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Maximal Order Block Trigonometrically Fitted Scheme for the Numerical Treatment of Second Order Initial Value Problem with Oscillating Solutions

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Abstract
A Maximal Order Block Trigonometrically Fitted Method (MBTFM) whose coefficients are functions of frequency and step size specially designed for the solution of second order Initial Value Problems (IVPs) with oscillatory solution is proposed in this paper. The MBTFM is obtained from one discrete formulae with two complementary formula which are provided by Continuous Trigonometrically Fitted Block Method (CTFBM). The convergence of the MBTFM is discussed and the performance of the method is demonstrated on some numerical examples to show accuracy and efficiency of the method.

Keywords: Collocation, Continuous Form, Convergence, Maximal order, Trigonometrically Fitted.
MSC2010: 65L05, 65L06, 65L20

1 Introduction

Consider the second order initial value problem given by

$$y'' = f(x, y, y'), y(x_0) = y_0, y'(x_0) = y'_0 \quad x_0 \leq x \leq x_N \tag{1.1}$$

with oscillating solutions where $f: R \times R^m \longrightarrow R^m$ are smooth and satisfy Lipschitz condition. Oscillatory initial value problems frequently arise in areas such as chemical kinetics, classical mechanics orbital dynamics, process vessels, control theory, biological sciences and theoretical physics (Ngwane and Jator [1], Ramos *et al.* [2], Martin-Vaquero and Vigo-Aguiar [3]). The numerical integration of (1.1) has received much attention during the past few decades and is still receiving attention because of its importance in applied science and engineering both in theory and practice.

Quite a number of work has been done in literature to numerically approximate the solution to (1.1). Such methods include polynomial interpolations (Lambert [4], Akinfenwa *et al.,* [5], Ngwane and Jator [6]), Mixed interpolation methods (Duxbury [7], Coleman and Duxbury [8]) exponential fitting methods (Ixaru *et al.*, [9], Vanden Berghe *et al.*, [10], Simos [11-12], Martin-Vaquero and Vigo-Aguiar [3], You and Chen [13], Franco and Gomez [14], Franco [15-16], Konguetsof and Simos [17], Franco [18], Vanden Berghe *et al.*, [19-20]), Piecewise Linearized methods (Ramos [21]), trigonometrically fitted methods based on Multistep collocation techniques (Ngwane and Jator [22-25], Jator *et al.*, [26]).

This paper presents a three-step second derivative block trigonometrically fitted method of order $2k+2$ based on Multistep collocation technique which integrates the IVP (1.1) where the solutions span $\{1, x, x^2, x^3, x^4, x^5, x^6, \sin(\omega x), \cos(\omega x)\}$. This basis function is inspired by its simplicity to analyse (Ngwane and Jator [22]) and to provide better approximation for initial value problems with oscillatory solutions (Coleman and Duxbury [8]); other possible basis functions are listed in Nguyen et al., [27]. The rest of this paper is organized as follows. In section 2 we construct the Maximal Order Block Trigonometrically Fitted Method (MBTFM). The analyses of the method which include its stability are discussed in section 3. Numerical experiments are presented in section 4. Finally, section 5 concludes the paper

2 Derivation of the MBTFM

Our objective in this section is to construct a Continuous Trigonometrically Fitted Block Method (CTFBM) which produces one main method and two discrete complementary methods as by product. The main method has the form

$$y_{n+3} - y_{n+1} = h\sum_{j=0}^{3}\beta_j(u)f_{n+j} + h^2\sum_{j=0}^{3}\gamma_j(u)g_{n+j} \tag{2.1}$$

and the two complementary methods are given by

$$y_n - y_{n+1} = h\sum_{j=0}^{3}\overline{\beta_{j,1}}(u)f_{n+j} + h^2\sum_{j=0}^{3}\overline{\gamma_{j,1}}(u)g_{n+j} \tag{2.2}$$

$$y_{n+2} - y_{n+1} = h\sum_{j=0}^{3}\overline{\beta_{j,2}}(u)f_{n+j} + h^2\sum_{j=0}^{3}\overline{\gamma_{j,2}}(u)g_{n+j} \tag{2.3}$$

where $y_{n+k} = y(x_n + kh)$, $f_{n+j} = y'(x_n + jh)$, $g_{n+j} = y''(x_n + jh)$, $u = \omega h$, ω is the frequency, x_n is a node point and $\beta_j, \overline{\beta_{J,1}}, \overline{\beta_{J,2}}, \gamma_j, \overline{\gamma_{J,1}}$ and $\overline{\gamma_{J,2}}$, $j = 0,1,2,3$ are coefficients to be uniquely obtained from multistep collocation techniques and dependent on the step size and frequency.

To obtain equations (2.1)-(2.3), we seek initially a continuous local approximation given by equation (2.4) below on the interval $[x_n, x_{n+3}]$ as follows

$$\Gamma(x,u) = y_{n+1} + h\sum_{j=0}^{3}\beta_j(x,u)f_{n+j} + h^2\sum_{j=0}^{3}\gamma_j(x,u)g_{n+j} \tag{2.4}$$

which represents the CTFBM.

It is assumed that $\Gamma(x_{n+j},u) = y_{n+j}$, $\left.\frac{\partial(\Gamma(x,u))}{\partial x}\right|_{x=x_{n+j}} = f_{n+j}$ *and* $\left.\frac{\partial^2(\Gamma(x,u))}{\partial x^2}\right|_{x=x_{n+j}} = g_{n+j}$ are the numerical approximations to the exact values $y(x_{n+j}), y'(x_{n+j})$ and $y''(x_{n+j})$ respectively

Theorem 1

Let $\xi_i = \{1, x, x^2, x^3, x^4, x^5, x^6, \sin(\omega x), \cos(\omega x)\}$ be basis functions,
$\xi(x) = (\xi_0(x), \xi_1(x), \cdots, \xi_8(x))^T$ and $\Omega = (y_{n+1}, f_n, \cdots, f_{n+3}, g_n, \cdots g_{n+3})^T$ be vectors, where denotes transpose. Define the matrix Π as

$$\Pi=\begin{bmatrix} \xi_0(x_{n+1}) & \xi_1(x_{n+1}) & \xi_2(x_{n+1}) & \xi_3(x_{n+1}) & \xi_4(x_{n+1}) & \xi_5(x_{n+1}) & \xi_6(x_{n+1}) & \xi_7(x_{n+1}) & \xi_8(x_{n+1}) \\ \xi_0'(x_n) & \xi_1'(x_n) & \xi_2'(x_n) & \xi_3'(x_n) & \xi_4'(x_n) & \xi_5'(x_n) & \xi_6'(x_n) & \xi_7'(x_n) & \xi_8'(x_n) \\ \xi_0'(x_{n+1}) & \xi_1'(x_{n+1}) & \xi_2'(x_{n+1}) & \xi_3'(x_{n+1}) & \xi_4'(x_{n+1}) & \xi_5'(x_{n+1}) & \xi_6'(x_{n+1}) & \xi_7'(x_{n+1}) & \xi_8'(x_{n+1}) \\ \xi_0'(x_{n+2}) & \xi_1'(x_{n+2}) & \xi_2'(x_{n+2}) & \xi_3'(x_{n+2}) & \xi_4'(x_{n+2}) & \xi_5'(x_{n+2}) & \xi_6'(x_{n+2}) & \xi_7'(x_{n+2}) & \xi_8'(x_{n+2}) \\ \xi_0'(x_{n+3}) & \xi_1'(x_{n+3}) & \xi_2'(x_{n+3}) & \xi_3'(x_{n+3}) & \xi_4'(x_{n+3}) & \xi_5'(x_{n+3}) & \xi_6'(x_{n+3}) & \xi_7'(x_{n+3}) & \xi_8'(x_{n+3}) \\ \xi_0''(x_n) & \xi_1''(x_n) & \xi_2''(x_n) & \xi_3''(x_n) & \xi_4''(x_n) & \xi_5''(x_n) & \xi_6''(x_n) & \xi_7''(x_n) & \xi_8''(x_n) \\ \xi_0''(x_{n+1}) & \xi_1''(x_{n+1}) & \xi_2''(x_{n+1}) & \xi_3''(x_{n+1}) & \xi_4''(x_{n+1}) & \xi_5''(x_{n+1}) & \xi_6''(x_{n+1}) & \xi_7''(x_{n+1}) & \xi_8''(x_{n+1}) \\ \xi_0''(x_{n+2}) & \xi_1''(x_{n+2}) & \xi_2''(x_{n+2}) & \xi_3''(x_{n+2}) & \xi_4''(x_{n+2}) & \xi_5''(x_{n+2}) & \xi_6''(x_{n+2}) & \xi_7''(x_{n+2}) & \xi_8''(x_{n+2}) \\ \xi_0''(x_{n+3}) & \xi_1''(x_{n+3}) & \xi_2''(x_{n+3}) & \xi_3''(x_{n+3}) & \xi_4''(x_{n+3}) & \xi_5''(x_{n+3}) & \xi_6''(x_{n+3}) & \xi_7''(x_{n+3}) & \xi_8''(x_{n+3}) \end{bmatrix}$$

and Π_i is obtained by replacing the ith column of Π by Ω. Suppose the following conditions are satisfied

$$\Gamma(x_{n+j},u) = y_{n+j} \quad , \quad j = 1 \tag{2.5}$$

$$\left.\frac{\partial(\Gamma(x,u))}{\partial x}\right|_{x=x_{n+j}} = f_{n+j}\,, \qquad j = 0(1)3 \tag{2.6}$$

$$\left.\frac{\partial^2(\Gamma(x,u))}{\partial x^2}\right|_{x=x_{n+j}} = g_{n+j}\,, \qquad j = 0(1)3 \tag{2.7}$$

Then the continuous representation (2.4) is equivalent to

$$\Gamma(x,u) = \sum_{i=0}^{8} \frac{\det(\Pi_i)}{\det(\Pi)} \xi_i(x) \tag{2.8}$$

Proof

It is required that equation (2.4) be defined by the assumed basis function as follows

$$\alpha_j(x,u) = \sum_{i=0}^{8} \alpha_{i,j}(x,u)\, \xi_i(x) \quad j = 1 \tag{2.9}$$

$$h\beta_j(x,u) = \sum_{i=0}^{8} h\beta_{i,j}(x,u)\, \xi_i(x) \quad j = 0,1,2,3 \tag{2.10}$$

$$h^2\gamma_j(x,u) = \sum_{i=0}^{8} h^2\gamma_{i,j}(x,u)\, \xi_i(x) \quad j = 0,1,2,3 \tag{2.11}$$

Substituting equations (2.9)-(2.11) into equation (2.4) yield

$$\Gamma(x,u) = \sum_{j=0}^{1}\sum_{i=0}^{8} \alpha_{i,j}(x,u)\, \xi_i(x)\, y_{n+j} + \sum_{j=0}^{3}\sum_{i=0}^{8} h\beta_{i,j}(x,u)\, \xi_i(x)\, f_{n+j} + \sum_{i=0}^{8} h^2\gamma_{i,2}(x,u)\, \xi_i(x) g_{n+j}$$

$$\Gamma(x,u) = \sum_{i=0}^{8} \left\{ \sum_{j=0}^{1} \alpha_{i,j}(x,u)\, y_{n+j} + h\sum_{j=0}^{2} \beta_{i,j}(x,u)\, f_{n+j} + h^2\gamma_{i,2}(x,u) g_{n+j} \right\} \xi_i(x) \tag{2.12}$$

Letting

$$\Delta_i = \sum_{j=0}^{1} \alpha_{i,j}(x,u)\, y_{n+j} + h\sum_{j=0}^{2} \beta_{i,j}(x,u)\, f_{n+j} + h^2\gamma_{i,2}(x,u) g_{n+j},$$

equation (2.12) becomes

$$\Gamma(x,u) = \sum_{i=0}^{8} \Delta_i \, \xi_i(x) \tag{2.13}$$

Imposing the conditions in equations (2.5)-(2.7) on equation (2.13), we obtain a system of 9 equations which is expressed as $\Pi\Delta = V$ where $\Delta = (\Delta_0, \Delta_1, \cdots, \Delta_8)^T$ is a vector form of 9 undetermined coefficients that are determined by applying Crammer's rule to obtain

$$\Delta_{\mathrm{i}} = \frac{\det(\Pi_{\mathrm{i}})}{\det(\Pi)}, \qquad \mathrm{i} = 0(1)(8) \tag{2.14}$$

Π_{i} is obtained by replacing the ith column of Π by Ω.
Substitute equation (2.14) into equation (2.13) to obtain

$$\Gamma(x,u) = \sum_{i=0}^{8} \frac{det(\Pi_i)}{det(\Pi)} \xi_i(x) \qquad \blacksquare \tag{2.15}$$

2.1 Specification of MBTFM

It worth noting that the continuous method in equation (2.15) which is equivalent to equation (2.4) is used to generate one main method and two discrete complementary methods. Both the main method and the discrete complementary methods are then applied in their power series form as MBTFM for solving equation (1.1).

Evaluating equation (2.15) at $x = x_{n+3}$ gives the discrete method $y_{n+3} = \Gamma(x_n + 3h)$ which takes the form of the main method. Evaluating equation (2.15) at $x = x_n$ and $x = x_{n+2}$ respectively, give the complementary methods $y_n = \Gamma(x_n)$ and $y_{n+2} = \Gamma(x_n + 2h)$.

The MBTFM whose coefficients are in trigonometric form is presented in equations (2.16)-(2.18). According to Lambert [4], to avoid heavy cancellation that may occur as $u \to 0$, series form of the coefficients is used. Thus, the corresponding converted series form of the trigonometric coefficients are given by equations (2.19)-(2.21) respectively.

$$\begin{aligned} y_{n+3} - y_{n+1} = {} & h\big(\beta_0(\sin u, \cos u) f_n + \beta_1(\sin u, \cos u) f_{n+1} + \beta_2(\sin u, \cos u) f_{n+2} \\ & + \beta_3(\sin u, \cos u) f_{n+3}\big) \\ & + h^2\big(\gamma_0(\sin u, \cos u) g_n + \gamma_1(\sin u, \cos u) g_{n+1} + \gamma_2(\sin u, \cos u) g_{n+2} \\ & + \gamma_3(\sin u, \cos u) g_{n+3}\big). \end{aligned} \tag{2.16}$$

$$\begin{aligned} y_n - y_{n+1} = {} & h\big(\overline{\beta_{0,1}}(\sin u, \cos u) f_n + \overline{\beta_{1,1}}(\sin u, \cos u) f_{n+1} + \overline{\beta_{2,1}}(\sin u, \cos u) f_{n+2} \\ & + \overline{\beta_{3,1}}(\sin u, \cos u) f_{n+3}\big) \\ & + h^2\big(\overline{\gamma_{0,1}}(\sin u, \cos u) g_n + \overline{\gamma_{1,1}}(\sin u, \cos u) g_{n+1} + \overline{\gamma_{2,1}}(\sin u, \cos u) g_{n+2} \\ & + \overline{\gamma_{3,1}}(\sin u, \cos u) g_{n+3}\big). \end{aligned} \tag{2.17}$$

$$\begin{aligned} y_{n+2} - y_{n+1} = {} & h\big(\overline{\beta_{0,2}}(\sin u, \cos u) f_n + \overline{\beta_{1,2}}(\sin u, \cos u) f_{n+1} + \overline{\beta_{2,2}}(\sin u, \cos u) f_{n+2} \\ & + \overline{\beta_{3,2}}(\sin u, \cos u) f_{n+3}\big) \\ & + h^2\big(\overline{\gamma_{0,2}}(\sin u, \cos u) g_n + \overline{\gamma_{1,2}}(\sin u, \cos u) g_{n+1} + \overline{\gamma_{2,2}}(\sin u, \cos u) g_{n+2} \\ & + \overline{\gamma_{3,2}}(\sin u, \cos u) g_{n+3}\big). \end{aligned} \tag{2.18}$$

$$\left.\begin{aligned}
\beta_0 &= \frac{20}{567}+\frac{257}{357210}u^2+\frac{5039}{198037224}u^4+\frac{6090451}{8109624322800}u^6+\frac{82999919}{4087250658691200}u^8+\cdots\\
\beta_1 &= \frac{13}{21}-\frac{22}{6615}u^2-\frac{107}{1833678}u^4-\frac{84211}{75089114100}u^6-\frac{9925753}{416294048570400}u^8+\cdots\\
\beta_2 &= \frac{20}{21}+\frac{1}{2646}u^2-\frac{85}{7334712}u^4-\frac{178721}{300356456400}u^6-\frac{31335527}{1665176194281600}u^8+\cdots\\
\beta_3 &= \frac{223}{567}+\frac{398}{178605}u^2+\frac{2203}{49509306}u^4+\frac{1957451}{2027406080700}u^6+\frac{22841851}{1021812664672800}u^8+\cdots\\
\gamma_0 &= \frac{8}{945}+\frac{17}{59535}u^2+\frac{299}{33006204}u^4+\frac{347143}{1351604053800}u^6+\frac{51187033}{7493292874267200}u^8+\cdots\\
\gamma_1 &= \frac{19}{105}-\frac{1}{2646}u^2+\frac{85}{7334712}u^4+\frac{178721}{300356456400}u^6+\frac{31335527}{1665176194281600}u^8+\cdots\\
\gamma_2 &= \frac{16}{105}-\frac{22}{6615}u^2-\frac{107}{1833678}u^4-\frac{84211}{75089114100}u^6-\frac{9925753}{416294048570400}u^8+\cdots\\
\gamma_3 &= -\frac{43}{945}-\frac{83}{119070}u^2-\frac{941}{66012408}u^4-\frac{852409}{2703208107600}u^6-\frac{110741551}{14986585748534400}u^8+\cdots
\end{aligned}\right\}\qquad (2.19)$$

$$\left.\begin{aligned}
\overline{\beta_{0,1}} &= -\frac{6893}{18144}-\frac{4260}{22861440}u^2-\frac{102479}{31685955840}u^4-\frac{631398127}{1038031913318400}u^6-\frac{149734579}{11890183734374400}u^8+\cdots\\
\overline{\beta_{1,1}} &= -\frac{313}{672}-\frac{629}{846720}u^2-\frac{683}{1173553920}u^4+\frac{9142333}{38445626419200}u^6+\frac{482609377}{53285638217011200}u^8+\cdots\\
\overline{\beta_{2,1}} &= -\frac{89}{672}+\frac{2507}{846720}u^2-\frac{4927}{106686720}u^4+\frac{29382077}{38445626419200}u^6+\frac{750368897}{53285638217011200}u^8+\cdots\\
\overline{\beta_{3,1}} &= -\frac{397}{18144}-\frac{1621}{4572288}u^2-\frac{420599}{31685955840}u^4-\frac{408760943}{1038031913318400}u^6-\frac{1379320849}{130792021078118400}u^8+\cdots\\
\overline{\gamma_{0,1}} &= -\frac{1283}{30240}-\frac{877}{1524096}u^2-\frac{107711}{10561985280}u^4-\frac{6173317}{31455512524800}u^6-\frac{152565167}{36890057227161600}u^8+\cdots\\
\overline{\gamma_{1,1}} &= \frac{851}{3360}-\frac{2507}{846720}u^2-\frac{4927}{106686720}u^4-\frac{29382077}{38445626419200}u^6-\frac{750368897}{53285638217011200}u^8+\cdots\\
\overline{\gamma_{2,1}} &= \frac{269}{3360}-\frac{629}{846720}u^2-\frac{683}{1173553920}u^4+\frac{9142333}{38445626419200}u^6+\frac{482609377}{53285638217011200}u^8+\cdots\\
\overline{\gamma_{3,1}} &= \frac{163}{30240}+\frac{1249}{7620480}u^2+\frac{52831}{10561985280}u^4+\frac{47666743}{346010637772800}u^6+\frac{1715587651}{479570743953100800}u^8+\cdots
\end{aligned}\right\}\qquad (2.20)$$

$$\left.\begin{aligned}
\overline{\beta_{0,2}} &= \frac{3}{224}+\frac{103}{282240}u^2+\frac{529}{43464960}u^4+\frac{33911}{94927472640}u^6+\frac{583757}{59804307763200}u^8+\cdots\\
\overline{\beta_{1,2}} &= \frac{109}{224}-\frac{103}{282240}u^2-\frac{529}{43464960}u^4-\frac{33911}{94927472640}u^6-\frac{583757}{59804307763200}u^8+\cdots\\
\overline{\beta_{2,2}} &= \frac{109}{224}-\frac{103}{282240}u^2-\frac{529}{4346960}u^4-\frac{33911}{94927472640}u^6-\frac{583757}{59804307763200}u^8+\cdots\\
\overline{\beta_{3,2}} &= \frac{109}{224}-\frac{103}{282240}u^2-\frac{529}{43464960}u^4+\frac{33911}{94927472640}u^6-\frac{583757}{59804307763200}u^8+\cdots\\
\overline{\gamma_{0,2}} &= \frac{31}{10080}+\frac{103}{846720}u^2+\frac{529}{130394880}u^4+\frac{33911}{284782417920}u^6+\frac{583757}{179412933289600}u^8+\cdots\\
\overline{\gamma_{1,2}} &= \frac{113}{1120}+\frac{103}{282240}u^2+\frac{529}{43464960}u^4+\frac{3391}{94927472640}u^6+\frac{583757}{59804307763200}u^8+\cdots\\
\overline{\gamma_{2,2}} &= -\frac{113}{1120}-\frac{103}{282240}u^2-\frac{529}{43464960}u^4-\frac{3391}{94927472640}u^6-\frac{583757}{59804307763200}u^8+\cdots\\
\overline{\gamma_{3,2}} &= -\frac{31}{10080}-\frac{103}{846720}u^2-\frac{529}{130394880}u^4-\frac{3391}{284782417920}u^6-\frac{583757}{179412923289600}u^8+\cdots
\end{aligned}\right\}\qquad (2.21)$$

In order to avoid the heavy cancellations which might occur when h is small, the use of the power series expansion of the parameters is preferable (Lambert [4]). It is interesting to note that as either $u \to 0$ method based on polynomial basis is recovered.

3 Analysis of MBTFM

3.1 Local Truncation Error of MBTFM

Theorem 2

The **MBTFM** has a Local Truncation Error (LTE) of $C_9h^9\left(\omega^2y^{(7)}(x_n)+y^{(9)}(x_n)\right)+O\left(h^{(10)}\right)$.

Proof:

The proof of the theorem is in the spirit of Ngwane and Jator [22].
Consider the Taylor series expansion of $y_{n+j}, y(x_n+jh), y'_{n+j}, y'(x_n+jh), y''_{n+j}$ and $y''(x_n+jh), j=0(1)3$. Also, assume that $y(x_{n+j})=y_{n+j}$, $y'(x_{n+j})=f_{n+j}$, $y''(x_{n+j})=g_{n+j}$. Then by substituting these into method in equation (2.16) and simplifying, we have that

$$\begin{aligned} LTE &= y(x_{n+2})-y_{n+2} \\ &= C_9h^9\left(\omega^2y^{(7)}(x_n)+y^{(9)}(x_n)\right)+O\left(h^{(10)}\right) \quad \blacksquare \end{aligned} \tag{3.1}$$

Consequently, the Local Truncation Error (LTE) of equations (2.16)-(2.18) are respectively obtained as

$$\text{LTE} = \begin{bmatrix} -\dfrac{313h^9}{25401600}\left(y^{(9)}(x_n)+\omega^2y^{(7)}(x_n)\right)+O(h^{10}) \\ \dfrac{103h^9}{25401600}\left(y^{(9)}(x_n)+\omega^2y^{(7)}(x_n)\right)+O(h^{10}) \\ \dfrac{13h^9}{793800}\left(y^{(9)}(x_n)+\omega^2y^{(7)}(x_n)\right)+O(h^{10}) \end{bmatrix} \tag{3.2}$$

According to Butcher [28], a linear $k-$step method of order p is said to be of maximal order if $p=2k+2$. Since the block method given by equation (2.16)-(2.18) are of order $p=(8,8,8)^T$ with error constants
$c_9=\left(\frac{-313}{25401600},\frac{103}{25401600},\frac{13}{25401600}\right)^T$, we therefore remark that **MBTFM** is a maximal order method.
Also, following the definition of Lambert [4] and Fatunla [29], a numerical method is consistent if its order is greater than one. We therefore remark that MBTFM is consistent.

3.2 Convergence of MBTFM

The convergence of the MBTFM is discussed in the following theorem.

Theorem 2

Let $\bar{\mathrm{Y}}$ be an approximation of the solution vector Y for the system obtained from the derived methods (2.16)-(2.18). If $e_n=\left|y(x_n)-y_n\right|$, where the exact solution is several times differentiable on $[a,b]$ and if $\|\mathrm{E}\|=\|\bar{\mathrm{Y}}-\mathrm{Y}\|$, then for sufficiently small h, MBTFM is an 8th order convergent method. In other words, $\|\mathrm{E}\|=O(h^8)$.

Proof

Let the matrices obtained from the MBTFM be defined as follows:

$$A_{11}=\begin{bmatrix}\overline{\alpha_{1,1}} & \overline{\alpha_{2,1}} & 0\\ \overline{\alpha_{1,2}} & \overline{\alpha_{2,2}} & 0\\ \alpha_1 & \alpha_2 & 1\end{bmatrix},\quad A_{12}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},\quad A_{21}=\begin{bmatrix}0&0&\overline{\alpha_{0,1}}\\0&0&\overline{\alpha_{0,2}}\\0&0&\alpha_0\end{bmatrix},\quad A_{22}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$B_{11}=h\begin{bmatrix}\overline{\beta_{1,1}} & \overline{\beta_{2,1}} & \overline{\beta_{3,1}}\\ \overline{\beta_{1,2}} & \overline{\beta_{2,2}} & \overline{\beta_{3,2}}\\ \beta_1 & \beta_2 & \beta_3\end{bmatrix}, B_{12}=h\begin{bmatrix}\overline{\gamma_{1,1}} & \overline{\gamma_{2,1}} & \overline{\gamma_{3,1}}\\ \overline{\gamma_{1,2}} & \overline{\gamma_{2,2}} & \overline{\gamma_{2,3}}\\ \gamma_1 & \gamma_2 & \gamma_3\end{bmatrix}, B_{21}=h\begin{bmatrix}0&0&\overline{\beta_{0,1}}\\0&0&\overline{\beta_{0,2}}\\0&0&\beta_0\end{bmatrix}, B_{22}=h\begin{bmatrix}0&0&\overline{\gamma_{3,1}}\\0&0&\overline{\gamma_{3,2}}\\0&0&\gamma_3\end{bmatrix}$$

In compact form, we write

$$A=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix} \text{ and } B=\begin{bmatrix}B_{11} & B_{12}\\ B_{21} & B_{22}\end{bmatrix}$$

where A and B are respectively $2\mathrm{N}\times 2\mathrm{N}$ matrices, P_{ij} and Q_{ij} are $\mathrm{N}\times\mathrm{N}$ matrices, A_{12} is null matrices while A_{22} is an Identity matrix.
We further define the following vectors:

$$\mathrm{Y}=(y(x_1),y(x_2),\cdots,y(x_N))^T, \mathrm{F}=(f_1,f_2,\cdots,f_N,hg_1,\cdots,hg_N)^T, L(h)=(l_1,l_2,\ldots,l_N)^T$$

where $L(h)$ is the Local truncation error.
The exact form of the system formed by equations (2.16)-(2.18) is given by

$$AY-BF(Y)+C+L(h)=0 \tag{3.3}$$

and the approximate form of the system is given by

$$A\bar{Y}-BF(\bar{Y})+C=0 \tag{3.4}$$

Subtracting (3.3) from (3.4), we have

$$A(\bar{Y}-Y)-B\left(F(\overline{Y})-F(Y)\right)=L(h) \tag{3.5}$$

Letting $E=\bar{Y}-Y=(e_1,e_2,\ldots,e_N)^T$, in equation (3.5), we can write

$$F(\overline{Y})=F(Y)+JE+o(\|\bar{Y}-Y\|) \tag{3.6}$$

Using mean value theorem, equation (3.6) can be approximated thus

$$\frac{F(\overline{Y})-F(Y)}{\bar{Y}-Y}=\frac{F(\overline{Y})-F(Y)}{E}=J$$

$$(A-BJ)E=L(h) \tag{3.7}$$

Where the Jacobian matrix and its entries $J_{11}, J_{12}, J_{21}, J_{22}$ are defined as follows

$$J_{11}=\begin{bmatrix}\frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_N}\\ \vdots & \ddots & \vdots\\ \frac{\partial f_N}{\partial y_1} & \cdots & \frac{\partial f_N}{\partial y_N}\end{bmatrix}, J_{12}=\begin{bmatrix}\frac{\partial f_1}{\partial y_1} & \cdots & \frac{\partial f_1}{\partial y_N}\\ \vdots & \ddots & \vdots\\ \frac{\partial f_N}{\partial y_1} & \cdots & \frac{\partial f_N}{\partial y_N}\end{bmatrix}, J_{21}=h\begin{bmatrix}\frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_N}\\ \vdots & \ddots & \vdots\\ \frac{\partial g_N}{\partial y_1} & \cdots & \frac{\partial g_N}{\partial y_N}\end{bmatrix}, J_{22}=h\begin{bmatrix}\frac{\partial g_1}{\partial y_1} & \cdots & \frac{\partial g_1}{\partial y_N}\\ \vdots & \ddots & \vdots\\ \frac{\partial g_N}{\partial y_1} & \cdots & \frac{\partial g_N}{\partial y_N}\end{bmatrix}$$

Let $M=-BJ$ be a $2N\times 2N$ matrix, we have $(A+M)E=L(h)$, and for sufficiently small h, $A+M$ is a monotone and lower triangular matrix and thus invertible (Jain and Aziz [30]). Therefore,
$(A+M)^{-1}=D=\left(d_{i,j}\right)\geq 0$ and $\sum_{j=1}^{2N} d_{i,j}=O(h^{-2}) \Rightarrow E=D\,L(h)$. If $\|E\|=\max_i|e_i|$; then $\|E\|=\|D\,L(h)\|=O(h^{-2})\,O(h^{10})=O(h^8)$. This shows that MBTFM is convergent and the global error is of order $O(h^8)$ ■

3.3 Stability of MBTFM

Following Akinfenwa *et al.*, [5], MBTFM can be represented by a block matrix finite difference equation given by

$$A^{(1)}Y_{w+1}=A^{(0)}\mathrm{Y}_w+hB^{(1)}F_{w+1}+hB^{(0)}F_w+h^2D^{(1)}G_{w+1}+h^2D^{(0)}G_w$$

(3.8)

where $Y_{w+1}=\left(y_{n+1},y_{n+2},y_{n+3}\right)^T$, $Y_w=(y_{n-2},y_{n-1},y_n)^T$, $F_{w+1}=\left(f_{n+1},f_{n+2},f_{n+3}\right)^T$,

$F_w = \left(f_{n-2}, f_{n-1}, f_n\right)^T$, $G_{w+1} = \left(g_{n+1}, g_{n+2}, g_{n+3}\right)^T$, $G_w = \left(g_{n-2}, g_{n-1}, g_n\right)^T$ and $A^{(0)}, A^{(1)}$, $B^{(0)}, B^{(1)}$, $D^{(0)}, D^{(1)}$ are 3×3 matrices specified as follows

$$A^{(1)} = \begin{bmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}, A^{(0)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, B^{(1)} = \begin{bmatrix} \overline{\beta_{1,1}} & \overline{\beta_{2,1}} & \overline{\beta_{3,1}} \\ \overline{\beta_{1,2}} & \overline{\beta_{2,2}} & \overline{\beta_{3,2}} \\ \beta_1 & \beta_2 & \beta_3 \end{bmatrix}, B^{(0)} = \begin{bmatrix} 0 & 0 & \overline{\beta_{0,1}} \\ 0 & 0 & \overline{\beta_{0,2}} \\ 0 & 0 & \beta_0 \end{bmatrix}, D^{(1)} = \begin{bmatrix} \overline{\gamma_{1,1}} & \overline{\gamma_{2,1}} & \overline{\gamma_{3,1}} \\ \overline{\gamma_{1,2}} & \overline{\gamma_{2,2}} & \overline{\gamma_{2,3}} \\ \gamma_1 & \gamma_2 & \gamma_3 \end{bmatrix},$$

$$D^{(0)} = \begin{bmatrix} 0 & 0 & \overline{\gamma_{0,1}} \\ 0 & 0 & \overline{\gamma_{0,2}} \\ 0 & 0 & \gamma_0 \end{bmatrix}$$

3.3.1 Zero Stability

According to Lambert [4] and Fatunla [29], MBTFM is zero stable if the roots of the first characteristic polynomial have modulus less than or equal to one and those of modulus one are simple. i.e.
$\rho(R) = \det\left[RA^{(1)} - A^{(0)}\right] = 0$ and $|R_i| \leq 1$. Hence MBTFM is zero stable since from our calculation $|R| = 0{,}0\ or\ 1$.

3.3.2 Linear Stability and Region of Absolute Stability of MBTFM

Applying the block method to the test equations $y' = \lambda y$ and $y'' = \lambda^2 y$ and letting $z = \lambda h$ yields $Y_{w+1} = \sigma(z)Y_w$, where $\sigma(z) = \frac{A^{(0)}+zB^{(0)}+z^2D^{(0)}}{A^{(1)}-zB^{(0)}-z^2D^{(1)}}$. The matrix $\xi(z)$ for MBTFM has eigenvalues given by $(\delta_1, \delta_2, \delta_3) = (0, 0, \delta_3)$, where $\delta_3(z,u) = \frac{\eta_3(z,u)}{\tau_3(z,u)}$ is called the stability function. According to Ndukum *et al.*, [31], having suitable values of u in a large interval means that the method can cope well for problems with estimated frequencies. It is observed that for MBTFM, the values of $u \epsilon [\pi, 2\pi)$ are satisfactory. The Region of Absolute Stability (RAS) of MBTFM is plotted for $u = \pi$ using the boundary locus method and is presented in figure 1

3.4 Definition

A Numerical scheme is said to be A(α) stable, with $\alpha \in \left(0, \frac{\pi}{2}\right)$ if its region of absolute stability contain the wedge $\{z: -\alpha < (\pi - \arg z) < \alpha\}$ and it is said to be A_0 stable if it is A(α) stable for some sufficiently small $\alpha \in \left(0, \frac{\pi}{2}\right)$. From figure 1 above, we conclude therefore that MBTFM is A_0 stable.

4 Implementation of Derived methods

In this section, the MBTFM is implemented in a block by block fashion without requiring starting values or/and predictors. The implementation was done with the aid of written codes in Maple 2016.2 software enhanced by the feature of *fsolve* for both linear and

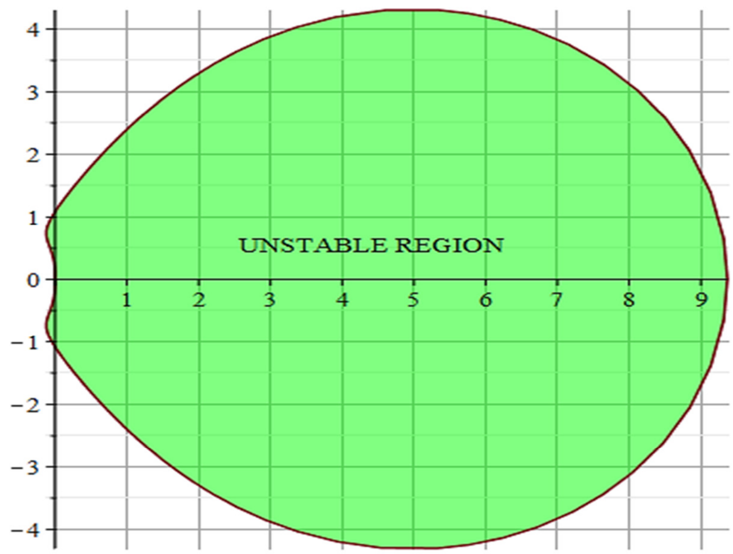

Figure 1: Region of Absolute Stability of MBTFM

nonlinear problems and executed on Windows 10 operating system. It is worth nothing that Maple 2016.2 can symbolically compute derivatives, hence the automatic generation of the entries of the Jacobian Matrices which involves the partial derivatives of both f and g. In particular, the MBTFM is applied to the considered oscillatory problems on the range of interest as follows:

1. Choose N, $h = \frac{b-a}{N}$ and the number of blocks $\Lambda = \frac{N}{k}$. For $n = 0$ *and* $w = 0$ the values of $(y_1, y_2, y_3)^T$ are simultaneously obtained over the subinterval $[x_0, x_3]$ as y_0 is known from the IVP under consideration.

2. For $n = 3$ and $w = 1$, the values of $(y_4, y_5, y_6)^T$ are simultaneously obtained over the subinterval $[x_3, x_6]$ as y_3 is known from the previous block.

3. The process is continued for $n = 6, \cdots, N-3$ and $w = 2, \cdots, \Lambda$ to obtain the numerical solution to the given IVP on the subinterval $[x_0, x_3], [x_3, x_6], \cdots [x_{N-3}, x_N]$.

4.1 Numerical Examples

In this section, the performance, efficiency and accuracy of the MBTFM on variety of well-known oscillatory IVPs is discussed. For each problem, the computational frequency is estimated by equating the local truncation error of the main methods to zero then solve for ω as described in Ramos and Vigo-Aguiar [32]. The absolute errors or maximum error of the approximate solutions are computed and compared with results from existing methods in the literature. We noted that the method developed in this paper can be implemented for all values of N. However, for purpose of comparison the N values used in the existing literature were used therein. For emphasis, except where specified, h the step length is defined as $h = \frac{b-a}{N}$.

Example 1 Highly Oscillatory Problem

As our first test, we consider a highly oscillatory problem given by $y'' = -100y + 99\sin x$, $y(0) = 1$, $y'(0) = 11$, $x \in [0, 2\pi]$ whose solution in closed form is given as $y =$

$\cos 10x + \sin 10x + \sin x$. According to Sallam and Anwar [33], the solution consists of rapid and slow function; the slowly varying function is due to the inhomogeneous term. This problem was solved by the 12th order Obrechkoff methods of Simos [34], Van Daele and Vanden Berghe [35] and Archar [36] in the interval $[0, 10\pi]$. The numerical results of **MBTFM** at $x = 10\pi$ and CPU time show that it is more accurate and efficient compared to the aforementioned methods as contained in Table 1 and Figure 2.

h	**MBTFM**		Archar		Daele		Simos	
	Error	CPU	Error	CPU	Error	CPU	Error	CPU
$\frac{\pi}{50}$	1.95×10^{-14}	0.578	5.79×10^{-13}	0.187	1.20×10^{-11}	0.250	3.05×10^{-11}	0.172
$\frac{\pi}{100}$	2.71×10^{-17}	1.211	5.79×10^{-13}	0.452	7.35×10^{-13}	0.530	2.28×10^{-13}	0.515
$\frac{\pi}{200}$	1.08×10^{-19}	2.516	1.32×10^{-12}	0.749	8.62×10^{-13}	0.827	4.40×10^{-13}	0.858
$\frac{\pi}{300}$	2.38×10^{-27}	3.891	1.96×10^{-12}	0.952	2.63×10^{-12}	1.154	2.11×10^{-12}	1.139
$\frac{\pi}{400}$	1.07×10^{-22}	5.331	4.78×10^{-12}	1.232	2.93×10^{-12}	1.404	1.38×10^{-12}	1.388
$\frac{\pi}{500}$	2.88×10^{-23}	6.594	7.50×10^{-12}	1.466	2.89×10^{-12}	1.778	6.47×10^{-12}	1.700

Table 1: Comparison of End Point Absolute Errors at $x = 10\pi$

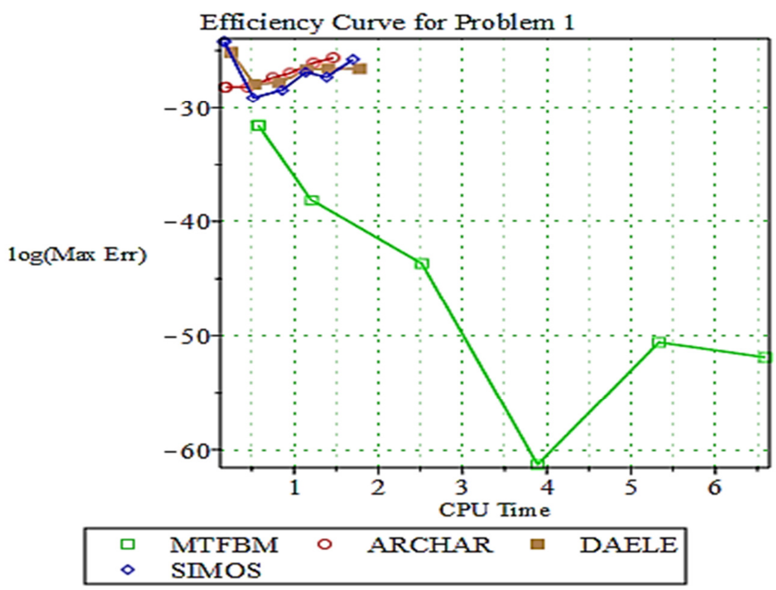

Figure 2: Efficiency curve for Problem 1

Example 2: Nonlinear Duffing Equation

Consider the nonlinear Duffing equation forced by a harmonic function given by $y'' + y + y^3 = B\cos(\Omega x)$, whose theoretical solution is unknown. A very accurate approximation of the theoretical solution of this equation is judged by comparison with a Galerkin approximation obtained by Van Dooren [37] and given by $y(x) = C_1\cos(\Omega x) + C_2\cos(3\Omega x) + C_3\cos(5\Omega x) + C_4\cos(7\Omega x)$ and the appropriate initial conditions are $y(0) = C_0$, $y'(0) = 0$, where $\Omega = 1.01$, $B = 0.002$, $C_0 = 0.200426728069$, $C_1 = 0.200179477536$, $C_2 = 0.246946143 \times 10^{-3}$, $C_3 = 0.304016 \times 10^{-6}$, $C_4 = 0.374 \times 10^{-9}$.
P-stable Obrechkoff methods of order 12 each were used by Simos [34], Wang *et al.*, [38] and Van Daele and Van Berghe [35] to solve the Nonlinear Duffing Equation in the

interval $\left[0, \frac{40.5\pi}{1.01}\right]$. Similarly, Archar [36] and Shokri and Saadat [39] the problem in the same interval with symmetric Obrechkoff method and trigonometrically fitted method each of order 12 respectively. The absolute errors of the MBTFM at $x = \frac{40.5\pi}{1.01}$, in comparison with the methods mentioned above are presented in the Table 2 while the CPU time for each of the methods are listed in Table 3.

h	Simos	Wang *et al.*	Daele & Berghe	Achar	Shokri & Saadat	**MBTFM**
$\frac{m}{500}$	3.15×10^{-4}	4.08×10^{-5}	4.06×10^{-5}	4.09×10^{-5}	6.09×10^{-12}	2.07×10^{-10}
$\frac{m}{1000}$	1.81×10^{-5}	1.27×10^{-6}	1.87×10^{-6}	1.27×10^{-6}	7.99×10^{-12}	1.64×10^{-12}
$\frac{m}{2000}$	1.08×10^{-6}	3.93×10^{-8}	3.84×10^{-8}	3.94×10^{-8}	5.52×10^{-12}	1.28×10^{-12}
$\frac{m}{3000}$	2.09×10^{-7}	5.17×10^{-9}	5.13×10^{-9}	5.18×10^{-9}	7.27×10^{-12}	1.77×10^{-12}
$\frac{m}{4000}$	6.55×10^{-8}	1.23×10^{-9}	3.19×10^{-9}	1.23×10^{-9}	6.99×10^{-12}	1.66×10^{-12}
$\frac{m}{5000}$	2.67×10^{-8}	4.07×10^{-10}	9.89×10^{-10}	4.09×10^{-10}	6.65×10^{-12}	1.59×10^{-12}

Table 2: Comparison of the End Point Absolute error for $m = \frac{40.5\pi}{1.01}$

h	Simos	Wang *et al.*	Daele & Berghe	Achar	Shokri & Saadat	**MBTFM**
$\frac{m}{500}$	1.437	1.406	1.484	1.188	1.453	4.849
$\frac{m}{1000}$	2.892	2.891	2.938	2.312	2.874	6.844
$\frac{m}{2000}$	6.233	6.236	6.360	4.812	6.267	8.792
$\frac{m}{3000}$	9.859	9.546	9.719	7.548	9.859	9.958
$\frac{m}{4000}$	13.548	13.063	13.390	9.986	13.424	13.365
$\frac{m}{5000}$	16.922	16.499	16.969	12.860	16.857	16.917

Table 3: Comparison of CPU Time

From Tables 2 and 3, and Figure 3 it is clear that **MBTFM** is more efficient than the methods of Simos [34], Wang *et al.*, [38], Van Daele and Van Berghe [35], Archar [36] and Shokri and Saadat [39].

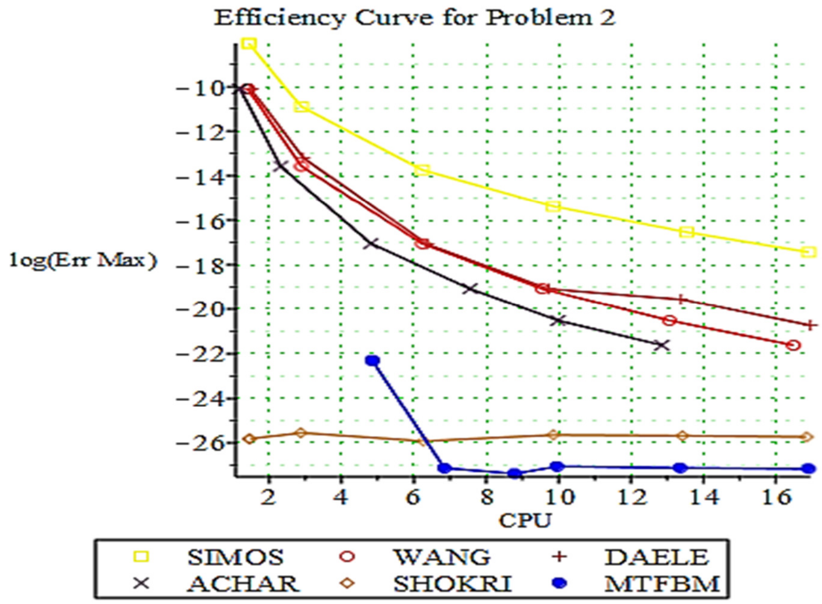

Figure 3: Efficiency curve for Problem 2

Example 3: Test like Equation

Consider the test-like equation with double frequencies given by
$y''(x) + \omega^2 y(x) = 12\cos(x)$, $y(0) = 1$, $y'(0) = 0$ whose solution in closed form is given by
$y(x) = \frac{\cos(5x)+\cos(x)}{2}$. This equation was solved by Wang [40] with $P-$stable linear symmetric multistep method of order 8 and hence a better comparison with **MBTFM** of the same order. The maximum absolute errors are compared with MBTFM in the interval $[0, 500\pi]$ and are presented in Table 4.

Method		N			
		1000	2000	3000	4000
MBTFM	Error	2.26×10^{-4}	4.90×10^{-10}	1.52×10^{-11}	1.49×10^{-12}
	NFE	2002	4002	6002	8002
Six-step order 8	Error	9.00×10^{-3}	1.00×10^{-5}	3.00×10^{-7}	3.00×10^{-8}
	NFE	6001	12001	18001	24001

Table 4: Comparison of Maximum Errors

It is obvious from Table 4 and Figure 4 that although MBTFM and Six-step method of Wang [40] are of the same order, MBTFM is a more efficient integrator for this problem.

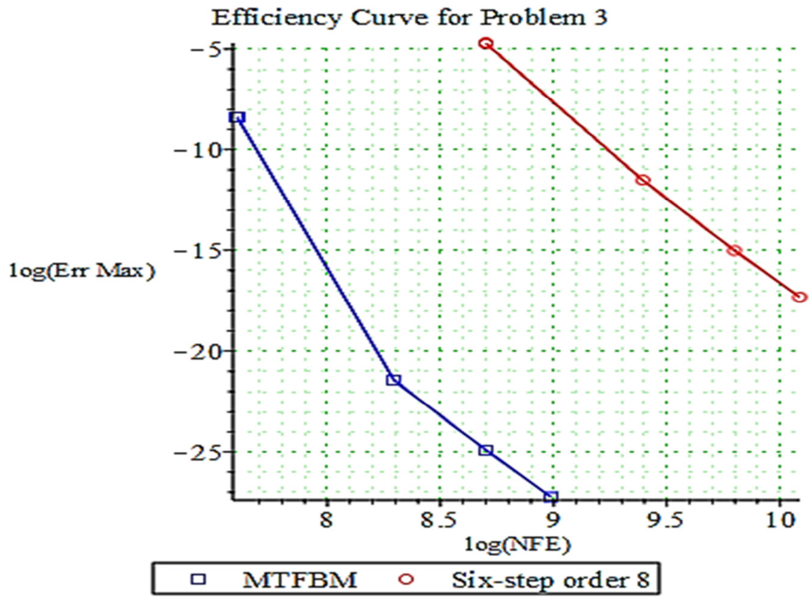

Figure 4: Efficiency curve for Problem 3

Example 4: Linear Kramarz Problem (Kramarz, 1980)

Consider the linear non autonomous stiff problem given by $y''(x) = \begin{bmatrix} 2498 & 4998 \\ -2499 & -4999 \end{bmatrix} y(x)$,
$y(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, y'(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $0 \le x \le 100$. The analytical solution is given by $y(x) = [2\cos(x), -\cos(x)]^T$. Nguyen *et al.* [27] considered an order 6 Trigonometric Implicit Runge-Kutta (TIRK3) for the numerical integration of the problem. The newly developed MBTFM is compared with TIRK3 and the end point global errors, Number of Functions Evaluation and the CPU time are presented in Table 5.

Methods	N	Errors	NFE	CPU Time	N	Errors	NFE	CPU Time	N	Errors	NFE	CPU Time
MBTFM	3	3.5E-28	20	0.046	6	1.7E-27	38	0.063	25	9.2E-28	152	0.297
TIRK3	3	3.3E-12	327	0.29	142	9.0E-12	707	0.501	170	3.7E-12	811	0.591

Table 5: Comparison of End Point Global Errors

In Table 5, MBTFM is not just accurate method for Linear Kramarz Problem but also efficient as the cost of implementation is very low compare to TIRK3. This is evident in Figures 5 and 6 respectively.

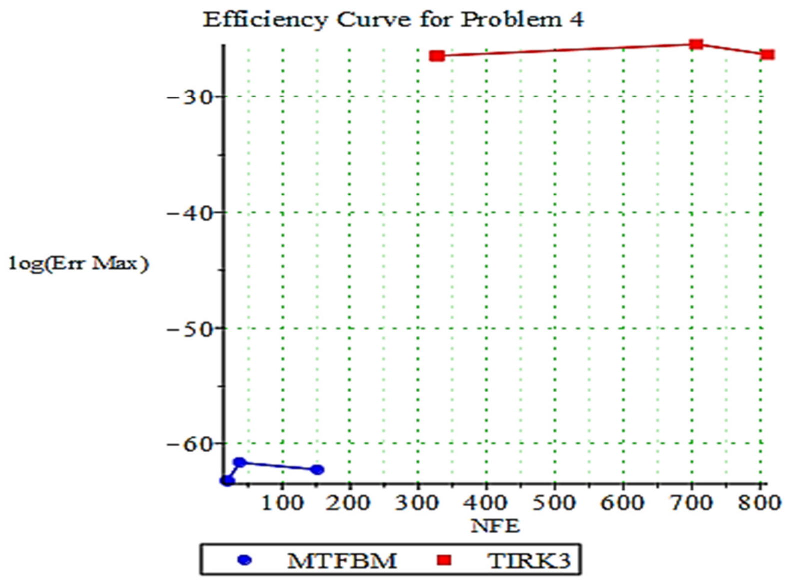

Figure 5: Efficiency curve for Problem 4

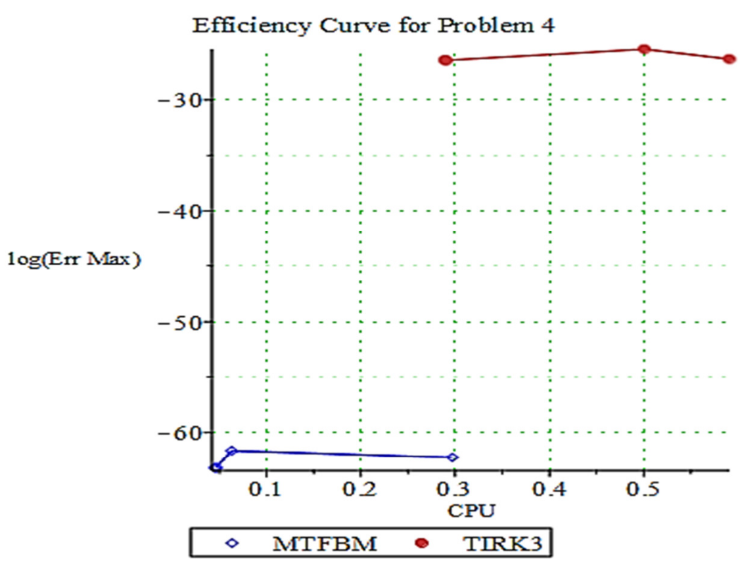

Figure 6: Efficiency curve for Problem 4

Example 5 (Franco, 2006)

We also consider the oscillatory linear system $y''(x) + \begin{bmatrix} 13 & -12 \\ -12 & 13 \end{bmatrix} y(x) = \begin{bmatrix} 9\cos 2x - 12\sin 2x \\ -12\cos 2x + 9\sin 2x \end{bmatrix}$ with initial conditions $y(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $y'(0) = \begin{bmatrix} -4 \\ 8 \end{bmatrix}$, whose exact solution is $y(x) = \begin{bmatrix} \sin x - \sin 5x + \cos 2x \\ \sin x + \sin 5x + \sin 2x \end{bmatrix}$.

This problem was solved by Franco [41] in the interval $[0,100]$ for $h = \frac{1}{2^i}$ $i \geq 2$ using order 6 Explicit Two Step Hybrid Method (ETSHM6). Table 6 displays the maximum absolute error of MBTFM in comparison with **ETSHM6**.

h	MBTFM		**ETSHM6**	
	Max Err.	NFE	Max Err.	NFE
$\frac{1}{4}$	7.15×10^{-7}	802	1.00×10^{-2}	3000
$\frac{1}{8}$	2.69×10^{-9}	1602	3.16×10^{-5}	4500
$\frac{1}{16}$	1.06×10^{-11}	3202	3.16×10^{-7}	6000
$\frac{1}{32}$	4.14×10^{-14}	6402	3.16×10^{-9}	13500

Table 6: Comparison of Maximum Errors and Number of Functions Evaluation

As expected, the MBTFM being a higher order method than **ETSHM6** of Franco [41] is more efficient as contained in Table 6 and Figure 7.

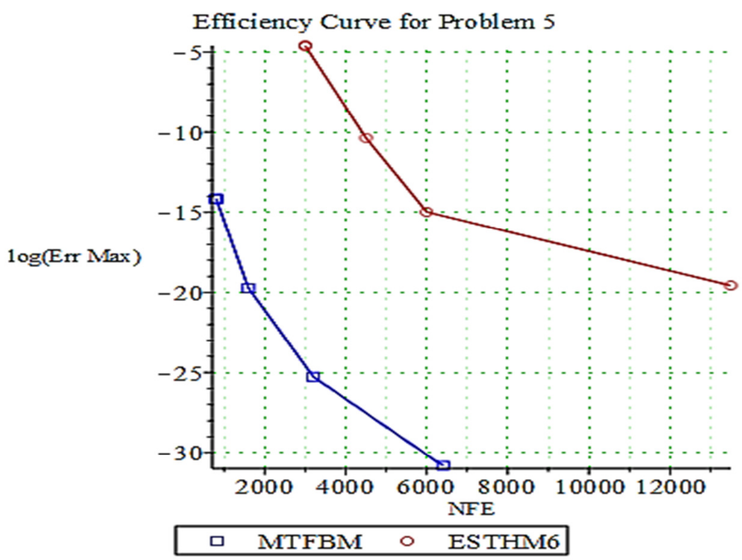

Figure 7: Efficiency curve for Problem 5

Problem 6: Non Linear Perturbed Systems (Fang *et al.* 2009)

Consider the nonlinear perturbed system on the range $[0, 10]$ with $\epsilon = 10^{-3}$.

$$y_1'' = \epsilon\varphi_1(x) - 25y_1 - \epsilon(y_1^2 + y_2^2) \quad y_1(0) = 1 \ , \ y_1'(0) = 0$$
$$y_2'' = \epsilon\varphi_2(x) - 25y_2 - \epsilon(y_1^2 + y_2^2) \quad y_2(0) = \epsilon \ , \ y_2'(0) = 5$$

where

$$\varphi_1(x) = 1 + \epsilon^2 + 2\epsilon\sin(5x + x^2) + 2\cos(x^2) + (25 - 4x^2)\sin(x^2)$$
$$\varphi_2(x) = 1 + \epsilon^2 + 2\epsilon\sin(5x + x^2) - 2\sin(x^2) + (25 - 4x^2)\cos(x^2)$$

The exact solution is given by $y_1(x) = \cos(5x) + \epsilon\sin(x^2)$, $y_2(x) = \sin(5x) + \epsilon\cos(x^2)$ which according for Fang *et al.* [42] represents a periodic motion of constant frequency with small perturbation of variable frequency. As selected by [42] and Ngwane and Jator [43], we choose $\omega = 5$ and the numerical results of the maximum global errors of MBTFM were compared with Block Hybrid Trigonometrically fitted BHT of Ngwane and Jator (2015) and Trigonometrically Fitted Adapted Runge-Kutta Nystrom TFARKN 5(3) of Fang *et al.* (2009) as presented in Table 7.

MBTFM		BHT		TFARKN 5(3)	
N	$-\log_{10}(Err)$	N	$-\log_{10}(Err)$	$N(rejected)$	$-\log_{10}(Err)$
50	10.95	50	3.42	29(6)	2.78
100	14.71	100	4.61	88(9)	5.33
260	17.26	260	7.52	262(8)	7.85
810	22.20	810	10.43	811(4)	10.38

Table 7: Comparison of log of Maximum Errors and Number Steps

From Table 7 and Figure 8 it can be seen that MBTFM outperformed BHT which is implemented in a corresponding fixed step size mode and TFARKN 5(3) which is implemented in variable step size mode respectively.

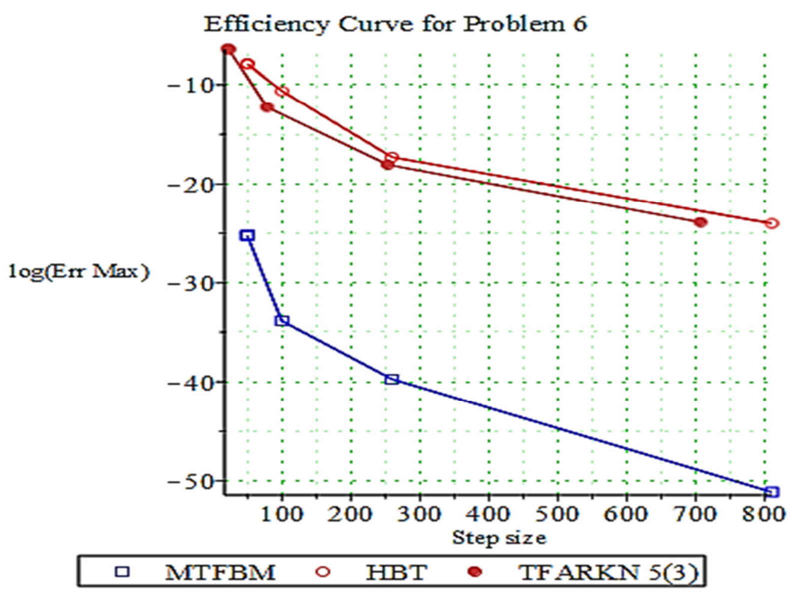

Figure 8: Efficiency curve for Problem 6

5 Conclusion

We considered a maximal order trigonometrically fitted method for the solution of second order initial value problems with oscillatory solution in this paper. The algorithm is self-starting, has good accuracy and required only six functions evaluation at each integration step except the first integration step that required eight functions evaluation. Representative numerical examples that are linear and nonlinear and highly oscillatory were presented. The numerical examples considered showed that **MBTFM** is an accurate and efficient integrator as presented in tables 1-7 and Figures 2-8.

Conflict of Interest: The authors declare that they have no conflict of interest.

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