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# BETA WEIGHTED EXPONENTIAL DISTRIBUTION: THEORY AND APPLICATION

Nofiu I. Badmus [1*], Adebayo T. Bamiduro[2],
Monsuru A. Rauf-Animasaun[3] and Akeem A. Akingbade[4]

[1,3 & 4] Department of Statistics, Yaba College of Technology, Lagos, Nigeria
[2]Department of Mathematical Sciences, Redeemer's University, Ede, Nigeria

* Corresponding author: idoscomus2011@gmail.com

**Abstract**

[10] modified the idea of [2] in which they introduced a shape parameter to an exponential model to obtain the weighted exponential distribution. In this article, we introduced two shape parameters to the existing weighted exponential distribution to develop the beta weighted exponential distribution using the logit of beta function by [12]. We studied the statistical properties of the new distribution. Parameter estimation was done by the method of maximum likelihood estimation with R software code. We then used a data set on survival times of guinea pigs injected with different amount of tubercle bacilli to compare properties of well-known distributions with those of the new distribution. Our comparison showed the new distribution as the much more flexible and versatile.

# 1 Introduction

Introducing shape parameter(s) into an exponential model is nothing new and there are various ways of achieving this. The use of logit of Beta distribution was introduced by [12] and quite a number research works have been based on this approach. . An extension of exponential distribution was provided by [14] also using the logit of Beta distribution. [9] explored the generalized exponential distribution to provide an alternative to exponential and Weibull distributions. [2] initiated a method of obtaining weighted distributions from independently identically distributed (*i.i.d.*) random variables $Y_1$ and $Y_2$ based on the expression

$$f_Y(y) = \frac{1}{P(\alpha X_1 > X_2)} f_Y(y) F_Y(\alpha y), \alpha > 0 \tag{1.1}$$

where f(y) and F(y) were the *pdf* and *cdf* of Y respectively and α was an unknown parameter. [10] slightly modified Azzalini's approach to obtain the

Weighted Exponential (WE) distribution defined as

$$f_Y(y) = \frac{1}{P\left(\alpha^{\frac{1}{\beta}} Y_1 > Y_2\right)} f_Y(y) F_Y\left(\alpha^{\frac{1}{\beta}} y\right), \alpha, \beta > 0 \tag{1.2}$$

where α and β were unknown parameters.

# 2 Material and Method

## 2.1 The Beta Weighted exponential (BWE) Distribution

## 2.2 The Probability Density Function (pdf) of BWE Distribution

We upgraded the model by [10] using the logit of beta link function method of [12] that has been used extensively in literature. [8] investigated on beta Weibull distribution, [1] proposed the beta pareto distribution, [4] studied some statistical properties of exponentiated weighted Weibull distribution, [3] also analyzed life length of components estimates with beta-weighted Weibull distribution among others; they all used the method of logit of beta link function in their work. The link function is given as

$$g(y) = \frac{1}{B(a,b)}[F(y)]^{a-1}[1-F(y)]^{b-1}f(y),\ \mathrm{a,b} > 0 \tag{2.2.1}$$

where

$$f(y) = \frac{\lambda(1+\alpha)}{\alpha}[\exp(-\lambda y)\,(1-\exp(-\alpha\lambda y)]$$

$$F(y) = \frac{[(1+\alpha)(1-\exp(-\lambda y)+\exp\big(-\lambda y(1+\alpha)\big)-1]}{\alpha}$$

where,

α,λ and $y > 0$ are shape and scale parameter; $f(y) and\ F(y)$ are *pdf* and *cdf* respectively of parent distribution, i.e. weighed exponential distribution

We substitute the $pdf\ f(y)$ and $cdf\ F(y)$ in (2.2.1) to obtain the pdf of beta weighted exponential probability density function as

$$f_{BWE}(y) = \frac{1}{B(a,b)}\left[\frac{\big[(1+\alpha)(1-\exp(-\lambda y)+\exp\big(-\lambda y(1+\alpha)\big)-1\big]}{\alpha}\right]^{a-1}$$

$$\left[1-\frac{\big[(1+\alpha)(1-\exp(-\lambda y))+\exp\big(-\lambda y(1+\alpha)\big)-1\big]}{\alpha}\right]^{b-1}$$

$$\left[\frac{\lambda(1+\alpha)\exp(-\lambda y)\,(1-\exp(-\lambda\alpha y))}{\alpha}\right]$$

$$a, b, \lambda, \alpha, y\ \approx BWE(\alpha, \lambda, a, b, y > 0) \tag{2.2.2}$$

where α was shape parameter, λ was scale parameter, $a\ and\ b$ were additional shape parameters. Expression (2.2.2) is the $pdf$ of the proposed BWE distribution.

We set

$$W(y) = F(y) = \frac{\left[(1+\alpha)(1-\exp(-\lambda y)) + \exp\left(-\lambda y(1+\alpha)\right) - 1\right]}{\alpha}$$

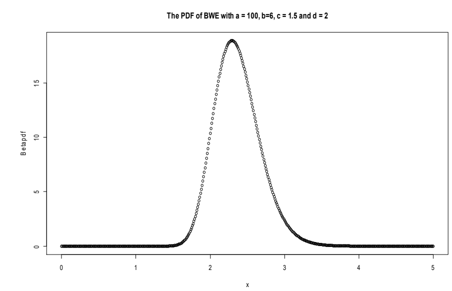
to obtain

$$\frac{dW}{dy} = \left[\frac{\lambda(1+\alpha)\,[\exp(-\lambda\ \ )(1-\exp(-\alpha\lambda y))]}{\alpha}\right]\left[\frac{\left[(1+\alpha)(1-\exp(-\lambda y))+\exp\left(-\lambda y(1+\alpha)\right)-1\right]}{\alpha}\right] \quad (2.2.3)$$

Then expression (2.2.2) becomes

$$f_{BWE}(y) = \frac{1}{B(a,b)}[W(y)]^{a-1}[1-W(y)]^{b-1}dW(y) \quad (2.2.4)$$

Figure 1 to 5 below show the PDF of BWE at values of a=100,50,20,10,5; b=6,5,3,2.5,2 ; λ=1.5; and α =2



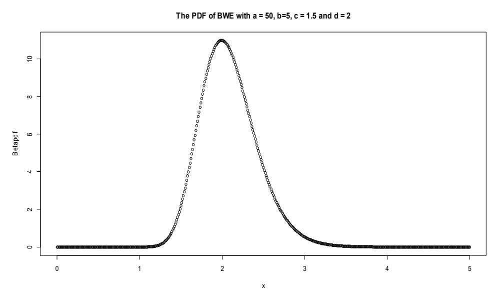
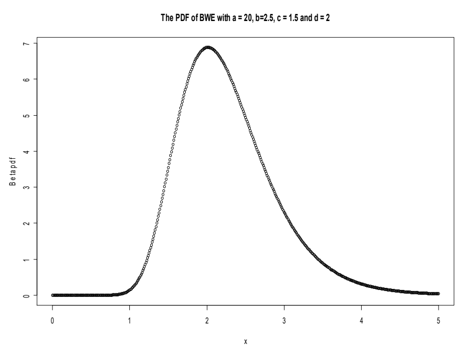




Fig 1. Plot of PDF of BWE at (a,b) = (100,6) Fig 2. Plot of PDF of BWE at (a,b) = (50,5)



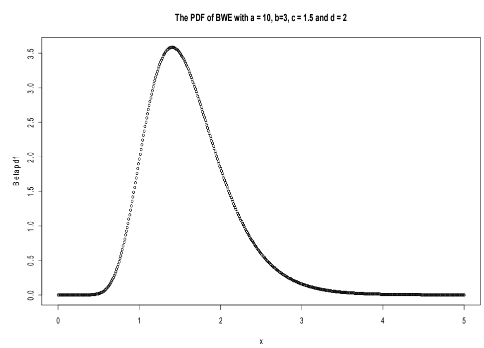




Fig 3. Plot of PDF of BWE at (a,b) = (20,3) Fig 4. Plot of PDF of BWE at (a,b) = (10,2.5)

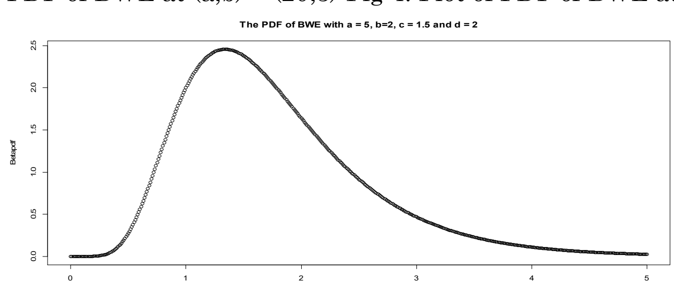



Fig 5 Plot of PDF of BWE at (a,b) = (5,2)

## 2.3 Cumulative Distribution Function (CDF)

The cumulative distribution function (cdf) of the Beta Weighted Exponential variable Y, derived from the pdf defined in equation (2.2.4) is given as

$$F_{BWE}(y) = \int_0^y f_{BWE}(t) = \frac{1}{B(a,b)} \int_0^y [W(t)]^{a-1}[1-W(t)]^{b-1} dW(t) \qquad (2.3.1)$$

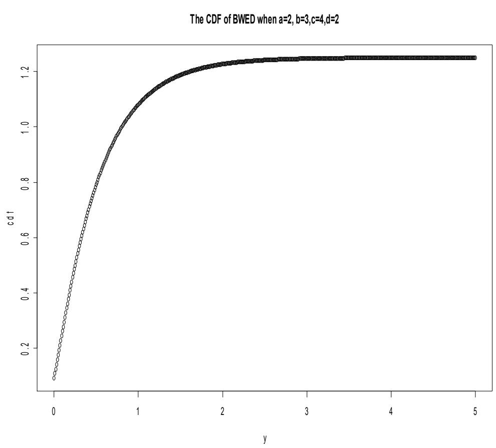
Hence,

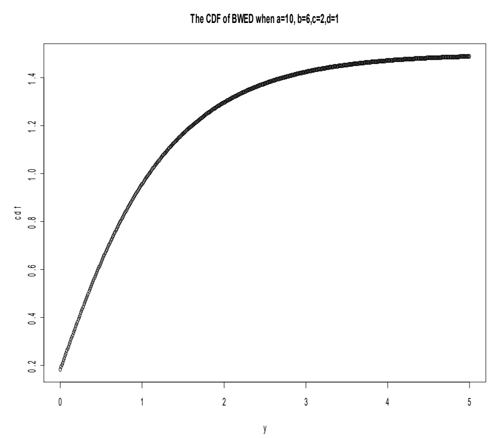
$$F_{BWE}(y) = \frac{B(y;a,b)}{B(a,b)} \qquad (2.3.2)$$

This is an Incomplete Beta function

Figures 6 and 7 show graphs of cumulative distribution function of the typical Beta weighted exponential distribution for some values of the parameters.

Figures 6 and 7 below are the graph of CDF of BWE Distribution at different values $[a = 2, b = 3, c = \alpha = 4 \; and \; d = \lambda = 2; a = 10, b = 6, c = \alpha = 2 \; and \; d = \lambda = 1]$




## 2.4 Survival Rate Function

The survival rate function of a random Beta weighted exponential variable Y with cumulative distribution function F(y) is given by

$$S_{BWE}(y) = 1 - F_{BWE}(y) \qquad (2.4.1)$$

From equation (2.3.2), the survival rate function is obtained as

$$S_{BWE}(y) = \frac{B(a,b) - B(y;a,b)}{B(a,b)} \qquad .(2.4.2)$$

Figure 8 below is a graph of an example of the survival rate function based on the Beta weighted exponential distribution

## 2.5 Hazard Rate Function

The hazard rate function of a random Beta weighted exponential variable Y is given as

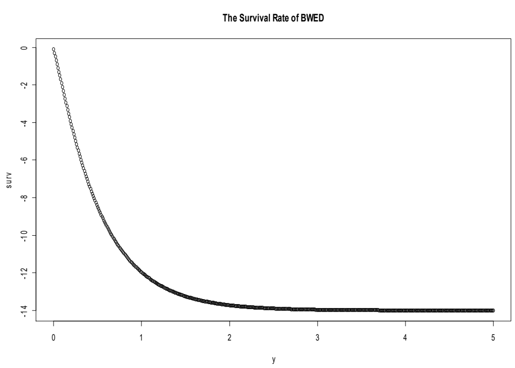
$$H_{BWE}(y) = \frac{f_{BWE}(y)}{1-F_{BWE}(y)} \tag{2.5.1}$$

Substituting from equations (6) and (8), the hazard rate function simplifies as

$$H_{BWE}(y) = \frac{[W(y)]^{a-1}[1-W(y)]^{b-1}dW(y)}{B(a,b)-B(y;a,b)} \tag{2.5.2}$$

where w(y) is as given in equation (2.2.3)

Figures 8 and 9 are the plots of the survival rate and Hazard Rate Function of BWE Distribution



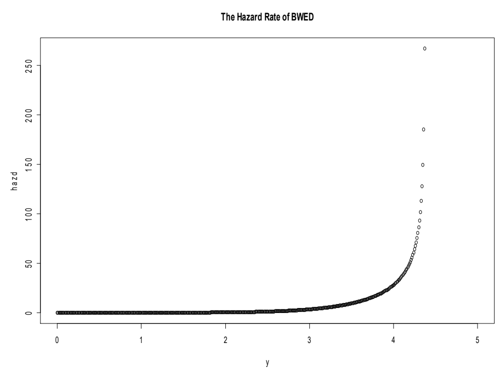




Fig. 8 The Survival Rate Function of BWE Distribution

Fig. 9 Hazard Rate Function of BWE Distribution

## 2.6 The Asymptotic Behaviour

The limits of the PDF are values of expression *(2.2.2)* as $y \to \infty$ and $y \to 0$; it is easy to show that the limits of the probability density function (PDF), $f_{BWE}(y)$, of the Beta weighted exponential variable y are given as $lim_{y\to\infty} f_{BWE}(y) = 0$ and $lim_{y\to 0} f_{BWE}(y) = 0$. This confirms that the Beta weighted exponential distribution has mode(s).

## 2.7 Special Sub-models

Expression (2.2.2) is important as a generalized model whose sub-models coincide with some extant distributions that correspond respectively to special values of the parameters. Some of the sub-models include the following

i. When, in eq. (2.2.2), $a = b = 1$ and $\alpha = \alpha^{\beta}$ $(\alpha = 0{,}1\ or\ \beta = 1)$, the Beta weighted distribution becomes the weighted Weibull distribution [13]

ii. Putting $a = 1$in (2.2.2), the generalized models give Lehmann type II exponential weighted distribution

iii. Setting $b = 1$in expression (2.2.2) reduces the generalized distribution to the exponentiated exponential weighted distribution

Also $a = b = 1$reduces the generalized distribution to the parent distribution (weighted exponential) distribution.

# 3 Moments and Generating Function (mgf)

We derived the moment generating function of our generalized Beta Weighted Exponential (BWE) distribution from the works of [11] which was also used in [3] to obtain the moment generating function (MGF) of beta generated distributions. The MGF $M(t) = E(e^{ty})$ was given as

$$M(t) = \frac{1}{B(a,b)}\sum_{i=0}^{\infty}(-1)^i \binom{b-1}{i} \int_{-\infty}^{\infty} e^{ty}[F(y)]^{a(i+1)-1} f(y)dy \qquad (3.1)$$

Substituting $pdf\ f(y)$ and $cdf\ \ F(y)$ as defined below

$$f(y) = \frac{\lambda(1+\alpha)\,[\exp(-\lambda y)\,(1-\exp(-\alpha\lambda y))]}{\alpha}$$

$$F(y) = \frac{[(1+\alpha)(1-\exp(-\lambda y)) + \exp\left(-\lambda y(1+\alpha)\right)]}{\alpha}$$

into the $MGF M(t)$ in equation (3.1) gave

$$M_{BWE}(t) = \frac{1}{B(a,b)}\sum_{i=0}^{\infty}(-1)^i \binom{b-1}{i} \int_{-\infty}^{\infty} e^{ty} \left[\frac{[(1+\alpha)(1-\exp(-\lambda y)+\exp\left(-\lambda y(1+\alpha)\right)]}{\alpha}\right]^{a(i+1)-1} \left[\frac{\lambda(1+\alpha)\,[\exp(-\lambda y)(1-\exp(-\alpha\lambda y)]}{\alpha}\right] dy \qquad (3.2)$$

[13] gave the r$^{th}$ noncentral moment of the class of Weighted Weibull distribution $WW(\alpha.\beta;\lambda)$ as

$$\mu'_{WWr} = E(Y^r) = \frac{\lambda^{\frac{r}{\beta}}}{\alpha\beta}(1+\alpha\beta)\Gamma(\tfrac{r+\beta}{\beta})\left(1-(1+\alpha\beta)^{\frac{r+\beta}{\beta}}\right) \qquad (3.3)$$

which, for $\beta = 1$, like in the case on hand, reduces to

$$\mu'_{WWr} = E(Y^r) = \frac{\lambda^r}{\alpha}(1+\alpha)\Gamma(r+1)[1-(1+\alpha)^{r+1}] \qquad (3.4)$$

The r$^{th}$ noncentral moment of the Beta Weighted Weibull distribution would be given as

$$\mu'_{BWE(r)} = \int_0^{\infty} y^r f_{BWE}(y)dy$$

i.e.

$$\mu'_{BWE(r)} = \int_0^{\infty} y^r \left\{\frac{1}{B(a,b)}[W(y)]^{a-1}[1-W(y)]^{b-1}dW(y)\right\}$$

where

$$W(y) = \frac{[\eta[1-\mathrm{c}(y)]) + [c(y)]^{\eta} - 1]}{\alpha},\ c(y) = e^{-\lambda y},\ \eta = (1+\alpha)$$

Then,

$$\mu'_{BWE(r)} = \left[\frac{\lambda^r \eta \Gamma(r+1)(1-\eta^{r+1})}{\alpha B(a,b)}\right] \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i}.$$

$$\left\{\int_0^{\infty} \left[\frac{\{\eta[1-\mathrm{c}(y)] + [c(y)]^{\eta} - 1\}}{\alpha}\right]^{a(i+1)-1} dy\right\}$$

$$= K[\lambda^r \eta \Gamma(r+1)(1-\eta^{r+1})] \qquad (3.5)$$

where $K = \frac{\sum_{i=0}^{\infty}(-1)^i \binom{b-1}{i} \int_0^{\infty}[[\eta(1-c(y)) + c(y)^{\eta} - 1]]^{a(i+1)-1} dy}{\alpha B(a\,,b)}$,

We obtained the first four non-central moments $\mu'_r$ by putting $r = 1, 2, 3$ *and* $4$ respectively in eq. 3.5; e.g. $\mu'_1$ is given as

$$\mu'_1 = E_{BWE}(y) = \left[\frac{\lambda \eta \Gamma(2)(1-\eta^2)}{\alpha B(a,b)}\right]\left[\sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i}\right]$$

Also, central moments $\mu_r, r = 1,2,3,4, \dots \dots$ are related to noncentral moments $\mu'_r$ as

$$\mu_r = \sum_{j=0}^{r} \binom{r}{j} \mu'_{r-j} \mu_j'^{j}, \text{ where } \mu'_1 = \mu \text{ and } \mu'_0 = 1 \qquad (3.6)$$

Consequently, the mean and 2nd, 3rd, and 4th moments of the BWE distribution are given as

$$\mu = \mu'_1$$
$$\mu_2 = \mu'_2 - \mu^2$$
$$\mu_3 = \mu'_3 - 3\mu\mu'_2 + 2\mu^3 \text{ and}$$
$$\mu_4 = \mu'_4 - 4\mu\mu'_3 + 6\mu^2\mu'_2 - 3\mu^4$$

where,

$$\mu'_1 = K[\lambda\eta(1-\eta^2)] \qquad (3.7)$$

$$\mu'_2 = K[\lambda^2\eta\Gamma(3)(1-\eta^3)] = 2K[\lambda^2\eta(1-\eta^3)] \qquad (3.8)$$

$$\mu'_3 = K[\lambda^3\eta\Gamma(4)(1-\eta^4)] = 6K[\lambda^3\eta(1-\eta^4)] \qquad (3.9)$$

$$\mu'_4 = K[\lambda^4\eta\Gamma(5)(1-\eta^5)] = 24K[\lambda^4\eta(1-\eta^5)] \qquad (3.10)$$

Moments measures of Skewness, $\gamma_1$ and of excess kurtosis, $\gamma_2$, are respectively given as

$$\gamma_1 = \frac{\mu_3}{\sqrt[2]{\mu_2^3}} \qquad (3.11)$$

$$\gamma_2 = \frac{\mu_4}{\mu_2^2} - 3 \qquad (3.12)$$

Figure 10 below shows plot of Skewness-kurtosis of Survival Time of Guinea Pig Data set

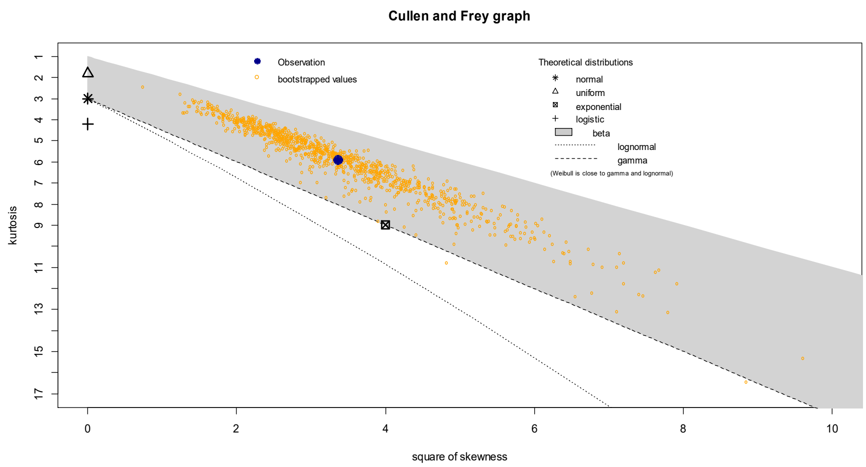



Figure 10: Skewness-kurtosis plot (Survival Time of Guinea Pig Data set)

# 4 Parameter Estimation

## 4.1 Maximum Likelihood Estimation (MLEs)

The maximum likelihood estimation (MLEs) of the parameter of $BWE(a,b,\lambda,\alpha)$ distribution has been derived from the study of [6] involving the log-likelihood function and from [3] and [4]; setting $\varphi = (a,b,c,\omega)$, where $\omega = (\lambda,\alpha)$ where $\omega$ is a vector of parameters. We had likelihood

$$L_{BWE}(\varphi) = nlogc - nlog[B(a,b)] + \sum_{i=1}^{n} \log[f(y;\varphi)] + (a-1)\sum_{i=1}^{n} \log[F(y;\varphi)]\,(b-1)\sum_{i=1}^{n} \log[1-F(y;\varphi)] \quad (4.1.1)$$

$$L_{BWE}(\varphi) = Const - nlog[B(a,b)] + \sum_{i=1}^{n} \log[f(y;\varphi)] + (a-1)\sum_{i=1}^{n} \log[F(y;\varphi)](b-1)i=1n\log\ [1-Fy;\varphi] \quad (4.1.2)$$

Taking partial derivative of (4.1.2) with respect to $(a,b,\lambda,\alpha)$, we get

$$\frac{\partial L_{BWE}(\varphi)}{\partial a} = -nlog(a,b) + (a-1)\sum_{y=1}^{n} \log[F(y;\varphi] \quad (4.1.3)$$

$$\frac{\partial L_{BWE}(\varphi)}{\partial b} = -nlog(a,b) + (b-1)\sum_{y=1}^{n} \log[1-F(y;\varphi] \quad (4.1.4)$$

$$\frac{\partial L_{BWE}(\varphi)}{\partial \lambda} = \sum_{y=1}^{n} log\left[\frac{\frac{\partial}{\partial \lambda}[f(y;\varphi)]}{f[(y;\varphi)]}\right] + (\mathrm{a}-1)\sum_{y=1}^{n} \log\left[\frac{\frac{\partial}{\partial \lambda}[F(y;\varphi)]}{F(y;\varphi)}\right] + (\mathrm{b}-1)\sum_{y=1}^{n} \log\left[\frac{\frac{\partial}{\partial \lambda}[1-F(y;\varphi)]}{1-F(y;\varphi)}\right] \quad (4.1.5)$$

$$\frac{\partial L_{BWE}(\varphi)}{\partial \alpha} \sum_{y=1}^{n} log\left[\frac{\frac{\partial}{\partial \alpha}[f(y;\varphi)]}{f[(y;\varphi)]}\right] + (\mathrm{a}-1)\sum_{y=1}^{n} \log\left[\frac{\frac{\partial}{\partial \alpha}[F(y;\varphi)]}{F(y;\varphi)}\right] + (\mathrm{b}-1)\sum_{y=1}^{n} \log\left[\frac{\frac{\partial}{\partial \lambda}[1-F(y;\varphi)]}{1-F(y;\varphi)}\right] \quad (4.1.6)$$

Equations (4.1.3) to (4.1.6) can be solved using iteration method (Newton Raphson) to obtain $\hat{a}, \hat{b}, \hat{\alpha}, \hat{\lambda}$ the MLE of $(a,b,\lambda,\alpha)$ respectively.

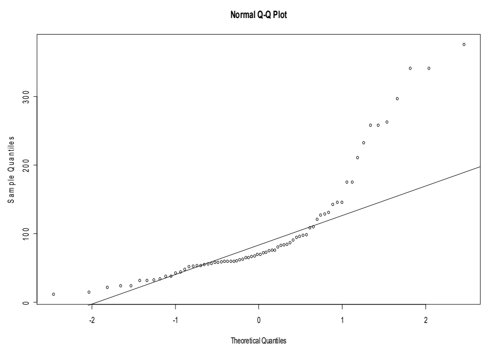
# 5 Result and Discussion

## 5.1 Result of Survival Time of Guinea Pig Data

This study used the data by [10] and studied by [5] on the survival times of guinea pigs injected with different amount of tubercle bacilli. The data are given below:

12 15 22 24 24 32 32 33 34 38 38 43 44 48 52 53 54 54 55 56 57 58 58 59 60 60 60 60 61 62 63 65 65 67 68 70 70 72 73 75 76 76 81 83 84 85 87 91 95 96 98 99 109 110 121 127 129 131 143 146 146 175 175 211 233 258 258 263 297 341 341 376.

Figures 11 – 14 shows Normal Quantile Plot, Density Plot, Histogram and Empirical Density and Cumulative Distribution plot of the data.



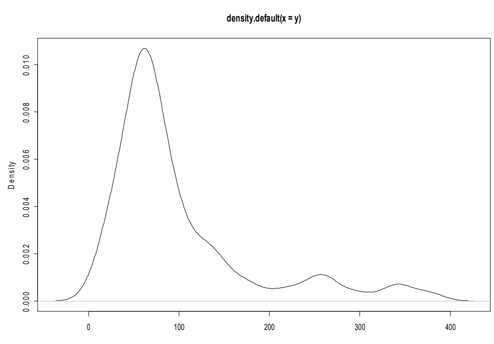




Fig. 11 Normal quantile of survival times of pigs Fig. 12 Density of survival times of pigs

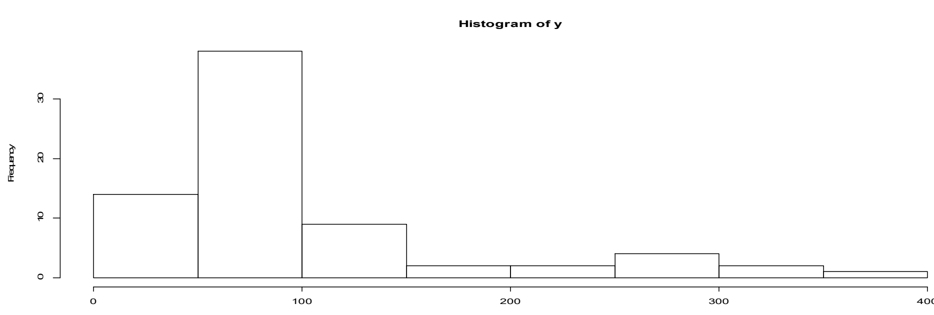


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Fig. 13:. Histogram of survival times of pigs

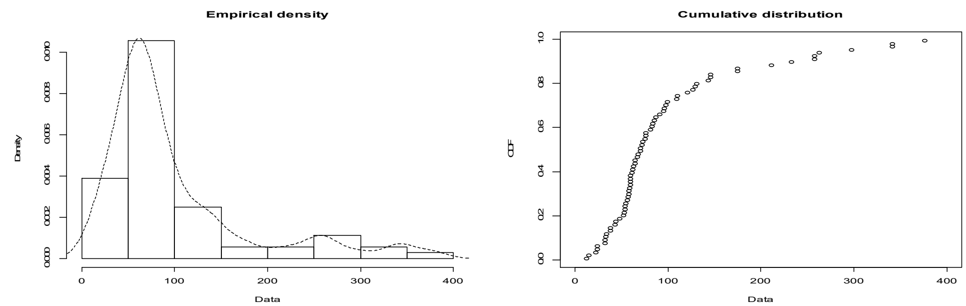


Fig. 14 Histogram and Empirical Density of survival times of pigs

| Min | 1st Q | Med | Mean | 3rd Q | Max | Skewness | Kurtosis |
|---|---|---|---|---|---|---|---|
| 12.00 | 54.75 | 70.00 | 99.82 | 112.80 | 376.00 | 1.7962 | 5.6144 |

Table 1: Descriptive Statistics for survival time of Guinea pigs in days.

| | Estimate, Standard Error in Parentheses and p-value in asterisks | | | | | | |
|---|---|---|---|---|---|---|---|
| Model | $\hat{a}$ | $\hat{b}$ | $\hat{\lambda}$ | $\hat{\alpha}$ | AIC | BIC | LogLik |
| BWE | 1.000e+00 (5.318e-04) <2e-16 *** | 9.619e-01 (7.038e-02) <2e-16 *** | 2.127e-02 (1.774e-03) <2e-16 *** | 5.210e-04 (5.343e-05) <2e-16 *** | 812.839 | 821.9457 | -402.4195 |
| EWE when b=1 | 1.11031 (5.68100) <2e-16 *** | 1 | 2.12286 (1.01403) < 2e-16 *** | 8.59040 (2.96594) 0.00378 ** | 834.7892 | 841.6192 | -414.3946 |
| LWE at a=1 | 1 | 1.11031 (5.68100) <2e-16 *** | 2.12286 (1.01403) < 2e-16 *** | 8.59040 (2.96594) 0.00378 ** | 834.7892 | 841.6192 | -414.3946 |
| WE When a= =1 | 1 | 1 | 1.10281 (1.00401) < 2e-16 *** | 8.73212 (2.96672) 0.01412** | 846.4306 | 850.9839 | -421.2153 |

Table 2: MLEs of the model parameters, the corresponding Standard Error and p-value

## 5.2 Discussion

We estimate parameters of the BWE, EWE, LWE and WE respectively using R software codes. Table 1 contains the descriptive statistics of the data set. The skewness value clearly reveals some asymmetry in the empirical distribution while the kurtosis measures the weight of tails in relation to the normal distribution [7]; the result in the Table 1 also shows excess of kurtosis; hence, the data calls for a more robust distribution/model that can accommodate all types of risk function as embedded in BWE distribution. Figure 10 shows the skewness-kurtosis feature of the data set. However, Figures 1 – 5 show the plot of the PDF of BWE distribution for different values of the parameters; the plots indicate positive and negative skewness i.e. the bathtub shape. Figures 6 and 7 show the plots of CDF of the BWE distribution at given values of the parameters and Figures 8 and 9 are plots of survival and hazard rates respectively. The hazard function is unimodal.

Table 2 above features the estimate of parameters $(\hat{a}, \hat{b}, \hat{\lambda}, \hat{\alpha})$ the log-likelihood, Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). In a nutshell, the values of AIC and BIC for the BWE distribution model were smaller than for other models; on this score, the BWE model provides a better representation of the Guinea pig data set. The asymptotic covariance matrix of the maximum likelihood estimates for the BWE distribution, which was generated from the inverse of Fisher's information matrix and is given as

$$\sum = \begin{bmatrix} 2.827714e-07 & -1.020188e-07 & 8.075292e-08 & -7.664858e-06 \\ -1.020188e-07 & 4.953495e-03 & -4.946238e-06 & -4.955421e-06 \\ 8.075292e-08 & -4.946238e-06 & 3.145951e-06 & 1.704223e-07 \\ -7.664858e-06 & -4.955421e-06 & 1.704223e-07 & 2.855199e-09 \end{bmatrix}$$

# 6 Conclusion

This article introduced the Beta Weighted Weibull Exponential distribution as a generalized model where some extant models are sub-models. The statistical properties of the proposed distribution were derived mathematically; these included the moments, moment generating function, skewness, kurtosis and maximum likelihood estimation. Thus PDF, CDF, survival rate and hazard rate function were plotted using R – software code. The Log-Likelihood of beta weighted exponential (BWE) distribution is -402.4195 and the Log-Likelihood of weighted exponential (WE) distribution is -421.2153, the selection criteria i.e. AIC and BIC of the BWE were 812.839 and 821.9457 while AIC and BIC of the WE were 846.4306 and 850.9839. The conclusion is that BWE is more efficient than WE distribution. The BWE distribution and its natural competitors were applied to guinea pig data; the result showed that BWE was a closer and more flexible representation of the distribution of the Guinea pig data.

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# Competing financial interests

We declared that no competing financial interests.

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