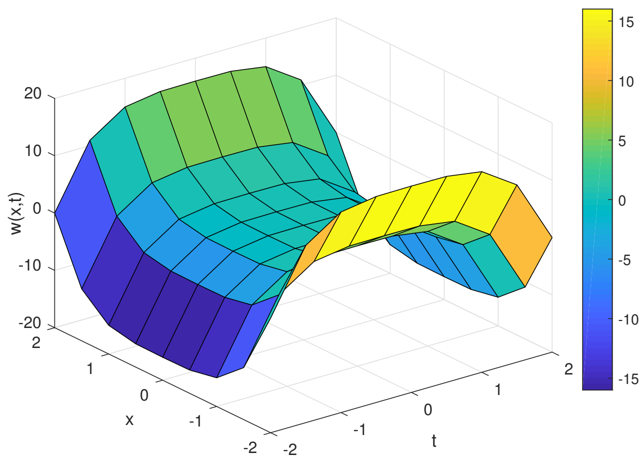

Figure 2: The graph of Exact Solution to Problem 2 via PDTM.

Problem 3.

If $k=3$, $\alpha=1$, $Nw = w\dfrac{\partial^2 w}{\partial t^2} - w\dfrac{\partial^2 w}{\partial x^2}$ and $h(x,t)=0$, then eqn. (1.1) and (1.2) becomes

$$\frac{\partial^6 w}{\partial t^6} - 3\frac{\partial^6 w}{\partial t^4 \partial x^2} + 3\frac{\partial^6 w}{\partial t^2 \partial x^4} - \frac{\partial^6 w}{\partial x^6} = w\frac{\partial^2 w}{\partial t^2} - w\frac{\partial^2 w}{\partial x^2}, \tag{5.20}$$

Subject to initial conditions

$$w(x,0) = \frac{\partial^2 w(x,0)}{\partial t^2} = \frac{\partial^4 w(x,0)}{\partial t^4} = 0, \tag{5.21}$$

and

$$\frac{\partial w(x,0)}{\partial t} = \cos(x), \quad \frac{\partial^3 w(x,0)}{\partial t^3} = -\frac{1}{6}\cos(x), \quad \frac{\partial^5 w(x,0)}{\partial t^5} = \frac{1}{120}\cos(x). \tag{5.22}$$

which is the Cauchy problem for the hyperbolic equation of sixth order [1].
Taking the **PDT** of eqn (5.20) with respect to t, we have

$$\begin{aligned}&(p+1)(p+2)(p+3)(p+4)(p+5)(p+6)W(x,p+6) - 3(p+1)(p+2)(p+3)(p+4)\times\\ &\quad \frac{\partial^2}{\partial x^2}W(x,p+4) + 3(p+1)(p+2)\frac{\partial^4}{\partial x^4}W(x,p+2) - \frac{\partial^6 W(x,p)}{\partial x^6} = \sum_{s=0}^{p}(p-s+1)\times\\ &\quad (p-s+2)W(x,p-s+2)W(x,s) - \sum_{s=0}^{p} W(x,p-s)\frac{\partial^2}{\partial x^2}W(x,s).\end{aligned} \tag{5.23}$$

Simplifying eqn. (5.23), reduces to a recurrence relation as follows,

$$\begin{aligned}W(x,p+6) &= \frac{1}{(p+1)(p+2)(p+3)(p+4)(p+5)(p+6)}\Big(3(p+1)(p+2)(p+3)(p+4)\frac{\partial^2}{\partial x^2}W(x,p+4) -\\ &\quad 3(p+1)(p+2)\frac{\partial^4}{\partial x^4}W(x,p+2) + \frac{\partial^6 W(x,p)}{\partial x^6} + \sum_{s=0}^{p}(p-s+1)(p-s+2)W(x,p-s+2)W(x,s) -\\ &\quad \sum_{s=0}^{p} W(x,p-s)\frac{\partial^2}{\partial x^2}W(x,s)\Big).\end{aligned} \tag{5.24}$$

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Transforming the initial conditions eqn. (5.21) - (5.22) using eqn. (3.2) we have

$$W(x,0)=0, \tag{5.25}$$
$$W(x,1)=\cos(x), \tag{5.26}$$
$$W(x,2)=0, \tag{5.27}$$
$$W(x,3)=-\frac{1}{6}\cos(x), \tag{5.28}$$
$$W(x,4)=0, \tag{5.29}$$
$$W(x,5)=\frac{1}{120}\cos(x), \tag{5.30}$$

Substituting eqn. (5.25)-(5.30) into the recurrence relation eqn. (5.24), we have;
$W(x,6)=0,\quad W(x,7)=-\dfrac{1}{5040}\cos(x),\quad W(x,8)=0,\quad W(x,9)=\dfrac{1}{362880}\cos(x),$
$W(x,10)=0,\quad W(x,11)=\dfrac{1}{39916800}\cos(x),\ldots$
On using the inverse **PDT** eqn. (3.3), we get

$$\begin{aligned} w(x,t) &= \sum_{p=0}^{\infty} W(x,p)(t-\tilde{t})^p \\ &= W(x,0)+W(x,1)t+W(x,2)t^2+W(x,3)t^3+W(x,4)t^4+W(x,5)t^5+\ldots \end{aligned} \tag{5.31}$$

Putting the values of $W(x,p)$, into eqn. (5.31) and factoring $\cos(x)$, we obtain

$$w(x,t)=\cos(x)\left(t-\frac{1}{3!}t^3+\frac{1}{5!}t^5-\frac{1}{7!}t^7+\frac{1}{9!}t^9-\frac{1}{11!}t^{11}+\ldots\right). \tag{5.32}$$

Thus, eqn. (5.32) is written in closed form as

$$w(x,t)=\sin(t)\cos(x).$$

This can easily be shown by substitution to be the exact solution to problem 3 as also obtained using MDM by bougoffa2007cauchy and depicted graphically in Figure 3.

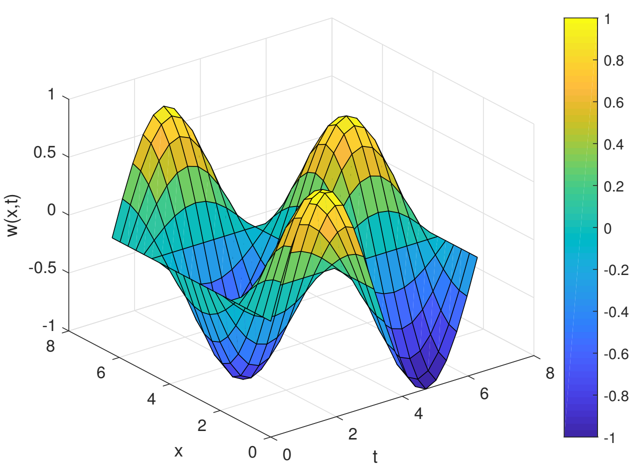

Figure 3: The graph of Exact Solution to Problem 3 via PDTM.

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5.2 Nonlinear Wave-Like Equations With Variable Coefficients

We now consider the case where $\alpha(x,t)$ is a function of x or/and t.

Problem 4. If $k = 1$, $Nw(x,t) = -\left(\dfrac{\partial w}{\partial x}\right)^2$, $\alpha(x,t) = x^2$ and $h(x,t) = xe^t + e^{2t}$, then eqn. (1.1)-(1.2) become

$$\frac{\partial^2 w}{\partial t^2} - x^2\frac{\partial^2 w}{\partial x^2} = -\left(\frac{\partial w}{\partial x}\right)^2 + xe^t + e^{2t}, \tag{5.33}$$

with initial conditions

$$w(x,0) = \frac{\partial w(x,t)}{\partial t} = x, \tag{5.34}$$

which is the Cauchy problem for nonlinear wave-like equation with variable coefficients [2,3]. Taking the **PDT** of eqn. (5.33) with regards to t, we have

$$(p+1)(p+2)W(x,p+2) - x^2\frac{\partial}{\partial x^2}W(x,p) = -\sum_{s=0}^{p}\left(\frac{\partial}{\partial x}W(x,s)\frac{\partial}{\partial x}W(x,p-s)\right) + \frac{x}{p!} + \frac{2^p}{p!}. \tag{5.35}$$

Simplifying eqn. (5.35), reduces to a recurrence relation as follows

$$W(x,p+2) = \frac{1}{(p+1)(p+2)}\left(x^2\frac{\partial}{\partial x^2}W(x,p) - \sum_{s=0}^{p}\frac{\partial}{\partial x}W(x,s)\frac{\partial}{\partial x}W(x,p-s) + \frac{x}{p!} + \frac{2^p}{p!}\right). \tag{5.36}$$

Transforming the initial conditions eqn. (5.34) using eqn. (3.2), we have

$$W(x,0) = W(x,1) = x. \tag{5.37}$$

Substituting eqn. (5.37) into the recurrence relation eqn.(5.36), the following values of $W(x,p)$ are obtained
$W(x,2) = \frac{1}{2}x, \quad W(x,3) = \frac{1}{6}x,, \quad W(x,4) = \frac{1}{24}x, \quad W(x,5) = \frac{1}{120}x, \quad W(x,6) = \frac{1}{720}x,$
$W(x,7) = \frac{1}{5040}x, \ldots$
On using the inverse PDT eqn. (3.3)

$$\begin{aligned} w(x,t) &= \sum_{p=0}^{\infty} W(x,p)(t-\tilde{t})^p \\ &= W(x,0) + W(x,1)t + W(x,2)t^2 + W(x,3)t^3 + W(x,4)t^4 + W(x,5)t^5 + \ldots \end{aligned} \tag{5.38}$$

and substituting the values of $W(x,p)$ into eqn. (5.38) and factoring x, we have

$$w(x,t) = x(1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \frac{1}{5!}t^5 + \frac{1}{7!}t^7 + \ldots) \tag{5.39}$$

Hence, eqn. (5.39) can be written in closed form as

$$w(x,t) = xe^{\mathrm{t}},$$

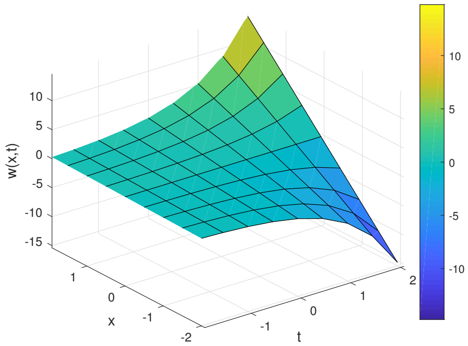
which by substitution, it can be clearly shown to be the exact solution to problem 4. Whereas the approximate-analytic solution obtained using HAA given in [3] is

$$w(x,t) = -\big(xe^t - x - tx - \frac{1}{2}t - \frac{1}{2}t^2 - \frac{1}{3}t^3 - \frac{1}{12}t^4 + \frac{1}{4}e^{2t} - \frac{1}{4}\big) \qquad for \quad x > t \tag{5.40}$$

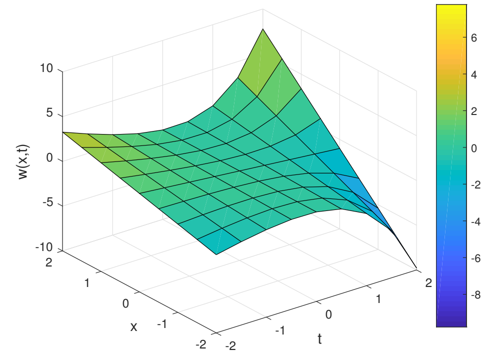
and for $x < t$,

$$w(x,t) = -\big(tx^2 - \frac{3}{2}x - t^2x - \frac{t}{3}x^3 - \frac{t^3}{3}x + \frac{t^2}{2}x^2 - \frac{e^{2t}}{4e^{2x}} - 2tx + xe^t + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{12} + \frac{e^{2t}}{4}\big) \tag{5.41}$$

Table 2 presents the absolute errors that arise from comparing the exact solution via PDTM and two-term approximated series solution via HAA [3] for select values of (x,t) in the domain $[0,1]\times[0,1]$. Additionally, the exact solution obtained via PDTM for problem 4 is depicted in Figure 4(a), while Figure 4(b) illustrates the approximate solution derived using HAA [3].



(a) Exact Solution via PDTM



(b) Two-terms approximate solution via HAA [3]

Figure 4: Comparing the exact solution via PDTM and approximate solution using HAA [3] to problem 4.

Table 2: Maximum Error of Problem 4 Using HAA [3].

$t \setminus x$	0.1	0.2	0.3	0.4	0.5	0.6	0.8	0.9
0.1	$9.0 \cdot 10^{-6}$	$9.0 \cdot 10^{-6}$	$9.0 \cdot 10^{-6}$	$9.0 \cdot 10^{-6}$	$9.0 \cdot 10^{-6}$	$9.0 \cdot 10^{-6}$	$9.0 \cdot 10^{-6}$	$9.0 \cdot 10^{-6}$
0.2	$1.4 \cdot 10^{-4}$	$1.5 \cdot 10^{-4}$	$1.5 \cdot 10^{-4}$	$1.5 \cdot 10^{-4}$	$1.5 \cdot 10^{-4}$	$1.5 \cdot 10^{-4}$	$1.5 \cdot 10^{-4}$	$1.5 \cdot 10^{-4}$
0.4	$2.0 \cdot 10^{-3}$	$2.7 \cdot 10^{-3}$	$2.9 \cdot 10^{-3}$	$2.9 \cdot 10^{-3}$	$2.9 \cdot 10^{-3}$	$2.9 \cdot 10^{-3}$	$2.9 \cdot 10^{-3}$	$2.9 \cdot 10^{-3}$
0.5	$4.7 \cdot 10^{-3}$	$6.8 \cdot 10^{-3}$	$7.5 \cdot 10^{-3}$	$7.6 \cdot 10^{-3}$	$7.6 \cdot 10^{-3}$	$7.6 \cdot 10^{-3}$	$7.6 \cdot 10^{-3}$	$7.6 \cdot 10^{-3}$
0.7	$1.7 \cdot 10^{-2}$	$2.6 \cdot 10^{-2}$	$3.1 \cdot 10^{-2}$	$3.3 \cdot 10^{-2}$	$3.4 \cdot 10^{-2}$	$3.4 \cdot 10^{-2}$	$3.4 \cdot 10^{-2}$	$3.4 \cdot 10^{-2}$
0.8	$2.9 \cdot 10^{-2}$	$4.6 \cdot 10^{-2}$	$5.5 \cdot 10^{-2}$	$6.0 \cdot 10^{-2}$	$6.2 \cdot 10^{-2}$	$6.3 \cdot 10^{-2}$	$6.3 \cdot 10^{-2}$	$6.3 \cdot 10^{-2}$
0.9	$4.6 \cdot 10^{-2}$	$7.5 \cdot 10^{-2}$	$9.2 \cdot 10^{-2}$	0.1020	0.10682	0.10888	0.10973	0.10974

Problem 5:

If $k = 1$, $Nw = w^2$, $\alpha(x,t) = t$ and $h(x,t) = -t^2x^4 - 2t^2x^3 - t^2x^2 - 2t^2$, then eqn. (1.1)-(1.2) become

$$\frac{\partial^2 w}{\partial t^2} - t\frac{\partial^2 w}{\partial x^2} = w^2 - t^2x^4 - 2t^2x^3 - t^2x^2 - 2t^2, \tag{5.42}$$

with initial conditions

$$w(x,0) = 0, \quad \frac{\partial w}{\partial t} = x + x^2, \tag{5.43}$$

which is the Cauchy problem for nonlinear wave-like equation with variable coefficients [3]. Taking the **PDT** of eqn. (5.42) with respect to t, we have

$$(p+1)(p+2)W(x,p+2) - \sum_{s=0}^{p} \delta(p-1)\frac{\partial}{\partial x^2}W(x,s) =$$
$$\sum_{s=0}^{p} W(x,s)W(x,p-s) - \delta(p-2)x^4 - 2\delta(p-2)x^3 - \delta(p-2)x^2 - 2\delta(p-2). \tag{5.44}$$

Simplifying eqn. (5.44), reduces to a recurrence relation as follows.

$$W(x,p+2) = \frac{1}{(p+1)(p+2)}\Big(\sum_{s=0}^{p} \delta(p-1)\frac{\partial}{\partial x^2}W(x,s) +$$
$$\sum_{s=0}^{p} W(x,s)W(x,p-s) - \delta(p-2)x^4 - 2\delta(p-2)x^3 - \delta(p-2)x^2 - 2\delta(p-2)\Big). \tag{5.45}$$

Transforming the initial conditions eqn. (5.43) using eqn. (3.2) we have

$$W(x,0)=0, \quad W(x,1)=x+x^2. \tag{5.46}$$

Substituting eqn. (5.46) into the recurrence relation eqn. (5.45), we have;
$W(x,2)=0, \quad W(x,3)=\frac{1}{3}, \quad W(x,4)=-\frac{1}{6}, \quad \cdots$
Thus from eqn. (3.3), it follows that

$$\begin{aligned} w(x,t) &= \sum_{p=0}^{\infty} W(x,p)(t-\tilde{t})^p, \\ &= W(x,0)+W(x,1)t+W(x,2)t^2+W(x,3)t^3+W(x,4)t^4+\cdots, \end{aligned} \tag{5.47}$$

and substituting the values of $W(x,p)$, we have

$$w(x,t)=xt+x^2t+\frac{1}{3}t^3-\frac{1}{6}t^4+\cdots. \tag{5.48}$$

Taking the two-term approximation of the series (5.48) yields the exact solution,

$$w(x,t)=(x+x^2)t.$$

Whereas the approximate-analytic solution obtain using HAA in [3] is as follows
for $x>t$

$$w(x,t)=-\frac{1}{180}t^8-\frac{2}{45}t^6x^2-\frac{2}{45}t^6x+\frac{2}{3}t^3, \tag{5.49}$$

and for $x<t$

$$\begin{aligned} w(x,t) = &\frac{79}{45}t^7x-\frac{64}{9}t^6x^2-\frac{1}{45}t^6x+\frac{473}{45}t^5x^3-\frac{253}{36}t^4x^4+\frac{8}{9}t^4x^3+ \\ &\frac{73}{45}t^3x^3-\frac{23}{6}t^3x^4+\frac{1}{3}t^3x+\frac{1}{45}t^2x^6+\frac{35}{6}t^2x^5+t^2x^2+t^2x+\frac{1}{45}tx^7-\frac{71}{18}tx^6- \\ &\frac{10}{3}tx^3+\frac{1}{360}x^8+\frac{125}{126}x^7+\frac{27}{6}x^4. \end{aligned} \tag{5.50}$$

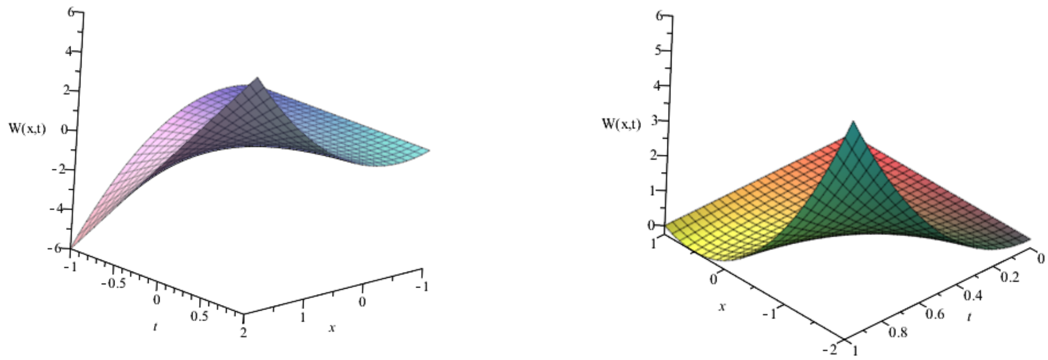
Table 3 displays the absolute errors resulting from comparing the exact solution via PDTM and two-term approximated series solution via HAA [3] for specific (x,t) values within the domain $[0,1]\times[0,1]$. Additionally, Figure 5(a) presents the exact solution obtained via PDTM for problem 5, while Figure 5(b) shows the approximate solution obtained through the HAA [3] method.

6 Conclusion

In this study, the PDTM has been applied successfully to solving higher-order nonlinear partial differential equations. We considered five numerical problems: three nonlinear higher-order hyperbolic equations and two nonlinear wave-like equations with variable coefficients type. The solutions obtained by PDTM are infinite power series for suitable initial conditions, which more often than not converge naturally to the exact solution of the differential equations. The obtained results demonstrate that the PDTM is a powerful mathematical tool for solving nonlinear PDEs of higher order. The nonlinear wave-like equations results have shown that such problems cannot be solved to obtain their exact solutions by HAA [3] but can easily be obtained via PDTM. The efficiency of the PDTM is in the ease of use and reduction in the size of computation, and it is proven to be super fast in converging to an exact solution which are the advantages of the method over MDM, HPM, and HAA.

7 Acknowledgment

All the references used in this paper are duly acknowledged and appreciated.



(a) Exact Solution via PDTM

(b) Two-terms approximate solution via HAA [3]

Figure 5: Comparing the exact solution via PDTM and approximate solution using HAA [3] to problem 5.

Table 3: Maximum Error of Problem 5 Using HAA [3].

$t \setminus x$	0.1	0.2	0.3	0.4	0.5	0.6	0.8	0.9
0.1	$3.2 \cdot 10^{-4}$	$3.3 \cdot 10^{-4}$	$3.3 \cdot 10^{-4}$	$3.3 \cdot 10^{-4}$	$3.3 \cdot 10^{-4}$	$3.3 \cdot 10^{-4}$	$3.3 \cdot 10^{-4}$	$3.3 \cdot 10^{-4}$
0.2	$1.2 \cdot 10^{-3}$	$2.6 \cdot 10^{-3}$	$2.6 \cdot 10^{-3}$	$2.6 \cdot 10^{-3}$	$2.6 \cdot 10^{-3}$	$2.6 \cdot 10^{-3}$	$2.6 \cdot 10^{-3}$	$2.6 \cdot 10^{-3}$
0.4	$3.3 \cdot 10^{-3}$	$6.6 \cdot 10^{-3}$	$1.4 \cdot 10^{-2}$	$2.1 \cdot 10^{-2}$	$2.1 \cdot 10^{-2}$	$2.1 \cdot 10^{-2}$	$2.1 \cdot 10^{-2}$	$2.1 \cdot 10^{-2}$
0.5	$5.5 \cdot 10^{-3}$	$8.7 \cdot 10^{-3}$	$1.2 \cdot 10^{-2}$	$2.6 \cdot 10^{-2}$	$4.0 \cdot 10^{-2}$	$4.1 \cdot 10^{-2}$	$4.0 \cdot 10^{-2}$	$4.0 \cdot 10^{-2}$
0.7	$1.8 \cdot 10^{-2}$	$2.2 \cdot 10^{-2}$	$1.5 \cdot 10^{-2}$	$4.8 \cdot 10^{-3}$	$8.9 \cdot 10^{-3}$	$5.2 \cdot 10^{-2}$	0.10648	0.10507
0.8	$3.7 \cdot 10^{-2}$	$4.4 \cdot 10^{-2}$	$2.9 \cdot 10^{-2}$	$1.9 \cdot 10^{-3}$	$2.0 \cdot 10^{-2}$	$1.4 \cdot 10^{-2}$	0.19175	0.14981
0.9	0.07335	$9.0 \cdot 10^{-2}$	$6.5 \cdot 10^{-2}$	$1.5 \cdot 10^{-2}$	$4.1 \cdot 10^{-2}$	$8.0 \cdot 10^{-2}$	$1.6 \cdot 10^{-2}$	0.1997

Conflicts of Interest

The authors have disclosed no conflicts of interest.

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