International Journal of Mathematical Analysis and Optimization: Theory and Applications
Vol. 2017 , pp. 201 - 210

On Plane Strain-gradient Plasticity Problem Accounting for Energetic Microforces

Adebowale Borokinni*

1. Distance Learning Institute, University of Lagos, Nigeria
*Corresponding author: aborokini@unilag.edu.ng

Article Info
Received: 27 November, 2017. Revised: 17 January, 2017.
Accepted: 22 January, 2018. Available online: 30 January, 2018.

Abstract

This study investigates the effect of energetic microforces on the plane strain problems of viscoplastic bodies. The strain-gradient plasticity theory of the Gurtin-Anand model is considered and slightly modified to account for an energetic microforce whose energy-conjugate is the divergence of the plastic strain. The modified model, gives rise to a flow rule of viscoplastic bodies which accounts for two energetic length scales associated with polar and vector microstresses through quadratic form of the defect energy. It is shown that there are no volumetric changes accompanied by elastic deformation whenever the energetic length scales are equal.

Keywords: Plane strain, Strain gradient, Flow rule, Length scales
MSC2010: 74C10

1 Introduction

Key concepts such as strain hardening, yield points and failure stresses provide useful information as to what happens during material failure [1]. Material begins to fail as soon as yielding begins, so that processes such as *size-effects*, *dislocation* etc. during strain hardening are understood from point view that stresses and free energy are functions of internal variables which account for internal, irreversible, restructuring of materials during plastic behaviour [2,3].

The inability of the classical plasticity at apprehending *size effects* has led to the study of strain gradient plasticity. Studies have revealed that size effects contribute to the strength of materials [4,5]. For instance, experimental works have shown that the strength of metallic components of metals undergoing inhomogeneous plastic flow is size dependent [6,7]. This is unaccounted for in the classical theory of plasticity. Thus to account for size effects, researchers have introduced within the constitutive theory and force balances, fields that are in themselves explicitly related to the strain gradient and hence necessitate the need for additional length scales for dimension consistency [8-11].

The strain gradient plasticity theory introduced by Gurtin [12] allows for more than one energetic material length scales associated with the Burgers tensor or gradient of the plastic strain [13,14]. The material length scales play crucial role in the weakening and strengthening properties of materials.

IJMAO

INTERNATIONAL JOURNAL OF MATHEMATICAL ANALYSIS AND OPTIMIZATION: THEORY AND APPLICATIONS
VOL. 2017 , PP. 202 - 210

However, no study has been directed towards determining the compressibility properties of materials based on these intrinsic energetic material length scales.

The purpose of this work is to use a modified form of the Gurin-Anand model [13,15] to determine the effect of material length scales on the plastic flow of materials *under plane strain*.

2 The Gurtin-Anand model

The *microforce balance* for viscoplastic bodies is given as

$$T_o = T^p - divK^p, \tag{2.1}$$

where T_o is the deviatoric part of the Cauchy stress T, T^p is called the plastic microstress tensor and K^p is the polar microstress. The force balance (2.1) is obtained by recognizing the energy conjugates of each of the microstresses and using the principle of virtual power.

The *constitutive relations* between the force-like variables and their conjugates are given by

$$T = 2\mu E_o^e + \kappa(trE^e)I \tag{2.2}$$

$$T^p = s_o \left(\frac{d^p}{d_o}\right)^m \frac{\dot{E}^p}{d^p}; K^p_{dis} = l^2 s_o \left(\frac{d^p}{d_o}\right)^m \frac{\nabla\dot{E}^p}{d^p}, \tag{2.3}$$

where s_o is a constant denoting the initial yield strength

The constitutive relations used here, assume the absence of an hardening function used in Gurtin and Anand [15] so that, the internal stress dimensioned quantity S equals the initial yield strength s_o, l is called the dissipative length scale associated with the dissipative part of the polar microstress K^p_{dis}, m is the rate sensitivity parameter, d_o is the initial flow rate and d^p is the effective flow rate defined by

$$d^p = \sqrt{|\dot{E}^p|^2 + l^2|\nabla\dot{E}^p|^2}. \tag{2.4}$$

K^p_{dis} is the dissipative part of the polar microstress K^p. μ and κ are the elastic shear and bulk modulus respectively, and E^e and E^p are the elastic and plastic strains respectively. The energetic part K^p_{en} of the polar microstress K^p is defined in component form as;

$$(K^p_{en})_{jnq} = \mu L^2 \left[E^p_{jq,n} - \frac{1}{2}\left(E^p_{jn,q} + E^p_{qn,j}\right) + \frac{1}{3}\delta_{jq}E^p_{rn,r}\right], \tag{2.5}$$

where L is called energetic length scale associated with energetic part K^p_{en} of the polar microstress K^p.

Substituting constitutive relations (2.3) and (2.5) into (2.1), we have the Gurtin-Anand flow rule given as

$$T_o + \mu L^2\left(\Delta E^p - sym_o\left(\nabla divE^p\right)\right) = s_o\left(\frac{d^p}{d_o}\right)^m \frac{\dot{E}^p}{d^p}$$

$$-l^2 s_o div\left[\left(\frac{d^p}{d_o}\right)^m \frac{\nabla\dot{E}^p}{d^p}\right]. \tag{2.6}$$

$sym_o(A)$ is the symmetric deviatoric part of tensor A defined as

$$sym_o(A) = sym(A) - \frac{1}{3}tr(A)I = \frac{1}{2}\left(A + A^T\right) - \frac{1}{3}tr(A)I,$$

where $sym(A)$ is the symmetric part of tensor A.

International Journal of Mathematical Analysis and Optimization: Theory and Applications
Vol. 2017 , pp. 203 - 210

3 Plastic flow rule accounting for an energetic internal micro-force

The basic rate-like variables in the Gurtin-Anand Model are $\dot{\vec{u}}$, $\dot{H}^e$ and $\dot{E}^p$. These variables are not independent of each other because of the kinematic relation given as

$$\nabla\vec{u} = H^e + E^p, \quad \text{with} \quad trE^p = 0, \tag{3.1}$$

where $\vec{u}$ is the displacement vector of an arbitrary point $\vec{x}$ in the deformed body and H^e is the elastic part of the displacement gradient $\nabla\vec{u}$. The microforce balance (2.1) of the Gurtin-Anand model [15] are obtained through the principle of virtual power following the assumptions that

- An elastic macrostress T is energy-conjugate to H^e
- A plastic microstress T^p is energy-conjugate to E^p
- A polar plastic microstress K^p is energy-conjugate to ∇E^p.

These assumptions allow the power expenditure $W_{int}(P)$ due to internal agencies over a sub-region P of the body B be written in the form

$$W_{int}(P) = \int_P \left(T : \dot{H}^e + T^p : \dot{E}^p + K^p \vdots \nabla\dot{E}^p \right) dV. \tag{3.2}$$

The operations $:$ and $\vdots$ are the inner products on second-order and third-order tensors respectively.

Suppose there exists an *energetic internal microforce* $\vec{\chi}$ assumed energy-conjugate to the divergence $divE^p$ of the plastic strain so that the microforce balance of Gurtin -Anand model takes the *modified* form[1] [13]

$$T_o = T^p - sym_o(\nabla\vec{\chi}) - divK^p \tag{3.3}$$

with

$$\vec{\chi} = \frac{\partial\psi}{\partial divE^p}, \tag{3.4}$$

where ψ is the free energy measured per unit volume. The free energy assumes the form [13]

$$\psi = \mu|E^e_o|^2 + \frac{1}{2}\kappa|trE^e|^2 + \frac{1}{2}\mu L^2|\nabla\times E^p|^2 + \frac{1}{2}\mu Q^2|divE^p|^2, \tag{3.5}$$

where Q is the energetic length scale due to the presence of the energetic internal microforce $\vec{\chi}$.

Following from equations (3.4) and (3.5) we have

$$\vec{\chi} = \mu Q^2 divE^p. \tag{3.6}$$

By substituting equations (2.3), (2.5) and (3.6) in (3.3), we have the flow rule accounting for an energetic dependence on the internal microforce $\vec{\chi}$ given as

$$T_o + \mu\left(L^2\Delta E^p + (Q^2 - L^2)sym_o(\nabla divE^p)\right) = s_o\left(\frac{d^p}{d_o}\right)^m \frac{\dot{E}^p}{d^p}$$

$$-l^2 s_o div\left[\left(\frac{d^p}{d_o}\right)^m \frac{\nabla\dot{E}^p}{d^p}\right]. \tag{3.7}$$

The microscopic boundary conditions following that of Gurtin and Anand [15] assume the form

$$\dot{E}^p = 0 \quad \text{on} \quad \Gamma_h \quad \text{and} \quad K^p\vec{n} + sym_o(\vec{\chi}\otimes\vec{n}) = \vec{0} \quad \text{on} \quad \Gamma_f, \tag{3.8}$$

where $\vec{n}$ is the outward unit normal on the surface Γ_f. Γ_h represents part of the boundary Γ that is micro-hard and Γ_f represents part of the boundary Γ that is micro-free with $\Gamma = \Gamma_h \cup \Gamma_f$ and $\Gamma_h \cap \Gamma_f$ is a smooth curve.

[1] This microforce balance is obtained via the principle of virtual power and the use of divergence theorem

IJMAO

International Journal of Mathematical Analysis and Optimization: Theory and Applications
Vol. 2017 , pp. 204 - 210

4 Plane strain-gradient formulation of visco-plastic media

Here we shall consider the *plane strain-gradient plasticity* theory of the modified Gurtin-Anand model given by (3.7). This will enable us to determine the effect of the energetic internal microforce on the flow rule of a visco-plastic material. As an aid to finite element implementation, the weak formulation of the flow rule for the following cases would be derived:

- $m = 1$
- $m \geq 0$ and $Q = L$.

Let x_1-coordinate axis be represented by the x-axis and the x_2-axis be represented as $y-$axis. Suppose a body undergoes a *plane strain* so that the displacement vector $\vec{u}$ can be written as

$$\vec{u} = u_1(x,y)\vec{e}_1 + u_2(x,y)\vec{e}_2, \tag{4.1}$$

where $\vec{e}_1$ and $\vec{e}_2$ are unit vectors parallel to the x and y- axes respectively. The non-vanishing components of the strain are defined by

$$E_{11} = \frac{\partial u_1}{\partial x}, \quad E_{22} = \frac{\partial u_2}{\partial y} \quad \text{and} \quad E_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right). \tag{4.2}$$

By plastic incompressibility given in $(3.1)_2$ we have

$$E^p_{11} + E^p_{22} = 0. \tag{4.3}$$

Since the non-vanishing components of the plastic strain are E^p_{11}, E^p_{22} and E^p_{12} then, the components of the plastic strain can be written as a column matrix in the form

$$E^p = \begin{pmatrix} E^p_{11} \\ \\ E^p_{22} \\ \\ E^p_{12} \end{pmatrix}, \tag{4.4}$$

so that the laplacian of the plastic strain is defined as

$$\Delta E^p = \begin{pmatrix} \frac{\partial^2 E^p_{11}}{\partial x^2} + \frac{\partial^2 E^p_{11}}{\partial y^2} \\ \\ \frac{\partial^2 E^p_{22}}{\partial x^2} + \frac{\partial^2 E^p_{22}}{\partial y^2} \\ \\ \frac{\partial^2 E^p_{12}}{\partial x^2} + \frac{\partial^2 E^p_{12}}{\partial y^2} \end{pmatrix} \tag{4.5}$$

and the symmetric deviatoric part of gradient of the divergence of the plastic strain is given by

$$sym_o(\nabla div E^p) = \begin{pmatrix} \frac{1}{3}\left(2\frac{\partial^2 E^p_{11}}{\partial x^2} + \frac{\partial^2 E^p_{12}}{\partial x \partial y} - \frac{\partial^2 E^p_{22}}{\partial y^2}\right) \\ \\ \frac{1}{3}\left(2\frac{\partial^2 E^p_{22}}{\partial y^2} + \frac{\partial^2 E^p_{12}}{\partial x \partial y} - \frac{\partial^2 E^p_{11}}{\partial x^2}\right) \\ \\ \frac{1}{2}\left(\frac{\partial^2 E^p_{11}}{\partial x \partial y} + \frac{\partial^2 E^p_{12}}{\partial x^2} + \frac{\partial^2 E^p_{12}}{\partial y^2} + \frac{\partial^2 E^p_{22}}{\partial x \partial y}\right) \end{pmatrix}. \tag{4.6}$$

The components of the macrostress in (2.2) are defined by

$$T_{11} = \kappa\theta + 2\mu\left(E^e_{11} - \frac{1}{3}(E^e_{11} + E^e_{22})\right)$$

International Journal of Mathematical Analysis and Optimization: Theory and Applications
Vol. 2017 , pp. 205 - 210

so that

$$T_{11} = \left(\frac{3\kappa - 2\mu}{3}\right)\theta + 2\mu E^e_{11}, \tag{4.7}$$

where $\theta = E^e_{11} + E^e_{22} = E_{11} + E_{22}$ with E_{11} and E_{22} the components of the total strain.

Similarly,

$$T_{22} = \left(\frac{3\kappa - 2\mu}{3}\right)\theta + 2\mu E^e_{22}, \tag{4.8}$$

and

$$T_{12} = 2\mu E^e_{12}. \tag{4.9}$$

Recall that under plane strain T_{33} is in general non-zero, where

$$T_{33} = \left(\frac{3\kappa - 2\mu}{3}\right)\theta. \tag{4.10}$$

Let $M = \frac{3\kappa-2\mu}{3}$ and $N = \frac{3\kappa+4\mu}{3}$. The components of the macrostress in terms of M and N are defined by

$$T_{11} = N\frac{\partial u_1}{\partial x} + M\frac{\partial u_2}{\partial y} - 2\mu E^p_{11}. \tag{4.11}$$

$$T_{22} = N\frac{\partial u_2}{\partial y} + M\frac{\partial u_1}{\partial x} - 2\mu E^p_{22}. \tag{4.12}$$

$$T_{12} = 2\mu\left(\frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) - E^p_{12}\right). \tag{4.13}$$

and

$$T_{33} = M\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right). \tag{4.14}$$

Recall that the deviatoric part of the macrostress T_o is defined by

$$T_o = T - \frac{1}{3}tr(T)I, \tag{4.15}$$

where I is the rank-two unit tensor. Clearly

$$T_{o11} = T_{11} - \frac{1}{3}(T_{11} + T_{22} + T_{33})$$

$$T_{o11} = \frac{2}{3}\left(N\frac{\partial u_1}{\partial x} + M\frac{\partial u_2}{\partial y} - 2\mu E^p_{11}\right) - \frac{1}{3}\left(N\frac{\partial u_2}{\partial y} + M\frac{\partial u_1}{\partial x} - 2\mu E^p_{22}\right)$$
$$-\frac{1}{3}M\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right).$$

By simplifying it is obvious that

$$T_{o11} = 2\mu\left(\frac{2}{3}\frac{\partial u_1}{\partial x} - \frac{1}{3}\frac{\partial u_2}{\partial y} - E^p_{11}\right). \tag{4.16}$$

Similarly,

$$T_{o22} = 2\mu\left(\frac{2}{3}\frac{\partial u_2}{\partial y} - \frac{1}{3}\frac{\partial u_1}{\partial x} - E^p_{22}\right), \tag{4.17}$$

$$T_{o12} = 2\mu\left(\frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) - E^p_{12}\right). \tag{4.18}$$

and

$$T_{o33} = -\frac{2}{3}\mu\left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y}\right). \tag{4.19}$$

Without loss of generality and for the purpose of determining the significance of length scale Q, assume that rate-sensitivity parameter $m = 1$, then the flow rule (3.7) becomes a linear partial differential equation given as

$$T_o + \mu\left[L^2\Delta E^p + (Q^2 - L^2)sym_o(\nabla div E^p)\right]$$

$$= \frac{s_o}{d_o}\dot{E}^p - \frac{l^2}{d_o}s_o div\nabla\dot{E}^p. \tag{4.20}$$

For this case of rate-sensitvity parameter the flow rule in component form after substituting (4.16)-(4.18) in (4.20) is given in component form as follows

$$\begin{aligned} 0 = 2\mu E^p_{11} + \frac{s_o}{d_o}\dot{E}^p_{11} - 2\mu\left(\frac{2}{3}\frac{\partial u_1}{\partial x} - \frac{1}{3}\frac{\partial u_2}{\partial y}\right) - \frac{\partial}{\partial x}\left(\frac{\mu}{3}(L^2 + 2Q^2)\frac{\partial E^p_{11}}{\partial x}\right) \\ -\frac{\partial}{\partial x}\left(\frac{\mu}{3}(Q^2 - L^2)\frac{\partial E^p_{12}}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\mu}{3}(2L^2 + Q^2)\frac{\partial E^p_{11}}{\partial y}\right) \\ -\frac{\partial}{\partial x}\left(l^2\frac{s_o}{d_o}\frac{\partial \dot{E}^p_{11}}{\partial x}\right) - \frac{\partial}{\partial y}\left(l^2\frac{s_o}{d_o}\frac{\partial \dot{E}^p_{11}}{\partial y}\right) \end{aligned} \tag{4.21}$$

$$\begin{aligned} 0 = 2\mu E^p_{22} + \frac{s_o}{d_o}\dot{E}^p_{22} - 2\mu\left(\frac{2}{3}\frac{\partial u_2}{\partial y} - \frac{1}{3}\frac{\partial u_1}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\mu}{3}(L^2 + 2Q^2)\frac{\partial E^p_{22}}{\partial y}\right) \\ -\frac{\partial}{\partial y}\left(\frac{\mu}{3}(Q^2 - L^2)\frac{\partial E^p_{12}}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\mu}{3}(2L^2 + Q^2)\frac{\partial E^p_{22}}{\partial x}\right) \\ -\frac{\partial}{\partial y}\left(l^2\frac{s_o}{d_o}\frac{\partial \dot{E}^p_{22}}{\partial y}\right) - \frac{\partial}{\partial x}\left(l^2\frac{s_o}{d_o}\frac{\partial \dot{E}^p_{22}}{\partial x}\right) \end{aligned} \tag{4.22}$$

and

$$\begin{aligned} 0 = 2\mu E^p_{12} + \frac{s_o}{d_o}\dot{E}^p_{12} - \mu\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\mu}{2}(L^2 + Q^2)\frac{\partial E^p_{12}}{\partial x}\right) \\ -\frac{\partial}{\partial y}\left(\frac{\mu}{2}(L^2 + Q^2)\frac{\partial E^p_{12}}{\partial y}\right) - \frac{\partial}{\partial x}\left(l^2\frac{s_o}{d_o}\frac{\partial \dot{E}^p_{12}}{\partial x}\right) - \frac{\partial}{\partial y}\left(l^2\frac{s_o}{d_o}\frac{\partial \dot{E}^p_{12}}{\partial y}\right). \end{aligned} \tag{4.23}$$

4.0.1 Effect of internal microforce on the flow rule

The flow rule presented in (4.21)-(4.23) for the special case of $Q = L$ will result to the following equations

$$\begin{aligned} 0 = 2\mu E^p_{11} + \frac{s_o}{d_o}\dot{E}^p_{11} - 2\mu\left(\frac{2}{3}\frac{\partial u_1}{\partial x} - \frac{1}{3}\frac{\partial u_2}{\partial y}\right) - \frac{\partial}{\partial x}\left(\mu L^2\frac{\partial E^p_{11}}{\partial x}\right) \\ -\frac{\partial}{\partial y}\left(\mu L^2\frac{\partial E^p_{11}}{\partial y}\right) - \frac{\partial}{\partial x}\left(l^2\frac{s_o}{d_o}\frac{\partial \dot{E}^p_{11}}{\partial x}\right) - \frac{\partial}{\partial y}\left(l^2\frac{s_o}{d_o}\frac{\partial \dot{E}^p_{11}}{\partial y}\right) \end{aligned} \tag{4.24}$$

$$0 = 2\mu E^p_{22} + \frac{s_o}{d_o}\dot{E}^p_{22} - 2\mu\left(\frac{2}{3}\frac{\partial u_2}{\partial y} - \frac{1}{3}\frac{\partial u_1}{\partial x}\right) - \frac{\partial}{\partial y}\left(\mu L^2\frac{\partial E^p_{22}}{\partial y}\right)$$

$$-\frac{\partial}{\partial x}\left(\mu L^2\frac{\partial E^p_{22}}{\partial x}\right)-\frac{\partial}{\partial y}\left(l^2\frac{s_o}{d_o}\frac{\partial \dot{E}^p_{22}}{\partial y}\right)-\frac{\partial}{\partial x}\left(l^2\frac{s_o}{d_o}\frac{\partial \dot{E}^p_{22}}{\partial x}\right) \tag{4.25}$$

and

$$0=2\mu E^p_{12}+\frac{s_o}{d_o}\dot{E}^p_{12}-\mu\left(\frac{\partial u_1}{\partial y}+\frac{\partial u_2}{\partial x}\right)-\frac{\partial}{\partial x}\left(\mu L^2\frac{\partial E^p_{12}}{\partial x}\right)$$

$$-\frac{\partial}{\partial y}\left(\mu L^2\frac{\partial E^p_{12}}{\partial y}\right)-\frac{\partial}{\partial x}\left(l^2\frac{s_o}{d_o}\frac{\partial \dot{E}^p_{12}}{\partial x}\right)-\frac{\partial}{\partial y}\left(l^2\frac{s_o}{d_o}\frac{\partial \dot{E}^p_{12}}{\partial y}\right). \tag{4.26}$$

By adding (4.24) and (4.25) with the fact that $E^p_{11}+E^p_{22}=0$ then

$$\left(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}\right)=0. \tag{4.27}$$

This implies that

$$tr(E)=0, \tag{4.28}$$

where E is the total strain defined by $E=E^e+E^p$ with $tr(E^p)=0$, so that we have as a consequence of $Q=L$, elastic incompressibility given by

$$tr(E^e)=0. \tag{4.29}$$

Thus (4.24) and (4.25) for fixed k $(k=1,2)$ become

$$0=2\mu E^p_{kk}+\frac{s_o}{d_o}\dot{E}^p_{kk}-2\mu E^e_{kk}-\frac{\partial}{\partial x}\left(\mu L^2\frac{\partial E^p_{kk}}{\partial x}\right)$$

$$-\frac{\partial}{\partial y}\left(\mu L^2\frac{\partial E^p_{kk}}{\partial y}\right)-\frac{\partial}{\partial x}\left(l^2\frac{s_o}{d_o}\frac{\partial \dot{E}^p_{kk}}{\partial x}\right)-\frac{\partial}{\partial y}\left(l^2\frac{s_o}{d_o}\frac{\partial \dot{E}^p_{kk}}{\partial y}\right) \tag{4.30}$$

By assuming that for $Q=L$

$$E^p_{11}=0,\quad E^p_{22}=0,\quad E^p_{12}\neq 0 \tag{4.31}$$

then

$$E^e_{11}=0,\quad E^e_{22}=0,\quad E^e_{12}\neq 0. \tag{4.32}$$

4.1 Weak formulation of the flow rule for $m=1$

Let $\omega_1(x,y)$ and $\omega_2(x,y)$ and $\omega_3(x,y)$ be weight functions satisfying the homogeneous boundary conditions of the flow rule. These weight functions have been chosen so that the force balances are satisfied for each approximate solution of the flow rule. The weight functions $\omega_1(x,y)$, $\omega_2(x,y)$ and $\omega_3(x,y)$ are chosen so that the weak formulation of the flow rule (4.21)-(4.23) are

$$0=\int_\Omega\left(2\mu\omega_1 E^p_{11}+\omega_1\frac{s_o}{d_o}\dot{E}^p_{11}\right)dxdy-\int_\Omega 2\mu\omega_1\left(\frac{2}{3}\frac{\partial u_1}{\partial x}-\frac{1}{3}\frac{\partial u_2}{\partial y}\right)dxdy$$

$$+\int_\Omega\left[\frac{\mu}{3}(L^2+2Q^2)\frac{\partial\omega_1}{\partial x}\frac{\partial E^p_{11}}{\partial x}+\frac{\mu}{3}(Q^2-L^2)\frac{\partial\omega_1}{\partial x}\frac{\partial E^p_{12}}{\partial y}\right]dxdy$$

$$+\int_\Omega\left[\frac{\mu}{3}(2L^2+Q^2)\frac{\partial\omega_1}{\partial y}\frac{\partial E^p_{11}}{\partial y}+l^2\frac{s_o}{d_o}\frac{\partial\omega_1}{\partial x}\frac{\partial \dot{E}^p_{11}}{\partial x}+l^2\frac{s_o}{d_o}\frac{\partial\omega_1}{\partial y}\frac{\partial \dot{E}^p_{11}}{\partial y}\right]dxdy$$

$$- \int_{\Gamma} \omega_1 \left[\left(\frac{\mu}{3}(L^2+2Q^2)\frac{\partial E^p_{11}}{\partial x} + \frac{\mu}{3}(Q^2-L^2)\frac{\partial E^p_{12}}{\partial y} + l^2 \frac{s_o}{d_o}\frac{\partial \dot{E}^p_{11}}{\partial x} \right) \eta_x \right] dS$$

$$- \int_{\Gamma} \omega_1 \left[\left(\frac{\mu}{3}(2L^2+Q^2)\frac{\partial E^p_{11}}{\partial y} + l^2 \frac{s_o}{d_o}\frac{\partial \dot{E}^p_{11}}{\partial y} \right) \eta_y \right] dS. \tag{4.33}$$

$$0 = \int_{\Omega} \left(2\mu\omega_2 E^p_{22} + \omega_2 \frac{s_o}{d_o}\dot{E}^p_{22} \right) dxdy - \int_{\Omega} 2\mu\omega_2 \left(\frac{2}{3}\frac{\partial u_2}{\partial y} - \frac{1}{3}\frac{\partial u_1}{\partial x} \right) dxdy$$

$$+ \int_{\Omega} \left[\frac{\mu}{3}(L^2+2Q^2)\frac{\partial \omega_2}{\partial y}\frac{\partial E^p_{22}}{\partial y} + \frac{\mu}{3}(Q^2-L^2)\frac{\partial \omega_2}{\partial y}\frac{\partial E^p_{12}}{\partial x} \right] dxdy$$

$$+ \int_{\Omega} \left[\frac{\mu}{3}(2L^2+Q^2)\frac{\partial \omega_4}{\partial x}\frac{\partial E^p_{22}}{\partial x} + l^2 \frac{s_o}{d_o}\frac{\partial \omega_2}{\partial y}\frac{\partial \dot{E}^p_{22}}{\partial y} + l^2 \frac{s_o}{d_o}\frac{\partial \omega_2}{\partial x}\frac{\partial \dot{E}^p_{22}}{\partial x} \right] dxdy$$

$$- \int_{\Gamma} \omega_2 \left[\left(\frac{\mu}{3}(L^2+2Q^2)\frac{\partial E^p_{22}}{\partial y} + \frac{\mu}{3}(Q^2-L^2)\frac{\partial E^p_{12}}{\partial x} + l^2 \frac{s_o}{d_o}\frac{\partial \dot{E}^p_{22}}{\partial y} \right) \eta_y \right] dS$$

$$- \int_{\Gamma} \omega_2 \left[\left(\frac{\mu}{3}(2L^2+Q^2)\frac{\partial E^p_{22}}{\partial x} + l^2 \frac{s_o}{d_o}\frac{\partial \dot{E}^p_{22}}{\partial x} \right) \eta_x \right] dS, \tag{4.34}$$

and

$$0 = \int_{\Omega} \left(2\mu\omega_3 E^p_{12} + \omega_3 \frac{s_o}{d_o}\dot{E}^p_{12} \right) dxdy - \int_{\Omega} \mu\omega_3 \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} \right) dxdy$$

$$+ \int_{\Omega} \left[\frac{\mu}{2}(L^2+Q^2)\frac{\partial \omega_3}{\partial x}\frac{\partial E^p_{12}}{\partial x} + \frac{\mu}{2}(L^2+Q^2)\frac{\partial \omega_3}{\partial y}\frac{\partial E^p_{12}}{\partial y} \right] dxdy$$

$$+ \int_{\Omega} \left[l^2 \frac{s_o}{d_o}\frac{\partial \omega_3}{\partial x}\frac{\partial \dot{E}^p_{12}}{\partial x} + l^2 \frac{s_o}{d_o}\frac{\partial \omega_3}{\partial y}\frac{\partial \dot{E}^p_{12}}{\partial y} \right] dxdy$$

$$- \int_{\Gamma} \omega_3 \left[\left(\frac{\mu}{2}(L^2+Q^2)\frac{\partial E^p_{12}}{\partial x} + l^2 \frac{s_o}{d_o}\frac{\partial \dot{E}^p_{12}}{\partial x} \right) \eta_x \right] dS$$

$$- \int_{\Gamma} \omega_3 \left[\left(\frac{\mu}{2}(L^2+Q^2)\frac{\partial E^p_{12}}{\partial y} + l^2 \frac{s_o}{d_o}\frac{\partial \dot{E}^p_{12}}{\partial y} \right) \eta_y \right] dS. \tag{4.35}$$

4.2 General Weak formulation of the flow rule for $m \geq 0$ and $Q = L$

Suppose $m \geq 0$ and $Q = L$, the flow rule takes the form

$$T_o + \mu L^2 \Delta E^p = s_o F \dot{E}^p - l^2 s_o div(F \nabla \dot{E}^p), \tag{4.36}$$

where $F = \left(\frac{d^p}{d_o}\right)^m \frac{1}{d^p}$.

Given that H is a test function satisfying homogeneous boundary condition of the flow rule then by divergence theorem, the weak formulation of the flow rule is given by

$$0 = \int_{\Omega} \left[l^2 s_o F \nabla \dot{E}^p \vdots \nabla H + \mu L^2 \nabla E^p \vdots \nabla H + s_o F \dot{E}^p : H - T_o : H \right] dV$$

$$- \int_{\Gamma} \left[l^2 s_o F H : (\nabla \dot{E}^p \vec{n}) + H : (\mu L^2 \nabla \dot{E}^p \vec{n}) \right] dS, \tag{4.37}$$

where $\vec{n}$ is the outward unit normal to the boundary Γ of Ω.

Suppose, under plane strain, E^p_{11} and E^p_{22} are neglected, then in component form, the weak formulation of the flow rule is given by

$$
\begin{aligned}
0 = &\int_\Omega \left[2\mu\omega_4 E^p_{12} + \omega_4 s_o F \dot{E}^p_{12}\right] dxdy - \int_\Omega 2\mu\omega_4 \left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) dxdy \\
&+ \int_\Omega \left[l^2 s_o F \frac{\partial \omega_4}{\partial x}\frac{\partial \dot{E}^p_{12}}{\partial x} + l^2 s_o F \frac{\partial \omega_4}{\partial y}\frac{\partial \dot{E}^p_{12}}{\partial y} + \mu L^2 \frac{\partial \omega_4}{\partial x}\frac{\partial \dot{E}^p_{12}}{\partial x} + \mu L^2 \frac{\partial \omega_4}{\partial y}\frac{\partial \dot{E}^p_{12}}{\partial y}\right] dxdy \\
&- \int_\Gamma \omega_4 \left[\left(l^2 s_o F \frac{\partial \dot{E}^p_{12}}{\partial x}\right)\eta_x + \left(l^2 s_o F \frac{\partial \dot{E}^p_{12}}{\partial y}\right)\eta_y\right] dS \\
&- \int_\Gamma \omega_4 \left[\left(\mu L^2 \frac{\partial \dot{E}^p_{12}}{\partial x}\right)\eta_x + \left(\mu L^2 \frac{\partial \dot{E}^p_{12}}{\partial y}\right)\eta_y\right] dS,
\end{aligned}
\tag{4.38}
$$

where η_x and η_y are the components of the outward unit normal $\vec{n}$.

5 Conclusion

The additional length scale Q associated with the divergence of the plastic strain gives rise to elastic incompressibility whenever the length scales Q and L are equal. Thus elastic incompressibility is achievable, even in the presence of the energetic length scale L associated with the gradient of the plastic strain.

Competing financial interests

The author declares no competing financial interests.

References

[1] Christensen, R. M. The theory of material failure. *Oxford University Press, Oxford. U.K*, (2013).

[2] Han, W. & Reddy, B. D. Plasticity: Mathematical theory and numerical analysis. *New York: Springer-Velag*, (2013).

[3] Christensen, R. M. A new theory of strain hardening and its consequences for yield stress and failure stress. *Failurecriteria.com*, (2015).

[4] Fleck, N., Muller, G. M., Ashby, M. F. & Hutchinson, J. W. Strain gradient plasticity: theory and experiment. *Acta Metall. Mater.* **42**. 475-487, (1994).

[5] Ebobisse, F., McBride, A. T. & Reddy, B. D. On mathematical formulations of a model of gradient plasticity. *IUTAM-Symposium on theoretical, modelling and computational aspects of inelastic media*, pp. 117-128. Cape town: Springer-Verlag, (2008).

[6] Hutchinson, J. W. Plasticity at the micron scale. *Int. J. Solids. Struct*, **37**, 225-238, (2000).

[7] Lloyd, D. J. Particle reinforced aluminium and magnesium matrix composites. *Int. metall. Rev.* **39**, 1-23, (1994).

[8] Gurtin, M. E. On framework for small-deformation viscoplasticity: free energy, microforce, strain gradients. *Int. J. Plast.* **19**, 47-90, (2003).

[9] Anand, L., Gurtin, M. E., Lele, S. P. & Gething, C. A one dimensional theory of strain-gradient plasticity: formulation, analysis, numerical results. *J. Mech. Phys. Solids.* **53**, 1789-1826, (2005).

[10] Reddy, B., Ebobisse, F. & McBride, A. Well-posedness of a model of strain gradient plasticity for plastically irrotational materials. *Int. J. Plast.* **24**, 55-73, (2008).

[11] Nielsen, K. F. & Niordson, C. F. A numerical basis for strain-gradient plasticity theory: Rate-independent and rate-dependent formulation. *J. Mech. Phys. Solids*, **63**, 113-127, (2014).

[12] Gurtin, M. E. A gradient theory of small- deformation isotropic plasticity that accounts for Burgers vector and dissipation due to plastic spin. *J. Mech. Phys. Solids*, **52**, 2545-2568, (2004).

[13] Borokinni, A. S. Finite element procedure for one-dimensional viscoplastic problem of an isotropic slab. *Z. Angew. Math. Phys.* **68:49** DOI: 10.1007/s00033-017-0793-9, (2017).

[14] Borokinni, A. S., Akinola, A. P., Layeni, O. P. & Fadodun, O. O. A new strain-gradient theory for an isotropic plastically deformed polycrystalline solid body. *Mathematics and Mechanics of Solid*, DOI: 10.1177/1081286517720842, (2017).

[15] Gurtin, M. E. & Anand, L. A theory of strain-gradient plasticity for isotropic, plastically irrotational materials. Part I: Small deformations. *J. Mech. Phys. Solids* **53**, 1624-1649, (2005).